Useful axioms

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Non-constructive principles for mathematics

A list of five (in some cases apparently unrelated) useful non-constructive principles:

1. The axiom of choice,
2. Baire's category theorem,
3. Large cardinal axioms,
4. Shoenfield's absoluteness,
5. Łoś Theorem for ultrapowers of first orders structures.

First aim: show that the language of forcing allows to bring out the analogies more or less evident between all these distinct principles and to express all of them as forcing axioms.

Second aim: formulate stronger and stronger non constructive principles leveraging on different aspects of the above analogies.

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The axiom of choice is a global forcing axiom

This observation has been handled to me by Stevo Todorčević.

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Definition

Let λ be an infinite cardinal. DC_λ holds if for all sets X and all functions $F : X^{<\lambda} \to P(X)$, there exists $g : \lambda \to X$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \lambda$.

Fact

The axiom of choice AC is equivalent over ZF to the assertion DC_λ holds for all λ.

This is a local statement, i.e. there is a level by level correspondance between the amount of choice and of dependent choice available in the universe.

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Let P be a partial order. $\mathrm{FA}_\lambda(P)$ holds if for all family $\{D_\alpha : \alpha < \lambda\}$ of dense subsets of P, there exists a filter $G \subset P$ which has non-empty intersection with all the D_α.

Let Γ be a class of partial orders. Then $\mathrm{FA}_\lambda(\Gamma)$ holds if $\mathrm{FA}_\lambda(P)$ holds for all $P \in \Gamma$.

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$\mathrm{DC}_{\aleph_0}$ *is equivalent over* ZF *to the assertion* $\mathrm{FA}_{\aleph_0}(P)$ *holds for all* P.

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Sketch of proof. I show just the direction I want to bring forward: Assume $F : X^{<\omega} \to P(X)$ is a function. Let T be the subtree of $X^{<\omega}$ given by finite sequences $s \in X^{<\omega}$ such that $s(i) \in F(s \restriction i)$ for all $i < |s|$. Consider the family given by the dense sets

$$D_n = \{s \in T : |s| > n\}.$$

If G is a filter on T meeting the dense sets of this family, $\cup G$ works.

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More generally:

Definition

A partial order P is $< \lambda$-closed if all chains in P of length less than λ have a lower bound.

Let $AC \upharpoonright \lambda$ abbreviate DC_γ *holds for all* $\gamma < \lambda$ and Γ_λ be the class of $< \lambda$-closed posets.

Fact

DC_λ *is equivalent to* $FA_\lambda(\Gamma_\lambda)$ *over the theory* $ZF + AC \upharpoonright \lambda$.

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Conclusion:

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The axiom of choice is equivalent over the theory ZF *to the assertion that* $\mathsf{FA}_\lambda(\Gamma_\lambda)$ *holds for all* λ.

The usual forcing axioms such as Martin's maximum or the proper forcing axiom are natural strenghtenings of the axiom of choice. They aim to isolate a maximal strengthening of $\mathsf{AC} \restriction \omega_2$ enlarging the family Γ for which $\mathsf{FA}_{\aleph_1}(\Gamma)$ holds.

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Baire's category theorem is a forcing axiom

Theorem (BCT)

Assume (X, τ) *is compact and Hausdorff. Let* $\{D_n : n \in \omega\}$ *be a family of dense open subsets of* X. *Then* $\bigcap_{n \in \omega} D_n$ *is non-empty.*

Fact

$\mathsf{FA}_{\aleph_0}(P)$ *for all forcing* P *entails* BCT.

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Proof.

Let (X, τ) compact Hausdorff and $\{D_n : n \in \omega\}$ a family of dense open subsets of X.
Let $(P, \leq_P) = (\tau \setminus \{\emptyset\}, \subseteq)$ and

$$E_n = \{A \in \tau : \overline{A} \subseteq D_n\}.$$

Each E_n is a dense subset of P. Let G be a filter on P with $G \cap E_n \neq \emptyset$ for all n. By compactness of X

$$\bigcap \{\mathrm{Cl}(A) : A \in G\} \neq \emptyset \in \bigcap_{n \in \omega} D_n.$$

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More general forcing axioms

Fact

Let G be a filter on a poset P and $X \subseteq P$. Then $G \cap X$ is non-empty iff $G \cap \downarrow X$ is non-empty.

Hence G meets a predense set A iff it meets the dense open set $\downarrow A$.

Definition

Given a poset P and a property ϕ, $\mathrm{FA}_\phi(P)$ holds if

> *For all $\mathcal{D}$ collection of predense subsets of P such that $\phi(\mathcal{D})$ holds, there exists G filter on P such that $G \cap X \neq \emptyset$ for all $X \in \mathcal{D}$.*

$\mathrm{FA}_\kappa(P)$ stands for $\mathrm{FA}_\phi(P)$ where
$\phi(\mathcal{D}) \equiv |\mathcal{D}| = \kappa$ *and each* $D \in \mathcal{D}$ *is predense.*

$\mathrm{BFA}_{\omega_1}(P)$ stands for $\mathrm{FA}_\phi(\mathrm{RO}(P))$ where
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Large cardinals as forcing axioms

Given a cardinal κ,

- I_κ is the ideal of bounded subsets of κ,
- $\mathcal{A}_\kappa$ is the family of maximal antichains of size less than κ in $\mathcal{P}(\kappa)/_{I_\kappa}$.

Definition

κ is measurable iff there is a ultrafilter $G \in \mathcal{P}(\kappa)/_{I_\kappa}$ such that $G \cap A \neq \emptyset$ for all $A \in \mathcal{A}_\kappa$.

I.e. κ is measurable if $\mathsf{FA}_\phi(\mathcal{P}(\kappa)/_{I_\kappa})$, where $\phi(\mathcal{D})$ stands for $\mathcal{D} = \mathcal{A}_\kappa$.
Cofinally many large cardinal properties of κ can be formulated as forcing axiom of the type $\mathsf{FA}_\phi(\mathcal{P}(\mathcal{P}(\lambda))/_{J_{\kappa,\lambda}})$, choosing ϕ and $J_{\kappa,\lambda}$ suitably.
for example supercompact, huge, extendible, n-huge, I_1, etc......

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Łoś theorem

Theorem

Let $\{\mathfrak{M}_l = (M_l, R_l) : l \in L\}$ be first oreder models for $\mathcal{L} = \{R\}$. Let $G \subseteq \mathcal{P}(L)$ be a ultrafilter on L. Set

- $[f]_G = [h]_G$ iff $\{l \in L : f(l) = h(l))\} \in G$,
- $\bar{R}([f_1]_G, \ldots, [f_n]_G)$ iff $\{l \in L : R_l(f_1(l), \ldots, f_n(l))\} \in G$.

Then:

1. *For all $\phi(x_1, \ldots, x_n)$ $(\prod_{l \in L} M_l/G, \bar{R}) \models \phi([f_1]_G, \ldots, [f_n]_G)$ if and only if $\{l \in L : \mathfrak{M}_l \models \phi(f_1(l), \ldots, f_n(l))\} \in G$.*
2. *If $\mathfrak{M}_l = \mathfrak{M}$ for all $l \in L$, $M \prec \prod_{l \in L} M_l/G$ as witnessed by the map $m \mapsto [c_m]_G$ (where $c_m : L \to M$ is constant with value m).*

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Recall on boolean algebras and Stone spaces

Given a boolean algebra B:

- St(B) is given by its ultrafilters G.
- St(B) is endowed with a *compact, Hausdorff* topology τ_B whose clopens are $N_b = \{G \in St(B) : b \in G\}$.
- The map $b \mapsto N_b$ defines a natural isomorphism of B with the boolean algebra CLOP(St(B)) of clopen subset of St(B).

- B is *complete* if and only if $CLOP(St(B)) = RO(St(B), \tau_B) \cong B$.
- Spaces X satisfying the property that its regular open sets are closed are *extremally (or extremely) disconnected.*
- $\mathcal{P}(X)$ is a complete boolean algebra, and $\beta(X) = St(\mathcal{P}(X))$ is the Stone-Cech compactification of X with discrete topology and is extremally disconnected.

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- Spaces X satisfying the property that its regular open sets are closed are *extremally (or extremely) disconnected*.
- $\mathcal{P}(X)$ is a complete boolean algebra, and $\beta(X) = \mathrm{St}(\mathcal{P}(X))$ is the Stone-Cech compactification of X with discrete topology and is extremally disconnected.

Recall on boolean algebras and Stone spaces

Given a boolean algebra B:

- St(B) is given by its ultrafilters G.
- St(B) is endowed with a *compact, Hausdorff* topology τ_B whose clopens are $N_b = \{G \in \mathrm{St}(B) : b \in G\}$.
- The map $b \mapsto N_b$ defines a natural isomorphism of B with the boolean algebra CLOP(St(B)) of clopen subset of St(B).

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Boolean valued models

Definition

Let B be a *cba* and a $\mathcal{L}$ be first order *relational* language.
A B-*valued model* for $\mathcal{L}$ is a tuple
$\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$ with

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\begin{aligned}
=^{\mathcal{M}} &: M^2 \to \mathsf{B} \\
&(\tau, \sigma) \mapsto [\![\tau = \sigma]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![\tau = \sigma]\!] ,
\end{aligned}
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\begin{aligned}
R^{\mathcal{M}} &: M^n \to \mathsf{B} \\
(\tau_1, \dots, \tau_n) &\mapsto [\![R(\tau_1, \dots, \tau_n)]\!]_{\mathsf{B}}^{\mathcal{M}} = [\![R(\tau_1, \dots, \tau_n)]\!]
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Forcing relations on boolean valued models

The constraints on $R^{\mathcal{M}}$ and $=^{\mathcal{M}}$ are the following:

- for $\tau, \sigma, \chi \in M$,
  1. $[\![\tau = \tau]\!] = 1_{\mathsf{B}}$;
  2. $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$;
  3. $[\![\tau = \sigma]\!] \wedge [\![\sigma = \chi]\!] \leq [\![\tau = \chi]\!]$;
- for $R \in \mathcal{L}$ with arity n, and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$,

$$\left(\bigwedge_{h \in \{1,\ldots,n\}} [\![\tau_h = \sigma_h]\!]\right) \wedge [\![R(\tau_1, \ldots, \tau_n)]\!] \leq [\![R(\sigma_1, \ldots, \sigma_n)]\!] .$$

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Boolean valued semantics

Definition

Let $\langle M, =^{\mathcal{M}}, R^{\mathcal{M}} \rangle$ be a B-valued model in the relational language $\mathcal{L} = \{R\}$, $\phi(x_1, \ldots, x_n)$ a $\mathcal{L}$-formula with displayed free variables, ν : free variables $\to M$.

$[\![\phi]\!]_{\mathsf{B}}^{\mathcal{M},\nu} = [\![\phi]\!]$, the *boolean value* of ϕ with the assignment ν is defined by recursion as follows:

- $[\![t = s]\!] = [\![\nu(t) = \nu(s)]\!]$,
  $[\![R(t_1, \ldots, t_n)]\!] = [\![R(\nu(t_1), \ldots, \nu(t_n))]\!]$;
- $[\![\neg\psi]\!] = \neg [\![\psi]\!]$;
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Soundness Theorem for B-valued semantics

Theorem (Soundness Theorem)

Assume $\mathcal{L}$ is a relational language and ϕ is a $\mathcal{L}$-formula which is syntactically provable by a $\mathcal{L}$-theory T.
Assume each formula in T has boolean value at least $b \in \mathsf{B}$ in a B-valued model $\mathcal{M}$ with valuation ν.
Then $[\![\phi]\!]^{\mathcal{M},\nu}_{\mathsf{B}} \geq b$ as well.

The completeness theorem is automatic given that 2 is a complete boolean algebra.

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Tarski quotient of B-valued models

Definition

Let B be a *cba*, $\mathcal{M}$ a B-valued model for $\mathcal{L}$, and G a ultrafilter over B. Consider the following equivalence relation on M:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

The first order (Tarski) model $\mathcal{M}/_G = \langle M/_G, R_i^{\mathcal{M}/G} : i \in I, c_j^{\mathcal{M}/G} : j \in J \rangle$ is defined letting:

- For any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \dots, [\tau_n]_G) \in (M/_G)^n : [\![R(\tau_1, \dots, \tau_n)]\!] \in G\}.$$

- For any constant symbol c in $\mathcal{L}$

$$c^{\mathcal{M}/G} = [c^{\mathcal{M}}]_G.$$

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Full B-valued models

Definition

A B-valued model $\mathcal{M}$ for the language $\mathcal{L}$ is *full* if for every $\mathcal{L}$-formula $\phi(x, \bar{y})$ and every $\bar{\tau} \in M^{|\bar{y}|}$ there is a $\sigma \in M$ such that

$$[\![\exists x \phi(x, \bar{\tau})]\!] = [\![\phi(\sigma, \bar{\tau})]\!]$$

Boolean valued Łoś Theorem — Forcing theorem

Theorem (B-valued Łoś's Theorem — Forcing theorem)

Assume $\mathcal{M}$ is a full B*-valued model for the relational language $\mathcal{L}$. Then for every formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \ldots, \tau_n) \in M^n$:*

1. *For all ultrafilters G over* B*, $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ if and only if $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$.*
2. *For all $a \in$ B the following are equivalent:*
   1. $[\![\phi(f_1, \ldots, f_n)]\!] \geq a$,
   2. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for all $G \in N_a$,*
   3. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for densely many $G \in N_a$.*

Łoś's Theorem versus boolean valued Łoś's Theorem

Fact

Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language $\mathcal{L}$. Then $N = \prod_{x \in X} M_x$ is a full $\mathcal{P}(X)$-model, letting for each n-ary relation symbol $R \in \mathcal{L}$,
$[\![R(f_1, \ldots, f_n)]\!]_{\mathcal{P}(X)} = \{x \in X : M_x \models R(f_1(x), \ldots, f_n(x))\}$.

Let G be any non-principal ultrafilter on X. Then the Tarski quotient N/G is the familiar ultraproduct of the family $(M_x : x \in X)$ by G.

> *The usual Łoś theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$-valued model N of the boolean valued Łoś theorem.*
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> *If N is an ultrapower of a model M, the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary.*

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Boolean ultrapowers of compact Hausdorff spaces

Let X be a set with the discrete topology.

- For $a \in X$, $G_a \in \mathrm{St}(\mathcal{P}(X))$ is the principal ultrafilter of supersets of $\{a\}$.
- The map $a \mapsto G_a$ embeds X as an open, dense, discrete subspace of $\mathrm{St}(\mathcal{P}(X))$.
- For any space (Y, τ), any $f : X \to Y$ is continuous. (since X has the discrete topology)

Moreover if Y is compact Hausdorff:

- $f : X \to Y$ induces a unique continuous extension $\bar{f} : \mathrm{St}(\mathcal{P}(X)) \to Y$. ($\mathrm{St}(\mathcal{P}(X))$ is also the Stone-Cech compactification of X).
- $C(X, Y) = Y^X$ is canonically isomorphic to $C(\mathrm{St}(\mathcal{P}(X)), Y)$.
- $C(\mathrm{St}(\mathcal{P}(X)), Y) \cong Y^X$ can be endowed of the structure of a $\mathcal{P}(X)$-valued *elementary* extension of Y for any first order structure on Y.

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Boolean ultrapowers of 2^ω

Let B be an arbitrary complete boolean algebra, and set $M = C(St(\mathsf{B}), 2^\omega)$.

Fix R a Borel (Universally Baire) relation on $(2^\omega)^n$. The continuity of an n-tuple $f_1, \ldots, f_n \in M$ grants that

$$\{G : R(f_1(G) \ldots, f_n(G))\} = (f_1 \times \cdots \times f_n)^{-1}[R]$$

has the Baire property in St(B), where $f_1 \times \cdots \times f_n(G) = (f_1(G), \ldots, f_n(G))$.
Define:

$$\begin{aligned} R^M : & M^n \to \mathsf{B} \\ & (f_1, \ldots, f_n) \mapsto \operatorname{Reg}(\{G : R(f_1(G), \ldots, f_n(G)\}) \end{aligned}$$

where $\operatorname{Reg}(A) = \operatorname{Int}(\operatorname{Cl}(A))$.
Also, since the diagonal is closed in $(2^\omega)^2$,

$$=^M (f, g) = \operatorname{Reg}(\{G : f(G) = g(G)\})$$

is well defined.

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where $\mathrm{Reg}\,(A) = \mathrm{Int}\,(\mathrm{Cl}\,(A))$.

Also, since the diagonal is closed in $(2^\omega)^2$,

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Let B be an arbitrary complete boolean algebra, and set $M = C(St(\mathsf{B}), 2^\omega)$.

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Let B be an arbitrary (even atomless) complete boolean algebra. The following holds:

- For any Borel (universally Baire) relation R on $(2^\omega)^n$, the structure $(M, =^M, R^M)$ is a *full* B-valued model.
- For $G \in \mathrm{St}(\mathsf{B})$,

$$\begin{aligned} i_G : 2^\omega &\to M/G \\ x &\mapsto [c_x]_G \end{aligned}$$

  (c_x is the constant function with value x) defines an injective morphism $(2^\omega, R)$ into $(M/G, R^M/G)$.

It is not clear whether this morphism is an elementary map or not:

- This is the case for $\mathsf{B} = \mathcal{P}(X)$, since in this case we are analyzing the standard embedding of the first order structure $(2^\omega, R)$ in its ultrapowers induced by ultrafilters on $\mathcal{P}(X)$.
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Shoenfield's absoluteness rephrased

Theorem (Cohen's absoluteness)

Assume B *is a complete boolean algebra and* $R \subseteq (2^\omega)^n$ *is a Borel (Universally Baire) relation. Let* $M = C(St(\mathsf{B}), 2^\omega)$ *and* $G \in \mathrm{St}(\mathsf{B})$*. Then*

$$(2^\omega, =, R) \prec_{\Sigma_2} (M/_G, =^M /_G, R^M/_G).$$

If one assumes the existence of a Woodin cardinal larger than $|\mathsf{B}|$

$$(2^\omega, =, R) \prec (M/_G, =^M /_G, R^M/_G).$$

Proof.

$C(St(\mathsf{B}), 2^\omega)$ is isomorphic to the B-names in V^{B} for elements of 2^ω (see next slide). Apply Shoenfield's (or Woodin's) absoluteness to V and $V[H]$ (for H V-generic for B) to infer the desired conclusion. □

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$C(\mathrm{St}(\mathrm{B}), 2^\omega)$ and V^{B}

Given $f \in C(\mathrm{St}(\mathrm{B}), 2^\omega) = M$, $\sigma \in V^{\mathrm{B}}$ with $[\![\sigma \in 2^\omega]\!] = 1_{\mathrm{B}}$ define:

- $\tau_f = \left\{\langle\langle n, i\rangle, f^{-1}[N_{n,i}]\rangle : n < \omega, i < 2\right\} \in V^{\mathrm{B}}$,
- $g_\sigma \in M$ by $g_\sigma(G)(n) = i$ iff $[\![\sigma(n) = i]\!] \in G$.

Then

- $g_{\tau_f} = f$,
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These identities allow to translate forcing relations from both sides.

The lift of a Universally Baire relation R to V^{B} is translated as the forcing relation (on M)

$$\begin{aligned} R^M : & M^n \to \mathrm{B} \\ & (f_1, \ldots, f_n) \mapsto \mathrm{Reg}\left(\{G : R(f_1(G), \ldots, f_n(G)\}\right). \end{aligned}$$

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Two questions

1. Where are forcing axioms playing a role in the above proof (and rephrasing) of Shoenfield's absoluteness?
2. What if $Y \neq 2^{\omega}$ is some other compact Hausdorff space?

1. Time permitting I'll give a proof of the above rephrasing of Shoenfield's absoluteness, which can be based on a Baire category argument and on Cohen's forcing theorem.
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Looking at 2^ω is the same as looking at H_{ω_1}

There exists a natural correspondence between the theory of projective subsets of 2^ω and the first order theory of H_{ω_1}. Any Σ^1_2-property of 2^ω corresponds to a Σ_1-property on H_{ω_1}.

Moreover 2^ω is a definable class in H_{ω_1}, hence the first order theory of H_{ω_1} interprets that of 2^ω with projective predicates.

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Cohen's absoluteness Lemma

Lemma

Assume that:

- $\phi(x, r)$ *is a* Δ_0*-formula with real parameter* r.
- $B \in V$ *is a Boolean algebra such that* $1_B \Vdash_B \exists x \phi(x, \check{i}_B(r))$.

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Assume $B \in V$ is a Boolean algebra such that

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Then $V_\lambda \models$ ZFC and

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Pick $N \prec V_\lambda$ countable such that $B \in N$.
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Proof continued

Since $\pi_N : N \to M$ is an isomorphism and $Q = \pi_N(B)$, $\pi_N(r) = r$,

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Now M is a countable and transitive model of ZFC with $Q \in M$. Thus there is $G \in V$ which is an M-generic filter for Q, hence for some $a \in M[G]$

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by Cohen's forcing theorem.

Proof continued

Since $\pi_N : N \to M$ is an isomorphism and $Q = \pi_N(B)$, $\pi_N(r) = r$,

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Since $M[G]$ is countable and transitive, $M[G] \in H_{\omega_1}$ and $M[G] \subset H_{\omega_1}$, thus $a, r \in H_{\omega_1}$.
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Boolean ultrapowers of H_κ

To analyze how to use forcing for the analysis of compact spaces other than 2^ω it is more convenient to move from H_{ω_1} to the analysis of H_κ for larger κ.

If we can define *elementary* boolean ultrapowers of H_κ, we can naturally define *elementary* boolean ultrapowers of any compact Hausdorff Y (or more generally any mathematical structure) definable in H_κ.

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How to get Woodin's absoluteness

There are two main ingredients in Woodin's arguments:

1. Coll$(\omega, <\delta)$ absorbs any B $\in V_\delta$ if δ is inaccessible. Moreover Coll$(\omega, <\delta)/K = \text{Coll}(\omega, <\delta)^{V[K]}$ for any K V-generic for B.
2. If δ is *large* (for example Woodin) and G is V-generic for Coll$(\omega, <\delta)$, then

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Hence

$$(H_{\omega_1}^{V}, \in) \prec (H_{\omega_1}^{V[K]}, \in).$$

Juggling with the notion of V-genericity and forcing we conclude that

$$(H_{\omega_1}, \in) \prec (H_{\omega_1}^{V^{\mathsf{B}}}/_K, \in /_K)$$

for all $K \in \mathrm{St}(\mathsf{B})$ in V (no requirement for K to be V-generic for B).

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What do Woodin's ingredients say

1. There are universal objects which absorbs all forcings of smaller size (i.e. the boolean completion of the posets $\mathrm{Coll}(\omega, < \delta)$) in the category poset

$$(\Omega, \leq_\Omega),$$

where Ω is the class of all cbas and B $\leq_\Omega$ C if there is a complete homomorpshim of C into B.

2. These universal objects Q are well behaved
   - $Q/H = Q^{V[H]}$ if H is any V-generic filter for some B absorbed by Q of smaller size.
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Forcing axioms as density properties of class posets.

Definition

Let Γ be a class of complete boolean algebras and Θ be a class of complete homomorphisms between elements of Γ and closed under composition and identity maps.

- $B \leq_\Theta Q$ if there is a complete homomorphism $i : B \to Q$ in Θ.
- $B \leq^*_\Theta Q$ if there is a complete and *injective* homomorphism $i : B \to Q$ in Θ.

With these definitions $(\Gamma, \leq_\Theta)$ and $(\Gamma, \leq^*_\Theta)$ are class partial orders.

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We can look at these class partial orders as forcing notions, and check whether they are interesting forcing notions.

In particular we look for universal objects satisfying both of Woodin's ingredients for some H_λ with $\lambda > \omega_1$.

The order $\leq^*_\Theta$ is the one we use to study iterated forcing and captures the notion of complete embedding for partial orders.

$\leq_\Theta$ has been neglected so far but is sufficient to grant that whenever $i : \mathsf{B} \to \mathsf{Q}$ witnesses $\mathsf{Q} \leq_\Theta \mathsf{B}$ and G is V-generic for Q, then $i^{-1}[G]$ is V-generic for B.

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Theorem

The following holds:

- ***Woodin:*** *Assume there are class many Woodin cardinals. Then Martin's maximum is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_\Omega)$.
- ***V.:*** *Assume there are class many Woodin cardinals Then* MM^{++} *(a strong form of* MM*) is equivalent to the assertion that the family of presaturated towers* $\mathcal{T}$ *is dense in* $(\mathsf{SSP}, \leq_{\mathsf{SSP}})$*, where* $\mathsf{B} \geq_{\mathsf{SSP}} \mathsf{Q}$ *iff there is* $i : \mathsf{B} \to \mathsf{Q}$ *complete homomorphism such that*

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The above theorem give ingredient 2 of Woodin for H_{ω_2} **and** SSP: *if* $\mathcal{T}$ *is a presaturated tower with critical point of generic embedding* ω_2,

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Strongest forcing axioms

Definition (V.)

MM^{+++} holds if the class of *strongly* presaturated towers is dense in $(SSP, \leq_{SSP})$.

Fact

$MM^{+++} \Rightarrow MM^{++} \Rightarrow MM$.

Theorem (V.)

MM^{+++} *is consistent relative to the existence of a huge cardinal.*

I postpone (or omit) the definition of rigidly presaturated tower.

Remark

MM^{+++} *will be forced by any of the standard iteration of length* δ *which yield* MM *provided that* δ *is superhuge.*

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Assume that δ is supercompact. Then $(\mathsf{SSP} \cap V_\delta, \leq_{\mathsf{SSP}} \restriction V_\delta)$ is an SSP partial order U_δ.
Moreover:

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- U_δ *forces* MM^{++}.

The first and second item gives ingredient 1 of Woodin's recipe.

Theorem (V.)

Assume δ is a reflecting cardinal and MM^{+++} holds. Then U_δ is itself a rigidly presaturated tower. Hence

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- $\mathsf{B} \geq_{\mathsf{SSP}} \mathsf{U}_\delta \restriction \mathsf{B}$ *for all* $\mathsf{B} \in \mathsf{SSP} \cap V_\delta$.
- $(\mathsf{U}_\delta \restriction \mathsf{B})/_G = \mathsf{U}_\delta^{V[G]}$ *whenever G is V-generic for* B.
- U_δ *forces* MM^{++}.

The first and second item gives ingredient 1 of Woodin's recipe.

Theorem (V.)

Assume δ is a reflecting cardinal and MM^{+++} holds. Then U_δ is itself a rigidly presaturated tower. Hence

$$(H_{\omega_2}, \in) \prec (H_{\omega_2}^{V^{\mathsf{U}_\delta}}, \in).$$

These items give both of Woodin's ingredient at the level of H_{ω_2} and allow to prove that

$$(H_{\omega_2}, \in) \prec (H_{\omega_2}^{V^{\mathsf{B}}}/_G, \in/_G)$$

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In general the following holds for suitable properties $\phi(x)$ for the category forcing U_δ:

> *$\phi(\mathsf{U}_\delta)$ holds if and only if the following set*
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Modular generic absoluteness and modular category forcing axioms

Definition

Let $\phi(x)$ be a Π_1-property.
Γ is ϕ-preserving if for all $\mathsf{B} \in \Gamma$ and all $S \in V$ such that $\phi(S)$ holds, we have that

$$V^{\mathsf{B}} \models \phi(\check{S}).$$

Properness, semiproperness, stationary set preserving forcings are all ϕ-preserving for suitable Π_1-properties $\phi(x)$.

- SSP: $\phi_{\mathrm{SSP}}(S) \equiv S$ is a stationary subset of ω_1
- **Properness:** $\phi_{\mathrm{proper}}(S) \equiv S$ is a stationary subset of $[X]^{\aleph_0}$ for some X.
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Lemma

Assume Γ is ϕ_Γ-preserving. Then Γ is closed under two step iterations, lottery sums and preimages of complete homomorphisms.

Γ-total rigidity

Definition

Assume Γ is closed under two-steps iterations.

$B \in \Gamma$ is Γ-totally rigid if for all $Q \leq_\Gamma B$ there exists only one $i : B \to Q$ witnessing it.

Remark

Any $B \in \Gamma$ which is Γ-rigidly presaturated is forcing equivalent to a presaturated tower and is also Γ-totally rigid. It is not clear if the converse holds. For this reason the definition I came up for a Γ-rigidly presaturated tower is more involved.

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Definition (V.)

CFA(Γ) holds if the class of Γ-rigidly presaturated towers which belong to Γ is dense in $(\Gamma, \leq_\Gamma)$.

Definition (V.,Asperó)

Γ is κ-suitable, if:

- it is ϕ-preserving for some Π_1-property $\phi(x)$ definable by a parameter in H_{κ^+},
- it is κ-iterable (essentially it has "nice" lower bounds in Γ for all "nice" $\leq^*_\Gamma$-descending sequences),
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Assume Γ is κ-suitable for some κ. Assume moreover that there are class many reflecting cardinals.
Then CFA(Γ) *entails that the theory of $L(\mathrm{Ord}^{\kappa}) \supseteq H_{\kappa^+}$ is invariant with respect to forcing in Γ which preserve* CFA(Γ).

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The following holds:

1. *Assume* Γ *is the intersection of the following family of classes given by the union of*
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   Then Γ *is* ω_1*-suitable.*
2. *There is a ninth* ω_1*-suitable class* Γ *such that* CFA(Γ) *implies* CH.

We obtain nine distinct classes Γ making the theory of $L(\mathrm{Ord}^{\omega_1})$ generically invariant with respect to the relevant forcings.

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Γ-correct filters

Definition

Let Γ be a κ-suitable class of forcings and ϕ_Γ be the Π_1-property preserved by Γ.
Let $M \prec H_\theta$ with $\mathrm{B} \in M \cap \Gamma$ and $\kappa \subseteq M$, $\mathrm{otp}(M \cap \theta) \leq \kappa^+$.
Let $\pi_M : M \to N_M$ be the transitive collapse map of $(M, \in)$.

$H \in \mathrm{St}(\mathrm{B} \cap M)$ is Γ-correct if

$$V \models \phi_\gamma(\pi_M(\dot{S})_{\pi_M[H]})$$

for all $\dot{S} \in M \cap V^{\mathrm{B}}$ such that $\llbracket \phi_\gamma(\dot{S}) \rrbracket = 1_{\mathrm{B}}$.

For example if $\Gamma = \mathrm{SSP}$,
Γ-correct filters for M and B are ultrafilters H for $\mathrm{B} \cap M$ which evaluate as stationary subsets of ω_1 in V all B-names for stationary subsets of ω_1 in M.

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Self-generic filters

Let $\mathcal{I} = \{I_X : X \in V_\delta\}$ be a tower of normal ideals and $\mathrm{T}_\mathcal{I}$ be the corresponding tower forcing.

For example if $\mathcal{I} = \{\mathrm{NS}_X : X \in V_\delta\}$, $\mathrm{T}_\mathcal{I}$ is Woodin's stationary tower.

$M \prec H_{\delta^+}$ is $\mathcal{I}$-self generic if

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We let $T_\mathcal{I}$ denote the set of such models M.

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Γ-rigidly presaturated towers

Definition

Let $\mathcal{I} = \{I_X : X \in V_\delta\}$ be a tower of normal ideals and Γ be a κ-suitable class of forcings.

$\mathrm{T}_{\mathcal{I}}$ is Γ-rigidly presaturated if:

- for all $M \prec H_{\delta^+}$ G_M is the unique possible Γ-correct M-generic filter for $\mathrm{T}_{\mathcal{I}}$.
- For all $S \in \mathrm{T}_{\mathcal{I}}$
$$T_{\mathcal{I}} \wedge S$$
is stationary.

Comments and open questions

- Category forcing axioms spring out from a natural inquire to strengthen as much as possible the nonconstructive tools.
- Most often BCT and AC suffice. In some cases (which are not resticted to set theory but occurs also in other parts of mathematics) generic absolutness arguments for projective sets are useful.
- This leads us to model theoretic considerations which show that forcing axioms yield a variety of canonical elementary superstructures of initial fragments of V.
- We now have a definite pattern which isolate a modular strategy to obtain forcing axioms (the axioms CFA(Γ) for a κ-suitable Γ) yielding more and more generic absoluteness for larger and larger fragments of the universe.
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Bibliography II

THANKS FOR YOUR PATIENCE AND ATTENTION