The McKay correspondence, noncommutative resolutions of singularities, and reflection groups

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These lectures should serve as an introduction to a fascinating research topic at the crossroads of representation theory, algebraic geometry, and homological (commutative) algebra.

In 1979 John McKay observed an astonishing direct connection between the resolution of singularities rational double points (aka ADE-surface singularities) and finite subgroups of SU(2). This correspondence was explained in geometric terms by Gonzales-Sprinberg–Verdier, Artin–Verdier, Esnault, Knörrer, and others. A few years later, 1986, Maurice Auslander found an algebraic explanation of the correspondence, which builds a bridge to the study of representations of Artin algebras and maximal Cohen–Macaulay modules over Cohen–Macaulay rings. Since then the correspondence has been generalized and interpreted in many different ways (e.g., as derived equivalence between certain categories). Auslander's result is the starting point for the flourishing area of noncommutative (crepant) resolutions of singularities, which were first formalized by Michel Van den Bergh in 2004.
We will review the classical McKay correspondence, then focus on Auslander's theorem and noncommutative resolutions of singularities. In particular, we will point out a surprising connection to finite complex reflection groups, that was found in recent joint work with Ragnar-Olaf Buchweitz and Colin Ingalls.

Below is a an extensive selection of references, which is only is only meant to motivate the reader to have a look at the relevant literature and become more familiar with the topics to be discussed. For the most part we will not assume familiarity with those references.

1. The classical McKay correspondence

1. Characterization of (extended) Dynkin diagrams: see [ARS97] and [Rei97].
2. Characterization of finite subgroups of SO(3) and SU(2), cf. [Cox91] and [BFI18a].
3. McKay graphs, see [FM81, McK80, GSV83]
4. Singularity theory: recall notion of quotient singularity, Kleinian singularity, (minimal) resolution of singularities, dual resolution graph. See [CLO07, dJP00, Lau71, Ném99]

2. Auslander's theorem and representation theory

1. Auslander's theorem, see [Aus86, Yos90, LW12].

2. Herzog's theorem and finite Cohen–Macaulay-type in dimension 2 (see [Aus86, Yos90, LW12], also see [Buc86] for maximal Cohen–Macaulay modules).
3. Preprojective algebras and McKay correspondence (see [CBH98], [RVdB89]).

3. McKay correspondence and NC(C)Rs

1. Some generalizations: The derived version [KV00], (homological) special McKay correspondence (see [IW10], [Wun88]), McKay correspondence for SL_3-quotients [BKR01].
2. Noncommutative crepant resolutions (NCCRs) á la Van den Bergh, cf. [VdB04, Leu12, Wem16].
3. NCRs after Dao–Iyama–Takahashi–Vial, cf. [DITV15, DFI15]. Connections to rational singularities.

4. More McKay correspondence

1. Reflection groups basics, cf. [Bou81, OT92].
2. McKay correspondence for reflection groups [BFI17, BFI18b].

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