Borel* Sets and Borel Reductions

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Generalised Descriptive Set Theory Basics

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- A set $A \subset 2^\kappa$ is *closed under isomorphism* if

$$\forall \eta \in A \forall \xi \in 2^\kappa (M_\xi \cong M_\eta \to \xi \in A)$$

Language $L_{\kappa^+\kappa}$

Disjunctions and conjunctions of size κ, quantification of size $< \kappa$.

(Lopez-Escobar-Vaught-Friedman-Hyttinen-Kulikov)

A set $A \subset 2^\kappa$ is Borel and closed under isomorphism if and only if the set $\{M_\eta \mid \eta \in A\}$ is definable in $L_{\kappa^+\kappa}$.

Trees

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- Both formulas in $L_{\kappa^+\kappa}$ and Borel sets have codes consisting of $\kappa^+\omega$-trees (blackoard.)

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- Borel sets have the property of Baire.

CUB-filter

(Halko-Shelah)

The CUB filter $\{\eta \in 2^\kappa \mid \eta^{-1}\{1\}$ *contains a cub*$\}$ *doesn't have the Property of Baire.*

Borel* and $M_{\kappa^+\kappa}$

A set A is Borel* if there is a $\kappa^+\kappa$-tree t with assignment $h\colon \ell(t) \to O(\kappa^\kappa)$ such that $\eta \in A$ iff Player *II* has a winning strategy in the game $G(t, h, \eta)$.

Borel* and $M_{\kappa^+\kappa}$

The language $M_{\kappa^+\kappa}$ consists of formulas defined by $\kappa^+\kappa$-trees in the same way as $L_{\kappa^+\kappa}$ is defined through $\kappa^+\omega$-trees.

Open Question

Is it consistent that a set A is Borel and closed under isomorphism iff it is definable in $M_{\kappa^+\kappa}$?*

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Δ_1^1, Σ_1^1

- (Friedman-Hyttinen-Kulikov) If $V = L$, then Borel*$= \Sigma_1^1$. There is a Wedge-reduction from every Σ_1^1-set to CUB.
- (Hyttinen-Kulikov) There is a $< \kappa$-closed κ^+-c.c. forcing such that in the extension the isomorphism on linear orders is not Borel*.

Open Question ($\kappa^{<\kappa} = \kappa$)

Is it consistent that Δ_1^1 = Borel?*

CUB-filter

The CUB-filter is a canonical example of a Borel* set that is not Borel.

Definition

Let $\lambda < \kappa$ and $A \subset \kappa$. In the cub-game $\mathrm{CUB}(\lambda, \kappa, A)$ the Players climb up the cardinal and *II* wins if they hit A at the limit.

Game characterisation

Huuskonen, Hyttinen, Rautila, (& Shelah)
We say that the λ-*game characterisation holds for* κ if for all $A \subset \kappa$

$$II \uparrow CUB(\lambda, \kappa, A) \iff A \text{ contains a } \lambda\text{-cub.}$$

Essentially λ*-game characterisation for* κ *can only fail if* $\kappa = \lambda^+$ *and* λ *is a Mahlo cardinal.*

CUB is a.a. Borel*

Suppose λ-game characterisation holds for κ. Then λ-CUB is a Borel* set with the tree of height $\lambda + 1$.

Conjecture

If $\kappa^{<\kappa} > \kappa$, then it is consistent that CUB is Borel, but not Σ^1_1. (Shelah-Väänänen proved that CUB can be definable in $L_{\kappa\kappa}$ when $\kappa^{<\kappa} > \kappa$).*

Borel Reductions

If E and E' are equivalence relations, a function $f\colon \kappa^\kappa \to \kappa^\kappa$ is a Borel reduction of E to E' if it is a Borel function and for all $\eta, \xi \in \kappa^\kappa$

$$(\eta, \xi) \in E \iff (f(\eta), f(\xi)) \in E'.$$

Research Program: Study the partial order $\leqslant_B$

Model theoretic motivation will become more clear in the next talk by Miguel Moreno.

Define E^{λ}_{μ} to be the equivalence relation on λ^{κ} where two functions are equivalent if they coincide in μ-cub.

Denote by $E^{\lambda}_{\mathrm{NS}}$ the same equivalence relation where they coincide on a cub (not μ-cub).

some results and questions

(Friedman-Hyttinen-Kulikov)

It is consistent that $E^2_\omega \leqslant E^2_{\omega_1}$

(Hyttinen-Kulikov)

If $V = L$*, then* E^κ_ω *is* Σ^1_1*-complete*

Open Question

Is it consistent that $E^3_\omega \leqslant_B E^2_\omega$ *or* $E^\kappa_\omega \leqslant_B E^2_\omega$*?*

Open Question

Is it consistent that $E^2_{\omega_1} \leqslant_B E^2_\omega$*?*

$E^{\kappa}_{\omega} \leqslant E^2_{\mathrm{NS}}$, $\kappa \geqslant \aleph_2$

Three independently discovered proofs of the consistency of $E^{\kappa}_{\omega} \leqslant E^2_{\mathrm{NS}}$ last weekend. I present one of them (mine).

Theorem

(Kulikov)

Suppose $V = L$ *and* κ *is weakly compact. Then*

$$E^{\kappa}_{\omega} \leqslant E^{2}_{\mathrm{NS}}$$

Corollary

If $V = L$ *and* κ *is weakly compact, then the equivalence on* $P(\kappa)$ *modulo* NS *is* Σ^1_1*-complete.*

Proof

Claim (Dual Diamond)

Suppose $V = L$ and κ is weakly compact. Then there is a sequence $\langle D_\alpha, f_\alpha \rangle_{\alpha<\kappa}$ such that

1. $D_\alpha \subset \alpha$ *is stationary in* α,
2. $f_\alpha \colon \alpha \to \alpha$,
3. *if* (Z, g) *is a pair such that* $Z \subset S^\kappa_\omega$ *is stationary and* $g \in \kappa^\kappa$, *then the set*

$$\{\alpha \in S^\kappa_{\text{reg}} \mid Z \cap \alpha = D_\alpha \text{ and } g \restriction \alpha = f_\alpha\}$$

is stationary.

Proof

Given dual diamond $\langle D_\alpha, f_\alpha \rangle_{\alpha<\kappa}$, define $f \colon \kappa^\kappa \to 2^\kappa$ by $f(\eta)(\alpha) = 1$ if $\alpha \in S^\kappa_{\mathrm{reg}}$ and $\eta \restriction \alpha$ is cub-equivalent to f_α; otherwise set to 0.

Blackboard.

Proof of the Claim

For the purpose of the proof we define triple $\langle D_\alpha, f_\alpha, C_\alpha\rangle$. Suppose $\langle D_\beta, f_\beta, C_\beta\rangle$ is already defined for $\beta < \alpha$.

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Define $\langle D, f, C\rangle$ to be the L-smallest such that

- $D \subset \alpha \cap S^\kappa_\omega$ is stationary,
- $f\colon \alpha \to \alpha$,
- $C \subset \alpha \cap S^\kappa_{\text{reg}}$ is cub,
- $\forall\beta < \alpha(D \cap \beta \neq D_\beta$ or $f \restriction \beta \neq f_\beta)$

and set $D_\alpha = D$, $f_\alpha = f$, $C_\alpha = C$ if such exists and otherwise $D_\alpha = f_\alpha = C_\alpha = \varnothing$.

Proof of the Claim

Counterassumption: (Z, g, C) is the L-least counterexample. Let M be elementary submodel of L_λ, $\lambda > \kappa$ s.t.

- $|M| < \kappa$
- $\alpha = M \cap \kappa \in C$,
- $Z \cap \alpha$ stationary in α,
- $Z, g, C, S^\kappa_\omega, S^\kappa_{\text{reg}}, \kappa \in M$

possible by the reflection property of weakly compact (as in the last lecture yesterday by Hazel Brickhill)

Proof of the Claim

Mostowski collapse $G\colon M \to L_\gamma$, $\gamma > \alpha$.
Now $G(Z) = Z \cap \alpha$, $G(g) = g \restriction \alpha$, $G(C) = C \cap \alpha$, $G(\kappa) = \alpha$.
The sequence $\langle D_\beta, f_\beta \rangle_{\beta<\alpha}$ is definable in L_γ

Proof of the Claim

Let $\varphi(Z, g, C, \kappa)$ say "(Z, g, C) is the L-least triple such that

- $Z \subset S^{\kappa}_{\omega}$ is stationary,
- $g \colon \kappa \to \kappa$,
- $C \subset \cap S^{\kappa}_{\text{reg}}$ is cub,
- $\forall \beta < \kappa (D \cap \beta \neq D_{\beta}$ or $f \restriction \beta \neq f_{\beta})$

But this formula relativises to L_{γ} and all notions are sufficiently absolute, so relativsed it says that (Z, g) reflects to $\alpha \in C$ which is a contradiction with the assumption that (Z, g, C) was a counterexample.