Approximate injectivity

J. Rosický

joint work with W. Tholen

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$$(\forall x)(\delta_A(x) \to (\exists y)(\delta_B(y) \wedge (x = yf))$$

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In structures, they correspond to classes axiomatizable by *regular* theories. These are theories consisting of sentences in L_κ

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Small injectivity classes are abundant – they include injective modules or Kan complexes.

In a *metric enriched category* $\mathcal{K}$, hom-sets $\mathcal{K}(A, B)$ are metric spaces. In the special case of categories *concrete* over metric spaces, objects of $\mathcal{K}$ are metric spaces as well. Examples include metric structures, in particular Banach spaces with their linear operators of norm at most 1.

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An object K is ε-*injective* to a morphism $f : A \to B$ in a metric enriched category $\mathcal{K}$ if, for every morphism $g : A \to K$, there exists a morphism $h : B \to K$ with $d(hf, g) \leq \varepsilon$.

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An object K is *approximately injective* to a morphism $f : A \to B$ in a metric enriched category $\mathcal{K}$ if it is ε-injective to f for every $\varepsilon > 0$.

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Let $\mathcal{F}$ consists of isometries between finitely dimensional Banach spaces. Then $\mathrm{Inj}\,\mathcal{F}$ and $\mathrm{Inj}_{\mathsf{ap}} \mathcal{F}$ are distinct.

Let λ be a regular cardinal. A morphism $f : K \to L$ in a category $\mathcal{K}$ is λ-*pure* if for any commutative square

$$\begin{array}{ccc} K & \xrightarrow{\ f\ } & L \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} \\ A & \xrightarrow[\ g\ ]{} & B \end{array}$$

with A and B λ-presentable in $\mathcal{K}$ there exists $t : B \to K$ such that $tg = u$.

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Theorem 1.(Adámek, Borceux, JR 2002) The λ-injectivity classes in structures are precisely the classes closed under products, λ-directed colimits and λ-pure subobjects.

Let λ be a regular cardinal. A morphism $f : K \to L$ in a metric enriched category $\mathcal{K}$ is λ-ε-*pure* if for any ε-commutative square

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Unlike λ-pure morphisms, λ-ε-pure morphisms are not necessarily monomorphisms. We say that $\mathcal{L}$ is *closed under λ-ε-pure morphisms* if $f : K \to L$ λ-ε-pure and $L \in \mathcal{L}$ implies that $K \in \mathcal{L}$.

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In metric structures, λ-ε-injectivity classes are closed under products, λ-directed colimits and λ-ε-pure morphisms. We do not know whether the converse is true.

Let λ be a regular cardinal. A morphism $f : K \to L$ in a metric enriched category $\mathcal{K}$ is *weakly* λ-ε-*pure* if for any ε-commutative square

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Theorem 2. Let λ be an uncountable regular cardinal. Then every class of metric structures closed under products, λ-directed colimits and weakly λ-ε-pure morphisms is a λ-ε-injectivity class.

A morphism in a metric enriched category is *(weakly)* λ-ap-*pure* if it is (weakly) λ-ε-pure for any $\varepsilon > 0$.

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Theorem 3. Let λ be an uncountable regular cardinal. Then the approximate λ-injectivity classes in metric structures are precisely the classes closed under products, λ-directed colimits and weakly λ-ap-pure subobjects.

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In fact, let $p : K \to L$ be λ-ap-pure and L belong to $\mathrm{Inj}_{\mathrm{ap}}\,\mathcal{F}$. Consider $f : A \to B$ in $\mathcal{F}$, $\varepsilon > 0$ and $u : A \to K$. There is $v : B \to L$ such that $d(pu, vf) \leq \frac{\varepsilon}{2}$. Since p is weakly λ-$\frac{\varepsilon}{2}$-pure, there exists $t : B \to K$ with $d(tf, u) \leq \varepsilon$. Thus K is ε-injective to f.

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A morphism $f : K \to L$ in a metric enriched category *ε-splits* if there is $g : L \to K$ with $d(gf, \mathrm{id}_K) \leq \varepsilon$.

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Theorem 5. Assuming Vopěnka's principle, the ap-injectivity classes in metric structures are precisely the classes closed under products and ap-split subobjects.