Generic absoluteness and supercompact cardinals

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Generic absoluteness and determinacy

Theorem (Martin, Solovay)

Suppose there is a proper class of measurable cardinals.

- *Then $\utilde{\Sigma}^1_3$-sentences are absolute between all generic extensions of V.*

Theorem (Jensen, Martin, Solovay)

The following are equivalent.

(1) *$\utilde{\Sigma}^1_3$-sentences are absolute between all generic extensions of V.*

(2) *$\utilde{\Sigma}^1_1$-determinacy holds in all generic extensions of V.*

$L(\mathbb{R})$ generic absoluteness

Theorem

Suppose there is a proper class of Woodin cardinals and that $V[G]$ is a generic extension of V.

- *Then there is an elementary embedding,*

$$j : L(\mathbb{R}) \to L(\mathbb{R}^{V[G]}).$$

Theorem

Suppose there is a proper class of strongly inaccessible cardinals. Then the following are equivalent.

(1) *$L(\mathbb{R})$-generic absoluteness.*

(2) $\mathrm{AD}^{L(\mathbb{R})}$*-holds in all generic extensions of V.*

(3) $L(\mathbb{R}) \not\models \text{Axiom of Choice}$ *in all generic extensions of V.*

What about projective generic absoluteness?

Theorem

Assume that δ is a limit of strong cardinals and $V[G]$ is a generic extension of V in which δ is countable.

- *Then projective generic absoluteness holds in $V[G]$.*

Theorem (Steel)

Suppose $L[\mathbb{E}]$ is an iterable fine-structure model in which δ is a limit of strong cardinals and then is no inner model of $L[\mathbb{E}]$ with a measurable limit of strong cardinals.

- *Suppose $L[\mathbb{E}][G]$ is a generic extension of $L[\mathbb{E}]$ in which δ is countable.*

Then in $L[\mathbb{E}][G]$:

- *Every projective subset of $\mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function;*
  - *In fact, every projective set has a projective scale.*

Definition

Suppose $x \in \mathbb{R}$. Then x-Projective Uniformization holds if:

- Every $\Sigma^1_n(x)$-set $A \subset \mathbb{R} \times \mathbb{R}$ can be uniformized by a function f whose graph is a $\Sigma^1_m(x)$-set for some $m < \omega$.

- If every projective subset of $\mathbb{R} \times \mathbb{R}$ can be uniformized by some projective function then there must exist $x \in \mathbb{R}$ such that x-Projective Uniformization holds.

Observation

Suppose that for some $x \in \mathbb{R}$, x-Projective Uniformization holds in every generic extension of V.

- *Then projective generic absoluteness holds.*

Two questions

Question

Suppose that generic projective absoluteness holds. Must projective uniformization hold?

Question

Assume generic projective absoluteness. Are the following equivalent?

1. 0-*Projective Uniformization.*
2. *Projective Determinacy.*

Conjecture

The answer to the second question is surely no since the answer is no for x-Projective Uniformization

- *by Steel's theorem.*

Uniformization in $L(\mathbb{R})$ and the necessity of determinacy

These questions are easily reformulated.

Question

Assume 0-Projective Uniformization. Are the following equivalent?

1. *Every projective set is Lebesgue measurable and has the property of Baire.*
2. *Projective Determinacy.*

Theorem

Suppose that uniformization holds in $L_\alpha(\mathbb{R})$ and that $\alpha = \omega_1 \cdot \beta$ for some limit ordinal β. Then the following are equivalent.

(1) *Every set $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is Lebesgue measurable and has the property of Baire.*

(2) *Every set $A \in L_\alpha(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is determined.*

Universally Baire sets: generalizing the projective sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is **universally** Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}^n$,

- the preimage of A by π has the property of Baire in the space Ω.

- Every universally Baire set is Lebesgue measurable and has the property of Baire.
- The universally Baire sets are closed under countable unions, complements, and continuous preimages.

Closure properties of the Universally Baire sets

Lemma

Assume $V = L$.

- *Then every set $A \subset \mathbb{R}$ is the projection of some universally Baire set $B \subset \mathbb{R} \times \mathbb{R}$.*

Theorem

Suppose that there is a proper class of Woodin cardinals. Then:

(1) *If $A \subset \mathbb{R}$ is Universally Baire then so is every set in $L(A, \mathbb{R})$.*

(2) *(Steel) Every Universally Baire set has a universally Baire scale.*

(3) *(Martin-Steel, Woodin) If A is Universally Baire then $L(A, \mathbb{R}) \models \mathrm{AD}^+$.*

The limits of generic absoluteness

Notation: Suppose $V[G]$ is a generic extension of V.

- $\mathbb{R}_G = \mathbb{R}^{V[G]}$,
- Γ^∞_G is the set of all the universally Baire sets of $V[G]$.

The Sealing Theorem

Assume δ is supercompact. Suppose that

$$V[G_1] \subseteq V[[G_2]$$

are generic extensions of V and $V_{\delta+1}$ is countable in $V[G_1]$. Then

- $\Gamma^\infty_{G_1} = \mathcal{P}(\mathbb{R}_{G_1}) \cap L(\Gamma^\infty_{G_1}, \mathbb{R}_{G_1})$.
- $\Gamma^\infty_{G_2} = \mathcal{P}(\mathbb{R}_{G_2}) \cap L(\Gamma^\infty_{G_2}, \mathbb{R}_{G_2})$.
- *There is an elementary embedding*

$$\pi : L(\Gamma^\infty_{G_1}, \mathbb{R}_{G_1}) \to L(\Gamma^\infty_{G_2}, \mathbb{R}_{G_2}).$$

- The Sealing Theorem generalizes the theorem on projective generic absoluteness after collapsing a limit of strong cardinals.

Question

Does some (consistent) large cardinal hypothesis imply that the conclusion of the Sealing Theorem must hold?

  - *Or even just that* $\Gamma^\infty = \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R})$*?*

- *Is there a generalization of the Martin-Steel Theorem on Projective Determinacy?*

- A positive answer would be a strong Anti-Inner-Model Theorem.

Conclusion

We cannot generalize generic absoluteness any further following this path.

The generic-multiverse of sets

Suppose that M is a countable transitive set and that

$$M \models \mathrm{ZFC}.$$

Let $\mathbb{V}_M$ be the smallest set of countable transitive sets such that $M \in \mathbb{V}_M$ and such that for all pairs, (M_1, M_2), of countable transitive sets, if

1. $M_1 \models \mathrm{ZFC}$,
2. M_2 is a generic extension of M_1,
3. $M_1 \in \mathbb{V}_M$ or $M_2 \in \mathbb{V}_M$,

then both M_1 and M_2 are in $\mathbb{V}_M$.

Definition

$\mathbb{V}_M$ is the **generic-multiverse** generated from M.

Evaluating truth in the generic-multiverse...

Theorem

For each sentence φ there is a sentence φ^* such that for all countable transitive sets M if

$$M \models \mathrm{ZFC}$$

then the following are equivalent.

1. $M \models \varphi^*$,
2. *For all* $N \in \mathbb{V}_M$, $N \models \varphi$.

The generic-multiverse view of truth

A Π_2*-sentence,* φ*, is a* **Generic-Multiverse truth** *if* φ *holds in each universe of the generic-multiverse generated by* V.

Ω-logic
(The logic of the Generic-Multiverse)

Definition

Suppose φ is a Π_2-sentence. Then

$$\models_\Omega \varphi$$

if φ holds in all generic extensions of V.

- One can easily allow parameters A which are universally Baire
  - provided one replaces A with its interpretation A_G in $V[G]$.

Theorem

Suppose there is a proper class of Woodin cardinals and that φ is a Π_2-sentence.

Then φ is a Generic-Multiverse truth if and only if $\models_\Omega \varphi$.

Universally Baire sets and strong closure

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and suppose that M is a countable transitive model of ZFC.

Then M is **strongly A-closed** if for all countable transitive sets N such that N is a generic extension of M,

$$A \cap N \in N.$$

Examples where M is necessarily strongly A-closed.

- A is finite.
- A and $\mathbb{R} \backslash A$ are Σ_1-definable over $V_{\omega+1}$:
  - A is Δ^1_1.
- (Shoenfield Absoluteness) A is Σ_1-definable over $V_{\omega+1}$:
  - A is Σ^1_1.

The definition of $\vdash_\Omega \varphi$

Definition

Suppose there is a proper class of Woodin cardinals. Suppose that φ is a Π_2-sentence.

Then $\vdash_\Omega \varphi$ if there exists a set $A \subset \mathbb{R}$ such that:

1. A is universally Baire,
2. for all countable transitive sets, M, if
   - $M \models \mathrm{ZFC}$ and M is strongly A-closed

   then $M \models$ "$\models_\Omega \varphi$".

- "$\vdash_\Omega \varphi$" is invariant across the Generic-Multiverse.
- One can allow universally Baire sets as parameters with the obvious adjustments in the definitions.

The Ω Conjecture

Theorem (Ω Soundness)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is Π_2-sentence.

If $\vdash_\Omega \varphi$ then $\models_\Omega \varphi$

Definition (Ω Conjecture)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is a Π_2-sentence.

Then $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

Conditional generic absoluteness

Theorem

Suppose there is a proper class of measurable Woodin cardinals.

- *Then Σ^2_1-sentences are absolute between all generic extensions of V in which CH holds.*

Theorem (Ω Conjecture)

Suppose that there is a proper class of measurable Woodin cardinals.

- *Then for each $A \in \Gamma^\infty$ and for each Σ^2_1-formula $\varphi(x)$ one of the following hold.*
  - $\vdash_\Omega \mathrm{CH} \to \varphi[A]$.
  - $\vdash_\Omega \mathrm{CH} \to (\neg\varphi)[A]$.

Question

Can this be characterized in terms of determinacy?

Universally Baire closed games of length ω_1

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and

$$X \subset \{0,1\}^{\omega_1}.$$

Then X is A-closed if there exists a formula $\varphi(x)$ such that

$$X = \{f \in \{0,1\}^{\omega_1} \mid (H(\omega_1), A, \in) \models \varphi[f|\alpha] \text{ for all } \alpha < \omega_1\}$$

- A-closed subsets of $\{0,1\}^{\omega_1}$ are closed in the natural sense for the space $\{0,1\}^{\omega_1}$.
  - But closure condition is specified by A and so more restrictive.

Σ^2_1-generic absoluteness and determinacy

Theorem

Suppose that there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Let Γ^∞ be the set of all $A \subseteq \mathbb{R}$ such that A is universally Baire. Then the following are equivalent.

(1) *For each $A \in \Gamma^\infty$, for each Σ^2_1-formula $\varphi(x)$, one of the following hold.*
   - $\vdash_\Omega \mathrm{CH} \rightarrow \varphi[A]$.
   - $\vdash_\Omega \mathrm{CH} \rightarrow (\neg\varphi)[A]$.

(2) *For each $A \in \Gamma^\infty$,*

$$\vdash_\Omega \text{“All } A\text{-closed sets } X \subseteq \{0,1\}^{\omega_1} \text{ are determined.”}$$

Question

Is there a version of this theorem for Σ^2_2-generic absoluteness conditioned on some form of $\diamond$?

A generic form of $\diamond$

Definition: $\diamond_G^\infty$

For each Σ_2-sentence, φ, and for each universally Baire set A,

$$(H(\omega_2), A, \in) \models \varphi$$

if and only if

$$(H(\omega_2), A, \in)^{V[G]} \models \varphi$$

where $G \subset \mathrm{Coll}(\omega_1, \mathbb{R})$ is V-generic.

The generalization of $\undertilde{\Sigma}^1_1$-determinacy to ω_1

Definition: $\Sigma_1(A)$-determinacy for $\{0,1\}^{\omega_1}$

Suppose $X \subseteq \{0,1\}^{\omega_1}$ and X is Σ_1-definable from (A, ω_1). Then X is determined.

Theorem

Suppose that ZFC is relatively consistent with the existence of a proper class of Woodin limits of Woodin cardinals.

- *Then ZFC is consistent with the existence of a proper class of Woodin cardinals together with:*
  - *The determinacy of* **all** *$X \subseteq \{0,1\}^{\omega_1}$ such that X is $\mathrm{OD}(A)$ for some $A \in \Gamma^\infty$.*

- So one cannot refute that $\Sigma_1(\mathbb{R})$-determinacy holds for $\{0,1\}^{\omega_1}$.

Conjecture

Suppose $\Sigma_1(\mathbb{R})$*-determinacy holds for* $\{0,1\}^{\omega_1}$*. Then* $\underset{\sim}{\delta}^1_2 < \omega_2$*.*

Lemma

Suppose $\Sigma_1(\mathbb{R})$*-determinacy holds for* $\{0,1\}^{\omega_1}$*.*

- *Then there exists* $A \subset \omega_1$ *such that if* G *is* $L[A]$*-generic for adding random reals then in* $L[A][G]$ *every* $\mathrm{OD}_{\mathbb{R}}$ *subset of* $\{0,1\}^{\omega_1}$ *is determined.*

Assume every $\mathrm{OD}_{\mathbb{R}}$ subset of $\{0,1\}^{\omega_1}$ is determined.

- How many consequences of CH must hold?

The hypothesis: IM

Definition (IM Hypothesis)

Suppose there is a proper class of measurable Woodin cardinals. Then for each universally Baire set A there is a countable transitive model M and $\mathcal{I}$ such that

1. $M \models \mathrm{ZFC} +$ "There is a measurable Woodin cardinal".
2. M is strongly A-closed.
3. $\mathcal{I}$ is an iteration strategy for M which is A-correct:
   - Iterates of M generated by $\mathcal{I}$ are strongly A-closed.
   - Iteration embeddings generated by $\mathcal{I}$ preserve the A-term relation.
4. $\mathcal{I}$ is universally Baire.

Σ^2_2-generic absoluteness and determinacy

Theorem (IM + Proper class of measurable Woodin cardinals)

The following are equivalent.

(1) *For each $A \in \Gamma^\infty$ and for each Σ^2_2-formula $\varphi(x)$ one of the following hold.*
  - $\vdash_\Omega \diamond^\infty_G \rightarrow \varphi[A]$.
  - $\vdash_\Omega \diamond^\infty_G \rightarrow (\neg\varphi)[A]$.

(2) *For each $A \in \Gamma^\infty$,*

$$\vdash_\Omega \diamond^\infty_G \rightarrow \text{“}\Sigma_1(A)\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{”}.$$

For each $A \in \Gamma^\infty$, the following are equivalent.

- $\vdash_\Omega$ “All A-closed sets $X \subseteq \{0,1\}^{\omega_1}$ are determined.”
- $\vdash_\Omega \mathrm{CH} \rightarrow$ “All A-closed sets $X \subseteq \{0,1\}^{\omega_1}$ are determined.”

- This explains the difference here in (2) versus the case for Σ^2_1.

A natural conjecture

Claim

The scenario that **some** *large cardinal hypothesis implies*

$$\vdash_\Omega \diamond_G^\infty \rightarrow \text{``}\Sigma_1(\mathbb{R})\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{''}.$$

is a compelling generalization of Martin's theorem that the existence of a measurable cardinal implies $\underset{\sim}{\Sigma}^1_1$-determinacy.

- (CH) The sets $X \subseteq \{0,1\}^{\omega_1}$ which are $\Sigma_1(\mathbb{R})$-definable are the generalization of the Σ^1_1 sets to $\{0,1\}^{\omega_1}$.

Conjecture

There is a (consistent) large cardinal hypothesis which proves that for each $A \in \Gamma^\infty$,

$$\vdash_\Omega \diamond_G^\infty \rightarrow \text{``}\Sigma_1(A)\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{''}.$$

Weak extender models for supercompactness

Definition

A transitive class $N \models \mathrm{ZFC}$ is a *weak extender model for δ is supercompact* if for every $\gamma > \delta$ there exists a δ-complete normal fine measure U on $\mathcal{P}_\delta(\gamma)$ such that

1. $N \cap \mathcal{P}_\delta(\gamma) \in U$,
2. $U \cap N \in N$.

- Weak extender models for the supercompactness of δ are *very* cose to V above δ.

Theorem

Suppose that N is a weak extender model for δ is supercompact and that $\gamma > \delta$ is a singular cardinal. Then γ is a singular cardinal in N and

$$(\gamma^+)^N = \gamma^+.$$

The Universality Theorem

Universality Theorem

Suppose that N is a weak extender model for δ is supercompact and that E is an N-extender of length η with critical point $\kappa_E \geq \delta$. Let

$$\pi_E : N \to M_E \cong \mathrm{Ult}_0(N, E)$$

be the ultrapower embedding. Then the following are equivalent.

(1) *For each $A \subset \eta$, $\pi_E(A) \cap \eta \in N$.*

(2) *$E \in N$.*

Theorem

Suppose that N is a weak extender model for δ is supercompact and that $\theta > \delta$ is a Woodin cardinal.

- *Then θ is a Woodin cardinal in N.*

Consequences of the Universality Theorem

Theorem

Let N be a weak extender model for δ is supercompact.

- *Then there is no nontrivial elementary embedding*

$$j : N \to N$$

  such that $\delta \leq \mathrm{CRT}(j)$.

Theorem

Suppose that δ is a supercompact cardinal.

- *Then there is a weak extender model N for the supercompactness of δ such that there is a non-trivial elementary embedding*

$$j : N \to N.$$

Σ^2_2-generic absoluteness and the form of inner models

The two principal current fine-structural hierarchies are:

1. The hierarchy of nonstrategic-extender models.
   - Key feature: comparison through iteration and least disagreement.
2. The hierarchy of strategic-extender models.

Lemma ($\diamond$)

Suppose that Σ^2_2-sentences are absolute between all generic extensions of V in which $\diamond$ holds.

- *Then there is no Σ^2_2-wellordering of the reals.*

Claim

Subject to very general conditions, the hierarchy of nonstrategic-extender models

- *yields models in which $\diamond$ holds and there is a Σ^2_2-wellordering of the reals.*

So where to look for generic Σ^2_2-absoluteness?

Speculation

Determine where hierarchy of nonstrategic-extender models must end.

- *This will identify the large cardinal hypothesis which implies*

$$\vdash_\Omega \diamond^\infty_G \rightarrow \text{“}\Sigma_1(\mathbb{R})\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{”}.$$

Claim

Subject to very general conditions on comparison:

- *There can be no nonstrategic-extender model in which there is a supercompact cardinal with a measurable cardinal above.*

- But maybe this hypothesis is already far too strong:
  - Identifying the optimal hypothesis might provide the key clues for actually proving conditional generic Σ^2_2-absoluteness.

An emerging picture

The detailed scenario to prove the Ultimate-L *Conjecture shows that if there is a large cardinal hypothesis which implies*

$$\vdash_\Omega \diamond_G^\infty \rightarrow \text{“}\Sigma_1(\mathbb{R})\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{”},$$

- *then that large cardinal hypothesis,*
    - *together with a proper class of huge cardinals,*

  must imply that there are iterable nonstrategic-extender models
    - $\mathcal{M} \models \mathrm{ZFC} +$ *“There is a supercompact cardinal”,*

  such that for cofinally many $\kappa < \mathrm{Ord}^{\mathcal{M}}$*, there is a normal* $\mathcal{M}$*-ultrafilter* U *on* κ *such that* $\mathrm{Ult}_0(\mathcal{M}, U)$ *is iterable.*

- This strongly suggests the upper bound previously identified for the extent of the hierarchy of nonstrategic-extender models is essentially optimal.