Completeness of infinitary intuitionistic logics

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1. $$\bigwedge_{\alpha<\gamma} \phi_\alpha, \qquad \bigvee_{\alpha<\gamma} \phi_\alpha$$
2. $$\forall \mathbf{x}_\gamma \phi, \qquad \exists \mathbf{x}_\gamma \phi$$

   (where $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$)

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  $$\bigwedge_{\alpha<\gamma} \forall_{\beta<\alpha} \mathbf{x}_\beta \exists \mathbf{x}_\alpha \psi_\alpha \to \exists_{\alpha<\gamma} \mathbf{x}_\alpha \bigwedge_{\alpha<\gamma} \psi_\alpha$$

  (for disjoint $\mathbf{x}_\alpha$ and such that no variable in $\mathbf{x}_\alpha$ is free in ψ_β for $\beta < \alpha$).

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  (for disjoint $\mathbf{x}_\alpha$ and such that no variable in $\mathbf{x}_\alpha$ is free in ψ_β for $\beta < \alpha$).
- Completeness theorem proved using Boolean algebraic methods and thus relies heavily in the use of the excluded middle axiom.

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As an example, it follows by results of de Jongh (1980) that the set of non-equivalent propositional formulas of κ-propositional intuitionistic logic with two atoms has cardinality κ.

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    - Substitution rule:

$$\frac{\phi \vdash_{\mathbf{x}} \psi}{\phi[\mathbf{s}/\mathbf{x}] \vdash_{\mathbf{y}} \psi[\mathbf{s}/\mathbf{x}]}$$

where **y** is a string of variables including all variables occurring in the string of terms **s**.

κ-first-order systems

- Cut rule:

$$\frac{\phi \vdash_{\mathbf{x}} \psi \qquad \psi \vdash_{\mathbf{x}} \theta}{\phi \vdash_{\mathbf{x}} \theta}$$

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  - $$\top \vdash_x x = x$$
  - $$(\mathbf{x} = \mathbf{y}) \wedge \phi \vdash_{\mathbf{z}} \phi[\mathbf{y}/\mathbf{x}]$$

    where **x**, **y** are contexts of the same length and type and **z** is any context containing **x**, **y** and the free variables of ϕ.

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- Conjunction axioms and rules:

$$\bigwedge_{i<\gamma} \phi_i \vdash_{\mathbf{x}} \phi_j$$

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- Universal rule:

$$\frac{\phi \vdash_{\mathbf{x},\mathbf{y}} \psi}{\phi \vdash_{\mathbf{x}} \forall \mathbf{y} \psi}$$

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κ-first-order systems

- Transfinite transitivity:

$$\frac{\begin{array}{ll} \phi_i \vdash_{\mathbf{y}_i} \bigvee_{j \in \gamma^{\beta+1}, j|_\beta = i} \exists \mathbf{x}_j \phi_j & \beta < \gamma, i \in \gamma^\beta \\ \phi_i \dashv\vdash_{\mathbf{y}_i} \bigwedge_{\alpha < \beta} \phi_{i|_\alpha} & \beta < \gamma, \text{ limit } \beta, i \in \gamma^\beta \end{array}}{\phi_\emptyset \vdash_{\mathbf{y}_\emptyset} \bigvee_{i \in \gamma^\gamma} \exists_{\beta < \gamma} \mathbf{x}_{i|_{\beta+1}} \bigwedge_{\beta < \gamma} \phi_{i|_\beta}}$$

for each cardinal $\gamma < \kappa$, where $\mathbf{y}_i$ is the canonical context of ϕ_i, provided that, for every $i \in \gamma^{\beta+1}$, $FV(\phi_i) = FV(\phi_{i|_\beta}) \cup \mathbf{x}_i$ and $\mathbf{x}_{i|_{\beta+1}} \cap FV(\phi_{i|_\beta}) = \emptyset$ for any $\beta < \gamma$, as well as $FV(\phi_i) = \bigcup_{\alpha<\beta} FV(\phi_{i|_\alpha})$ for limit β. Note that we assume that there is a fixed well-ordering of γ^γ for each $\gamma < \kappa$.

κ-Kripke semantics

A Kripke model for pure first-order logic over Σ is a quadruple $\mathcal{B} = (K, \leq, D, \Vdash)$, where $(K, \leq)$ is a tree, D is a set-valued functor on K and the forcing relation $\Vdash$ is a binary relation between elements of K and sentences of the language with constants from $\bigcup_{k \in K} D(k)$, defined for atomic formulas ϕ with the conditions that $k \nVdash \bot$ and that $k \Vdash \phi(\mathbf{d}) \implies l \Vdash \phi(D_{kl}(\mathbf{d}))$ for $\mathbf{d} \subseteq D(k)$, and recursively extended to arbitrary formulas as follows:

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- $k \Vdash \bigwedge_{i<\gamma} \phi_i(\mathbf{d}) \iff k \Vdash \phi_i(\mathbf{d})$ for every $i < \gamma$
- $k \Vdash \bigvee_{i<\gamma} \phi_i(\mathbf{d}) \iff k \Vdash \phi_i(\mathbf{d})$ for some $i < \gamma$

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- $k \Vdash \phi(\mathbf{d}) \to \psi(\mathbf{d}') \iff \forall k' \geq k(k' \Vdash \phi(D_{kk'}(\mathbf{d})) \implies k' \Vdash \psi(D_{kk'}(\mathbf{d}')))$

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A Kripke model for a theory $\mathcal{C}$ is a Kripke model forcing all the axioms of the theory.

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- $P_a \vdash P_b$ for each pair a, b such that a is a successor of b

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We can show, on the other hand, that the hypothesis of weak compactness is enough to derive a completeness theorem for κ-first-order theories of cardinality at most κ with respect to κ-Kripke semantics. We have:

Theorem (E., 2016)

Let κ be a weakly (resp. strongly) compact cardinal. Then κ-first-order theories of cardinality at most κ (resp. of arbitrary cardinality) are semantically complete with respect to κ-Kripke models.

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Theorem (E., 2016)

Let κ be an inaccessible cardinal. Then κ-first-order logic is sound and complete with respect to models in Grothendieck toposes with the transfinite transitivity property.

Applications

It is possible to improve the completeness theorem restricting ourselves to the case of Kripke models over trees. This semantics allows to prove the following:

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Corollary

Let κ be a weakly compact cardinal. Then κ-first-order logic over a language without function symbols has the disjunction property: if $\bigvee_{i<\gamma} \phi_i$ is provable (in the empty theory) then, for some i, ϕ_i is already provable.

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Corollary

Let κ be a weakly compact cardinal. Then κ-first-order logic over a language without function symbols and at least one constant symbol has the existence property: if $\exists \mathbf{x} \phi(\mathbf{x})$ is provable (in the empty theory) then, for some constants $\mathbf{c}$, $\phi(\mathbf{c})$ is already provable.

Applications

Another application is a characterization of weakly (resp. strongly) compact cardinals in terms of the existence of certain κ-complete κ-prime filters in κ-complete, κ-distributive lattices. The correct notion of distributivity is the given by the transfinite transitivity rule applied in the propositional case:

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Given a cardinal κ, κ-complete lattices are lattices that have joins and meets of less than κ elements (in particular they are bounded). We say that a lattice is κ-distributive if for every $\gamma < \kappa$ and all elements $\{a_f : f \in \gamma^\beta, \beta < \gamma\}$ such that

$$a_f \leq \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} a_g$$

for all $f \in \gamma^\beta, \beta < \gamma$, and

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for all limit β, $f \in \gamma^\beta, \beta < \gamma$, we have that

$$a_\emptyset \leq \bigvee_{f\in\gamma^\gamma} \bigwedge_{\beta<\gamma} a_{f|_\beta}.$$

In words, if for each $\gamma < \kappa$ we have in the lattice a tree of type γ^γ (whose partial order is the reverse order of the lattice) in which every element is below the join of its immediate successors and where at limits levels every element is the meet of its predecessors, then the root element is below the join over all cofinal branches of the elements that are intersections of elements in each cofinal branch of the tree.

Applications

κ-distributivity implies the distributivity property:

$$\bigwedge_{i<\gamma} \bigvee_{j<\gamma} a_{ij} = \bigvee_{f \in \gamma^{\gamma}} \bigwedge_{i<\gamma} a_{if(i)} \qquad (1)$$

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and in fact one can see that in a κ-complete Boolean algebra they are equivalent properties.
However κ-distributivity is stronger in general; for example the interval $[0, 1]$ satisfies (1) but is not κ-distributive.

Applications

A κ-complete filter in the lattice is a filter such that whenever $a_i \in \mathcal{F}$ for every $i \in I$, $|I| < \kappa$, then $\bigwedge_{i \in I} a_i \in \mathcal{F}$. A κ-prime filter in the lattice is a filter $\mathcal{F}$ such that whenever $\bigvee_{i \in I} a_i$ is in $\mathcal{F}$ for $|I| < \kappa$ then $a_i \in \mathcal{F}$ for some $i \in I$.

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Theorem (E., 2016)

Let κ be an inaccessible cardinal. Then κ is weakly (resp. strongly) compact if and only if every κ-complete, κ-distributive lattice of cardinality at most κ (resp. of arbitrary cardinality) has a κ-complete, κ-prime filter.

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- Establish conceptual completeness theorems, or to what extent the category of models determines the theory.
- Study the case of finite-quantifier theories over $\mathcal{L}_{\kappa,\omega}$.
- Call κ a *Heyting cardinal* if κ-first-order theories of cardinality strictly less than κ are complete for κ-Kripke semantics. Determine its strength within the large cardinal hierarchy.