On the absoluteness of ω-orbital stability

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A formula/sentence/property ϕ is *absolute* for a given collection of structures if all structures in the class agree about the truth value of ϕ.

Theorem 1 (Shoenfield). *If ϕ is a unary Σ^1_2 formula, $x \subseteq \omega$ and M is a model of ZF containing ω_1^V, then $M \models \phi(x)$ if and only if $\phi(x)$.*

So : Σ^2_1 formulas are absolute between models of ZF containing ω_1^V.

Quote from a famous model theorist : the fundamental notions of model theory are absolute.

We say that formula ϕ is *forcing-absolute* for a model if the truth value of ϕ is invariant over forcing extensions.

"There is a nonconstructible real" is Σ^1_3 and not forcing-absolute for models of

$$\mathsf{ZFC} + V = L.$$

Theorem 2 (Woodin). *If there exist proper class many Woodin cardinals, then all statements of the form* $L(\mathbb{R}) \models \phi(x)$*, for* $x \in \mathbb{R}$*, are forcing-absolute.*

In particular, projective formulas are forcing-absolute, if there exist proper class many Woodin cardinals.

3 Definition (Shelah). An *abstract elementary class* (AEC) $\boldsymbol{k}$ (over some vocabulary τ is a pair $(\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ such that the following hold.

- $\boldsymbol{K}$ is a class of τ-structures closed under isomorphisms.

- $\preceq_{\boldsymbol{k}}$ is a partial order on $\boldsymbol{K}$.

- For all $M, N \in K$, if $M \preceq_{\boldsymbol{k}} N$ then M is a substructure of N;

- (Coherence) if $M_1 \preceq_{\boldsymbol{k}} M_3$, $M_2 \preceq_{\boldsymbol{k}} M_3$ and $M_1 \subseteq M_2$ then $M_1 \preceq_{\boldsymbol{k}} M_2$.

- (Tarski-Vaught property) if γ is an ordinal and

$$\langle M_\alpha : \alpha < \gamma \rangle$$

is $\preceq_{\boldsymbol{k}}$-increasing then $\bigcup_{\alpha<\gamma} M_\alpha \in \boldsymbol{K}$,

$$M_0 \preceq_{\boldsymbol{k}} \bigcup_{\alpha<\gamma} M_\alpha$$

and, for any $N \in \boldsymbol{K}$, if

$$M_\alpha \preceq_{\boldsymbol{k}} N$$

for all $\alpha < \gamma$ then $\bigcup_{\alpha<\gamma} M_\alpha \preceq_{\boldsymbol{k}} N$.

- (Löwenheim-Skolem property) There is a cardinal

$$\mu \geq |\tau| + \aleph_0$$

such that for all $M \in \boldsymbol{K}$ and all $A \subseteq M$ there is an $N \in \boldsymbol{K}$ such that $A \subseteq N$,

$$|N| \leq |A| + \mu$$

and $N \preceq_{\boldsymbol{k}} M$. The cardinal μ is called the Löwenheim-Skolem number of k ($LS(\boldsymbol{k})$).

Fixing some coding of hereditarily countable sets by reals, we say that AEC $\boldsymbol{k}$ is *analytically presented* if $LS(\boldsymbol{k}) = \aleph_0$ and the two following sets are analytic :

- the set of codes for countable members of $\boldsymbol{K}$, and
- the set of codes for pairs (M, N), where M and N are both countable and $M \preceq_{\boldsymbol{k}} N$.

Example: for a countable vocabulary τ, then set of models of a sentence in $L_{\omega_1,\omega}(\tau)$, with the order of elementarity with respect to a fixed countable fragment of $L_{\omega_1,\omega}(\tau)$, is an analytically presented AEC. In this case both sets are Borel.

4 Question. For a countable vocabulary τ and a sentence ϕ of $L_{\omega_1,\omega}(\tau)$, must the statement that ϕ is $\aleph_1$-categorical be absolute (between models of ZFC containing the ordinals)?

Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq)$, a $\boldsymbol{k}$-embedding is a an injection

$$f\colon M \to N$$

(for some M, N in K) such that $f[M] \in \boldsymbol{K}$ and

$$f[M] \preceq_{\boldsymbol{k}} N.$$

An AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is said to have the joint embedding property (JEP) if $\preceq_{\boldsymbol{k}}$ is directed; i.e., if any two members of $\boldsymbol{K}$ $\preceq_{\boldsymbol{k}}$-embed into a third.

An AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is said to satisfy *amalgamation* if for all M, N, P in K such that

$$M \preceq_{\boldsymbol{k}} N$$

and

$$M \preceq_{\boldsymbol{k}} P,$$

there exist a $Q \in \boldsymbol{K}$ and $\preceq_{\boldsymbol{k}}$-embeddings of N and P into Q which are the identity function on M.

Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq)$, let T be the set of triples (M, a, N) where $M \preceq_{\boldsymbol{k}} N$ are in $\boldsymbol{K}$ and $a \in N \setminus M$. Given (M, a, N), (P, b, Q) in T, say that

$$(M, a, N) \sim (P, b, Q)$$

if $M = P$ and there exist a $Q \in \boldsymbol{K}$ and $\preceq$-embeddings of $f \colon N \to Q$ and $g \colon P \to Q$ which are the identity function on M such that $f(a) = g(b)$.

If k satisfies amalgamation, then $\sim$ is transitive. In any case, taking the transitive closure of $\sim$ gives an equivalence relation, and the equivalence classes are called orbital (or Galois) types. Fixing the first model M, we can talk of orbital types over M.

Theorem 5 (Burgess). *If E is an analytic equivalence relation on $\mathbb{R}$, then E has either countably many equivalence classes, exactly $\aleph_1$ many, or a perfect set of inequivalent reals.*

We call an analytically presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ *orbitally ω-stable* if there are just countably many orbital types over each countable M in K.

If over no countable $M \in K$ are there perfectly many types, we say that k is *orbitally almost ω-stable*.

In the remaining case, we say that k is *orbitally ω-unstable*.

We can generalize the notion of orbital types to finite tuples (i.e., sequences of the form $(M, a_1, \dots, a_n, N)$).

By a theorem of Boney, for a given AEC $\boldsymbol{k}$ the number of orbital types (in the sense of the previous slide) does not change if we do this.

The assertion that an analytically presented AEC k is orbitally ω-unstable is Σ^1_2 in a code for $\boldsymbol{k}$.

6 Question. Is orbital ω-stability absolute for analytically presented AEC's (between models of ZFC containing the ordinals)?

For a given analytically presented AEC, orbital ω-stability is a Π^1_4 statement in a real parameter.

Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$, a subclass $\boldsymbol{K}'$ of $\boldsymbol{K}$ and an $M \in \boldsymbol{K}'$, we say that M is $\preceq_{\boldsymbol{k}}$-*universal* for $\boldsymbol{K}'$ if for each $N \in \boldsymbol{K}'$ there is a $\preceq_{\boldsymbol{k}}$-embedding of N into M.

M is $\preceq_{\boldsymbol{k}}$-*maximal* for $\boldsymbol{K}'$ if there does not exist an $N \in \boldsymbol{K}'$ (other than M) such that $M \preceq_{\boldsymbol{k}} N$.

For an $\aleph_0$-presented AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ the following are absolute.

- The statement that $\boldsymbol{K}_{\aleph_0}$ is nonempty (Σ^1_1 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).

- The statement that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies amalgamation (Π^1_2 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).

- The statement that $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ satisfies joint embedding (Π^1_2 in a code for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$).

- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, M is a $\preceq_{\boldsymbol{k}}$-universal member of $\boldsymbol{K}_{\aleph_0}$ (Π^1_2 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ and M).

- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, M is a $\preceq_{\boldsymbol{k}}$-maximal member of $\boldsymbol{K}_{\aleph_0}$ (Π^1_1 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ and M).

- For a fixed $M \in \boldsymbol{K}_{\aleph_0}$, and a fixed countable set of pairs (a, N) with $N \in \boldsymbol{K}_{\aleph_0}$, $M \preceq_{\boldsymbol{k}} N$ and $a \in N \setminus M$, the statement that every orbital type over M contains a member of the set (Π^1_2 in codes for $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$, M and the set) .

Theorem 7 (Baldwin-Larson)**.** *If $\boldsymbol{k}$ is an analytically presented almost orbitally ω-stable AEC satisfying amalgamation (for countable structures) then the assertion that $\boldsymbol{k}$ is $\aleph_1$-categorical is Π^1_2 in a code for $\boldsymbol{k}$.*

Theorem 8 (Larson-Shelah). *If $\boldsymbol{k}$ is an analytically presented almost orbitally ω-stable AEC satisfying amalgamation and joint embedding (for countable structures) then the assertion that $\boldsymbol{k}$ is orbitally ω-stable is upwards absolute.*

Theorem 9 (Larson-Shelah). *There is an analytically presented AEC satisfying amalgamation but not joint embedding which is orbitally ω-stable if and only if either $\mathbb{R} \subseteq L$ or ω_1^L is countable.*

The Counterexample

Let T_L be the theory of the structure $\langle L_{\omega_1^L}, \in \rangle$.

10 Fact (H. Friedman). If M is a countable illfounded ω-model of T_L then the ordinals of M have ordertype $\alpha + (\mathbb{Q} \times \alpha)$ for some ordinal $\alpha < \omega_1^L$, where $\mathbb{Q} \times \alpha$ is given the lexicographical order.

Let τ be the vocabulary consisting of $=$, binary symbols E and $<$, and unary symbols W_n, for each $n \in \omega$.

Let $\boldsymbol{K}^\tau$ be the class of τ-structures M of the form

$$\langle |M|, E^M, <^M, W_n^M; n \in \omega \rangle$$

such that

- E^M is an equivalence relation on $|M|$ and $<^M$ is a subset of E^M;

- each W_n^M is either the empty set or all of $|M|$;

- for each $a \in |M|$, there exists an ω-model N of T such that
  - the ordinals of N are $[a]_{E^M}$, and
  $$<^M \cap([a]_{E^M})^2$$
  is the corresponding ordering,
  - $\{n \in \omega : W_n^M \neq \emptyset\}$ is not a member of N (i.e., for no $w \in N$ is it true that $N \models w \subseteq \omega$ and, for all $n \in \omega$, that $N \models$ "the n-th member of ω is in w" if and only if $W_n^M \neq \emptyset$).

Given $M, N \in \boldsymbol{K}^\tau$, let $M \preceq_{\boldsymbol{k}^\tau} N$ if $|M| \subseteq |N|$,
$$E^M = E^N \cap (|M|)^2,$$
$$<^M = <^N \cap(|M| \times |M|),$$
each E^N-equivalence class is either contained in or disjoint from $|M|$ and, for each $n \in \omega$, $W_n^N = \emptyset$ if and only if $W_n^M = \emptyset$.

Theorem 11. *Suppose that* $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ *is an abstract elementary class such that*

- $\boldsymbol{K}_{\aleph_0} \neq \emptyset$;
- $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ *satisfies the joint embedding property;*
- *for each* $M \in \boldsymbol{K}_{\aleph_0}$*, the set of orbital types over* M *(for finite tuples) is countable.*

Then $(\boldsymbol{K}_{\aleph_0}, \preceq_{\boldsymbol{k}})$ *has a universal element.*

If $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ is an abstract elementary class satisfying amalgamation, then whenever

$$M \preceq_{\boldsymbol{k}} N,$$

the orbital types over M inject into the orbital types over N.

Theorem 12. *Suppose that* $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ *is an AEC satisfying amalgamation and the joint embedding property, for which* $\boldsymbol{K}_{\aleph_0}$ *is nonempty. Then* $\boldsymbol{K}$ *is orbitally* ω*-stable if and only if* $\boldsymbol{K}_{\aleph_0}$ *has a* $\preceq_{\boldsymbol{k}}$*-universal member over which there are only countably many orbital types.*

The latter statement is Σ^1_3 in a real parameter.

Model-Theoretic Forcing

Given a class $\boldsymbol{K}$ and a cardinal λ, we write $\boldsymbol{K}_\lambda$ for the class of members of $\boldsymbol{K}$ of cardinality λ.

13 Definition. Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ with vocabulary τ, we define $N \Vdash_{\boldsymbol{k}} \phi(\bar{a})$, for $N \in \boldsymbol{K}_{\aleph_0}$, $\bar{a}$ a finite sequence of elements of N, and ϕ a formula of $L_{\aleph_1,\aleph_0}(\tau)$, as follows.

1. For atomic ϕ, $N \Vdash_{\boldsymbol{k}} \phi(\bar{a})$ if $N \models \phi(\bar{a})$.

2. If $\phi = \bigwedge_{i\in\omega} \phi_i$, then $N \Vdash_{\boldsymbol{k}} \phi(\bar{a})$ if $N \Vdash_{\boldsymbol{k}} \phi_i(\bar{a})$ for each $i \in \omega$.

3. If $\phi = \neg\psi$, then $N \Vdash_{\boldsymbol{k}} \phi(\bar{a})$ if there is no $N' \in \boldsymbol{K}_{\aleph_0}$ such that

$$N \preceq_{\boldsymbol{k}} N'$$

   and $N' \Vdash_{\boldsymbol{k}} \psi(\bar{a})$.

4. If $\phi = \exists x \psi$, then $N \Vdash_{\boldsymbol{k}} \phi(\bar{a})$ if for each

$$N_1 \in \boldsymbol{K}_{\aleph_0}$$

with $N \preceq_{\boldsymbol{k}} N_1$ there exist an $N_2 \in \boldsymbol{K}_{\aleph_0}$ and a $b \in N_2$ such that

$$N_1 \preceq_{\boldsymbol{k}} N_2$$

and $N_2 \Vdash_{\boldsymbol{k}} \psi(b, \bar{a})$.

14 Definition. Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$, we say that a set $S \subseteq \boldsymbol{K}_{\aleph_0}$ is *dense* if for every $M \in \boldsymbol{K}_{\aleph_0}$ there exists an $N \in S$ such that $M \preceq_{\boldsymbol{k}} N$, and *open* if whenever $M \preceq_{\boldsymbol{k}} N$ and $N \in S$, then $M \in S$.

15 Definition. Given $L \subseteq L_{\aleph_1,\aleph_0}(\tau)$, $M \in \boldsymbol{K}_{\aleph_0}$ and a finite tuple $\bar{a}$ from M, we say that M *decides the L-type of $\bar{a}$* if for each $|\bar{a}|$-ary formula $\phi \in L$, one of $\phi(\bar{a})$ and $\neg\phi(\bar{a})$ is forced by M.

If L is countable, densely many elements of $\boldsymbol{K}_{\aleph_0}$ decide the L-type of each of their finite tuples.

16 Definition. A function F on $\boldsymbol{K}_{\aleph_0}$ is a *genericity function* if the range of F consists of dense open subsets of $\boldsymbol{K}_{\aleph_0}$.

17 Definition. Given a genericity function F, a limit ordinal γ and a $\preceq_{\boldsymbol{k}}$ increasing sequence

$$\bar{M} = \langle M_\alpha : \alpha < \gamma \rangle$$

of elements of $\boldsymbol{K}_{\aleph_0}$, $\bar{M}$ is F-*generic* if for each $\alpha < \gamma$ there exists a $\beta \in [\alpha, \gamma)$ such that $M_\beta \in F(M_\alpha)$.

18 Definition. A structure M (in $\boldsymbol{K}_{\aleph_0} \cup \boldsymbol{K}_{\aleph_1}$) is F-*generic* if it is the union of an F-generic sequence.

The following is easily verified by induction on complexity of formulas.

19 Fact. Let $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ be an AEC over a vocabulary τ without maximal countable models, and let n be an element of ω. For each formula $\phi \in L_{\aleph_1,\aleph_0}(\tau)$, there is a genericity function F_ϕ on $\boldsymbol{K}_{\aleph_0}$ such that all F_ϕ-generic elements of $\boldsymbol{K}_{\aleph_1}$ satisfy ϕ.

20 Example. Let $\boldsymbol{K}$ be the class of linear orders, and define $\preceq_{\boldsymbol{K}}$ on $\boldsymbol{K}$ by setting $M \preceq_{\boldsymbol{k}} N$ if and only if M is a suborder of N. There is a genericity function F such that all F-generic models in $\boldsymbol{K}_{\aleph_1}$ are dense and without endpoint. Furthermore, every element of $K_{\aleph_0}$ forces the statement $\forall x \forall y\, x < y \to \exists z x < z \wedge z < y$, even though the set of structures which do not satisfy this statement is dense in $K_{\aleph_0}$.

If $\boldsymbol{K}_{\aleph_0}$ satisfies joint embedding, then for each sentence ϕ of $L_{\omega_1,\omega}(\tau)$, either all $M \in \boldsymbol{K}_{\aleph_0}$ force ϕ or all $M \in \boldsymbol{K}_{\aleph_0}$ force $\neg\phi$.

If $\boldsymbol{K}_{\aleph_0}$ satifies amalgamation, then each element of $\boldsymbol{K}_{\aleph_0}$ decides the $L_{\omega_1,\omega}(\tau)$-theory of all if its finite tuples.

21 Definition (Scott). Given an infinite τ-structure M over a relational vocabulary τ, we define for each finite ordered tuple

$$\bar{a} = \langle a_0, \ldots, a_{|\bar{a}|-1} \rangle$$

of distinct elements of M and each ordinal α the $|\bar{a}|$-ary $L_{\infty,\aleph_0}(\tau)$-formula $\phi^M_{\bar{a},\alpha}$, recursively on α, as follows.

1. Each formula $\phi^M_{\bar{a},0}$ is the conjunction of all expressions of the two following forms:

   - $R(x_{f(0)}, \ldots, x_{f(k-1)})$, for R a k-ary relation symbol from τ and f a function from k to $|a|$, such that

     $$M \models R(a_{f(0)}, \ldots, a_{f(k-1)}),$$

   - $\neg R(x_{f(0)}, \ldots, x_{f(k-1)})$, for R a k-ary relation symbol from τ and f a function from k to $|a|$, such that

     $$M \models \neg R(a_{f(0)}, \ldots, a_{f(k-1)}).$$

2. Each formula $\phi^M_{\bar{a},\alpha+1}$ is the conjunction of the following three formulas:

   - $\phi^M_{\bar{a},\alpha}$,
   - $\bigwedge_{c\in M\setminus\{a_0,\ldots,a_{|\bar{a}|-1}\}} \exists x_{|\bar{a}|} \phi^M_{\bar{a}^\frown\langle c\rangle,\alpha}$,
   - $\forall x_{|\bar{a}|} \notin \{x_0,\ldots,x_{|a|-1}\}$
     $\bigvee_{c\in M\setminus\{a_0,\ldots,a_{|\bar{a}|-1}\}} \phi^M_{\bar{a}^\frown\langle c\rangle,\alpha}$.

3. For limit ordinals β, $\phi^M_{\bar{a},\beta} = \bigwedge_{\alpha<\beta} \phi^M_{\bar{a},\alpha}$.

We call $\phi^M_{\bar{a},\alpha}$ the *Scott formula* of $\bar{a}$ *in* M *at level* α.

The Scott rank of M is the least ordinal M such that for all finite tuples $\bar{a}$, $\bar{b}$ from M, $\phi^M_{\bar{a},\alpha} = \phi^M_{\bar{b},\alpha}$ implies $\phi^M_{\bar{a},\alpha+1} = \phi^M_{\bar{b},\alpha+1}$.

If M is countable, the Scott rank of M is countable, and, if α is the Scott rank of M, then all models of $\phi^M_{\langle\rangle,\alpha+\omega}$ are isomorphic to M.

For each $\alpha < \omega_1$, let S_α be the set of formulas of the form $\phi^M_{\bar{a},\alpha}$, for some τ-structure M.

In each τ-structure M, each finite tuple $\bar{a}$ satisfies exactly one of these formulas.

For each $\alpha < \omega_1$, let S'_α be the set of formulas in S_α which are forced to hold (for some tuples of M) by an $M \in \boldsymbol{K}_{\aleph_0}$.

If $\boldsymbol{k}$ is orbitally ω-stable, each S'_α is countable.

Again assuming that $\boldsymbol{k}$ is orbitally ω-stable, any element N of $\boldsymbol{K}_{\aleph_1}$ which is sufficiently generic for $\bigcup_{\alpha<\omega_1} S'_\alpha$ has countable Scott rank.

To see this, note that if A is an uncountable subset of ω_1, and, for each $\alpha \in A$, $\bar{a}_\alpha$, $\bar{b}_\alpha$ are such that $\phi^N_{\bar{a},\alpha} = \phi^N_{\bar{b},\alpha}$ but $\phi^N_{\bar{a},\alpha+1} \neq \phi^N_{\bar{b},\alpha+1}$, then formulas $\phi^N_{\bar{a}^\frown\bar{b},\alpha+1}$ $(\alpha \in N)$ are each forced, but mutually incompatible.

It follows that there an $M \in \boldsymbol{K}_{\aleph_0}$ with the same Scott sentence as N, and M is desired universal model.

Can this approach be used to analyze the almost orbitally ω-stable case?

Let $\boldsymbol{k} = (\boldsymbol{K}, \preceq_{\boldsymbol{k}})$ be an analytically presented orbitally ω-stable AEC satisfying amalgamation and joint embedding. The question of the downwards absoluteness of orbital ω-stability for such $\boldsymbol{k}$ remains open.

Suppose that $\boldsymbol{k}$ is not orbitally ω-stable, but that in some forcing extension it is. Then it is in any further forcing extension as well, so there is a homogeneous forcing extension in which orbital ω-stability holds, and there is a universal countable model in this extension. The $L_{\omega_1,\omega}(\tau)$-theory of the universal model given by Theorem 11 is definable, and thus exists already in the ground model.

In the case where $Col(\omega, \omega_1)$ forces the existence of a universal countable model, a model of this theory exists already in the ground model, by an argument of Harrington.

In the ground model this model has cardinality $\aleph_1$, and there exists a $\preceq_{\boldsymbol{k}}$-increasing ω_1-chain of models in $\boldsymbol{K}_{\aleph_0}$ such that $\boldsymbol{K}_{\aleph_0}$ is the set of τ-structures which $\preceq_{\boldsymbol{k}}$-embed into a member of this sequence.