Strong DOP and the Borel hierarchy

Miguel Moreno

Department of Mathematics and Statistics
University of Helsinki

Set-theoretical aspects of the model theory of strong logics

Outline

Outline

The spectrum problem

Let $I(T,\alpha)$ denote the number of non-isomorphic models of T with cardinality α.

What is the behavior of $I(T,\alpha)$?

- **Löwenheim-Skolem Theorem:** $\exists\alpha \geq \omega \ I(T,\alpha) \neq 0 \Rightarrow \forall\beta \geq \omega \ I(T,\beta) \neq 0$.
- **Morley's categoricity:** $\exists\alpha > \omega \ I(T,\alpha) = 1 \Rightarrow \forall\beta > \omega \ I(T,\beta) = 1$
- **Shelah's Main Gap Theorem:** Either, for every uncountable cardinal α, $I(T,\alpha) = 2^{\alpha}$, or $\forall\alpha > 0 \ I(T,\aleph_\alpha) < \beth_{\omega_1}(\mid \alpha \mid)$.

Approaches

- Shelah's stability theory.
  Classify the models of T by cardinal invariants and clearly differentiate between the theories that can be classified and those that cannot.

- Descriptive set theory:
  It uses Borel-reducibility and the isomorphism relation to define a partial order on the set of all first-order complete countable theories.

The topology

κ is a cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set 2^κ with the bounded topology. For every $\zeta \in 2^{<\kappa}$, the set

$$[\zeta] = \{\eta \in 2^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The collection of Borel subsets of 2^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Reductions

A function $f\colon 2^\kappa \to 2^\kappa$ is *Borel*, if for every open set $A \subseteq 2^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of 2^κ.

Let E_1 and E_2 be equivalence relations on 2^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon 2^\kappa \to 2^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B E_2$.

Coding structures

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in 2^\kappa$ define the structure $\mathcal{A}_f$ with domain κ by: for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) = 1$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in 2^\kappa$ are $\cong_T^\kappa$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The complexity

We can define a partial order on the set of all first-order complete countable theories

$$T \leqslant^\kappa T' \textit{ iff } \cong^\kappa_T \leqslant_B \cong^\kappa_{T'}$$

Outline

Shelah's Main Gap Theorem

Theorem (Shelah)

If T is classifiable and T' is not, then T is less complex than T' and their complexity are not close.

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem in the space 2^κ?

ENI-DOP

For every $L_{\omega_1,\omega}$-sentence, φ, denote by iso^{ω}_{φ} the isomorphism relation of the models of φ with universe ω.

Definition

An ω-stable theory T has ENI-NDOP if the primary model over any independent triple of ω-saturated models is ω-saturated. We say T has ENI-DOP if it fails to have ENI-NDOP.

Definition

We call a theory T Borel complete if $iso^{\omega}_{\varphi} \leqslant_B \cong^{\omega}_{T}$ for every $L_{\omega_1,\omega}$-sentence φ.

Theorem (Laskowski, Shelah)

If T is ω-stable with ENI-DOP, then T is Borel complete.

Countable

$T = Th(\mathbb{Q}, \leqslant)$.
T', the theory of vector space over the field of rational numbers.

By the Borel-reducibility hierarchy:

$$T \leqslant^{\omega} T'$$

$$T' \not\leqslant^{\omega} T$$

By the stability theory T' is simpler than T.

Uncountable

Theorem (Shelah)

If T is classifiable, then T is Δ_1^1.

Theorem (Friedman, Hyttinen, Kulikov)

If T is unstable then T is not Δ_1^1.

Theorem (Friedman, Hyttinen, Kulikov)

If T is unstable and T' is classifiable, then $T \not\leq^\kappa T'$.

The Equivalence Modulo Non-stationary Ideals

Definition

For every $X \subset \kappa$ stationary, we define E_X as the relation

$$E_X = \{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\eta^{-1}[1] \triangle \xi^{-1}[1]) \cap X \text{ is not stationary}\}$$

where $\triangle$ denotes the symmetric difference.

When $X = \{\alpha < \kappa | cf(\alpha) = \lambda\}$, we will denote E_X by E_λ.

Looking above the Gap

Theorem (Friedman, Hyttinen, Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ *and* $\lambda^{<\lambda} = \lambda$.

- *If* T *is an unstable or superstable with OTOP, then* $E_\lambda \leqslant_B \cong^\kappa_T$.
- *If* $\lambda \geq 2^\omega$ *and* T *is a superstable with DOP, then* $E_\lambda \leqslant_B \cong^\kappa_T$.

Theorem (Friedman, Hyttinen, Kulikov)

Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ *and* T *is a stable unsuperstable.*
Then $E_\omega \leqslant_B \cong^\kappa_T$

Looking below the Gap

Theorem (Friedman, Hyttinen, Kulikov)

If T is a classifiable theory, then for all regular cardinal $\lambda < \kappa$, $E_\lambda \not\leqslant_B \cong^\kappa_T$

Theorem (Hyttinen, Kulikov, Moreno)

Denote by S^κ_λ the set $\{\alpha < \kappa | cf(\alpha) = \lambda\}$.
Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. If $\diamondsuit(S^\kappa_\lambda)$ holds, then $\cong^\kappa_T \leqslant_B E_\lambda$.

The stable unsuperstable theories

Theorem (Hyttinen, Kulikov, Moreno)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If T is a classifiable theory and T' is a stable unsuperstable theory, then $\cong^\kappa_T \leqslant_B E_\omega \leqslant_B \cong^\kappa_{T'}$ and $E_\omega \not\leqslant_B \cong^\kappa_T$.

Proof.

Shelah proved that if $\kappa = \lambda^+ = 2^\lambda$ and S is a stationary subset of $\{\alpha < \kappa \mid cf(\alpha) \neq cf(\lambda)\}$, then $\diamondsuit(S)$ holds. So, in this case $\diamondsuit(S^\kappa_\omega)$ holds and $\cong^\kappa_T \leqslant_B E_\omega$. The other reduction is [FHK] theorem. ■

Consistency

Let $H(\kappa)$ be the following property: If T is classifiable and T' is not, then $T \leqslant^\kappa T'$ and $T' \not\leqslant^\kappa T$.

Theorem (Hyttinen, Kulikov, Moreno)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. *If $V = L$, then $H(\kappa)$ holds.*
2. *It is consistent that $H(\kappa)$ holds and there are 2^κ equivalence relations strictly between $\cong^\kappa_T$ and $\cong^\kappa_{T'}$.*

Question:
Is there a Borel reducibility counterpart of the Main Gap Theorem that does not need to force diamonds?

It can be studied in two ways:

- Does it holds $E_\omega \leqslant_B \cong^\kappa_T$ for every theory T non-classifiable under some cardinal assumptions that imply $\diamondsuit(S^\kappa_\omega)$?
- Is there a Borel reducibility counterpart of the Main Gap Theorem in another space?

Outline

The generalized Baire space

Let κ be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$.

We equip the set κ^κ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set

$$[\zeta] = \{\eta \in \kappa^\kappa \mid \zeta \subset \eta\}$$

is a basic open set.

The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Reductions in GBS

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a continuous function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \leqslant_B E_2$.

Coding structures in GBS

Fix a language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $f \in \kappa^\kappa$ define the structure $\mathcal{A}_f$ with domain κ by: for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_f} \Leftrightarrow f(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition (The isomorphism relation)

Given T a first-order complete countable theory in a countable vocabulary, we say that $f, g \in \kappa^\kappa$ are $\cong^\kappa_T$ equivalent if

- $\mathcal{A}_f \models T, \mathcal{A}_g \models T, \mathcal{A}_f \cong \mathcal{A}_g$
  or
- $\mathcal{A}_f \not\models T, \mathcal{A}_g \not\models T$

The Equivalence Modulo Non-stationary Ideals in GBS

We say that $f, g \in \kappa^\kappa$ are E_λ equivalent if the set $\{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ contains an unbounded set that is closed under λ-limits.

Theorem (Hyttinen, Moreno)

Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. Then $\cong^\kappa_T \leqslant_B E_\lambda$.

Proof

Let $\mathcal{A}$ and $\mathcal{B}$ be structures with domain κ, and $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration for all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$.
The game $\mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B})$ is played by **I** and **II** as follows.
In the *n*-th turn **I** choose an ordinal $\beta_n < \kappa$ such that $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \kappa$ such that $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subset f_{\theta_n}$, the game starts with X_{β_0} and f_{θ_0} as empty sets. The game finish after ω moves.
The player **II** wins if $\cup_{i<\omega} f_{\theta_i} : A \to B$ is a partial isomorphism, otherwise the player **I** wins.

Proof

For every $\alpha < \kappa$, structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the game $\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha}, \mathcal{B}\restriction_{\alpha})$ is played by **I** and **II** as follows.
In the *n*-th turn **I** choose an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$.
The game starts with X_{β_0} and f_{θ_0} as empty sets, and finishes after ω moves. The player **II** wins if $\cup_{i<\omega} f_{\theta_i} : A\restriction_{\alpha} \to B\restriction_{\alpha}$ is a partial isomorphism, otherwise the player **I** wins.

Proof

Lemma

For every pair of strcutures, $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the following holds:

- $\mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.
- $\mathbf{I} \uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{I} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.

Definition

Given T a first order complete countable theory in a countable vocabulary and $\alpha \leqslant \kappa$. Define the relation $R^{\alpha}_{EF} \subseteq \kappa^{\kappa} \times \kappa^{\kappa}$ as $\eta\ R^{\alpha}_{EF}\ \xi$ if $\mathcal{A}_{\eta}\restriction_{\alpha} \not\models T$ and $\mathcal{A}_{\xi}\restriction_{\alpha} \not\models T$, or $\mathcal{A}_{\eta}\restriction_{\alpha} \models T$, $\mathcal{A}_{\xi}\restriction_{\alpha} \models T$ and the player $\mathbf{II}$ has a winning strategy for the restricted game $EF^{\kappa}_{\omega}(\mathcal{A}_{\eta}\restriction_{\alpha},\mathcal{A}_{\xi}\restriction_{\alpha})$.

Proof

Lemma

For every T first order complete countable theory in a countable vocabulary, there are club many α such that R^α_{EF} is an equivalence relation.

Define the reduction as follows.
For every $\eta \in \kappa^\kappa$ define the function f_η, as $f_\eta(\alpha)$ a code in $\kappa\backslash\{0\}$ for the R^α_{EF} equivalence class for $\mathcal{A}_\eta\restriction_\alpha$, when $cf(\alpha)=\lambda$, $\mathcal{A}_\eta\restriction_\alpha\models T$, and R^α_{EF} is an equivalence relation; $f_\eta(\alpha)=0$ in other case.

s-isolation

Definition

Denote by F^s_λ the set of pairs (p, A) with $|A| < \lambda_r(T)$, such that for some $B \supseteq A$, $p \in S(B)$, and $p \restriction_A \vdash p$.

1. *We say that C is s-constructible over A if there is a sequence $(a_i, B_i)_{i<\alpha}$ such that $C = A \cup \bigcup_{i<\alpha} a_i$, and for all $i < \alpha$, $(t(a_i, A_i), B_i) \in F^s_\lambda$, where $A_i = A \cup \bigcup_{j<i} a_j$.*
2. *We say that C is s-primary over A if it is s-constructible over A and $\lambda_r(T)$-saturated.*

The Orthogonal Chain Property

Definition

Given $p \in S(A)$ and $B \subseteq A$, we say $p \perp B$ if for every $q \in S(A)$ that doesn't fork over B the following holds; for every a, b, and $B' \supseteq A$, if a realizes p, b realizes q, $a \downarrow_A B'$ and $b \downarrow_A B'$ then $a \downarrow_{B'} b$.

Definition

A theory T has the property OCP if there exist $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, such that for every $i \leqslant j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$ and $a \notin \cup_{i<\omega}\mathcal{A}_i$ such that $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_i$.

Define $T_\omega = Th((\omega^\omega, R_n)_{n<\omega})$, where $\eta\ R_n\ \xi$ holds if $\eta \restriction_n = \xi \restriction_n$. T_ω has the OCP.

OCP and superstable

Lemma

If a theory T has the OCP, then T is not superstable.

Theorem (Hyttinen, Moreno)

Suppose T is a classifiable theory, T' is an stable theory with the OCP, and κ an inaccessible cardinal. Then $\cong^{\kappa}_{T} \leqslant_B \ E_{\omega} \leqslant_B \cong^{\kappa}_{T'}$

a-isolation

Definition

Denote by F^a_ω the set of pairs (p, A) with $|A| < \omega$, such that for some $B \supseteq A$, $p \in S(B)$, $a \models p$ and $stp(a, A) \vdash p$.

1. *We say that C is a-constructible over A if there is a sequence $(a_i, B_i)_{i<\alpha}$ such that $C = A \cup \bigcup_{i<\alpha} a_i$, and for all $i < \alpha$, $(t(a_i, A_i), B_i) \in F^a_\omega$, where $A_i = A \cup \bigcup_{j<i} a_j$.*
2. *We say that C is a-saturated if for all $B \subseteq C$ of power less than $|C|^+$ and $p \in S(B)$ the following holds: if for some A, $(p, A) \in F^a_\omega$, then p is realized in C.*
3. *We say that C is a-primary over A if it is a-constructible over A and a-saturated.*
4. *We say that C is a-atomic over A if for every $c \in C$, there is $B \subseteq A$ such that $(t(c, A), B) \in F^a_\omega$*

DOP

Definition

A theory T has the dimensional order property (DOP) if there are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, $M_1 \downarrow_{M_0} M_2$, and the a-primary model over $M_1 \cup M_2$ is not a-minimal over $M_1 \cup M_2$.

Lemma (Shelah)

Let $M_0 \subset M_1 \cap M_2$ be a-saturated models, $M_1 \downarrow_{M_0} M_2$, M a-atomic over $M_1 \cup M_2$ and a-saturated. Then the following conditions are equivalent:

1. *M is not a-minimal over $M_1 \cup M_2$.*
2. *There is a type $p \in S(M)$ orthogonal to M_1 and to M_2, p not algebraic.*

Strong DOP

Definition

We say that a theory T has the strong dimensional order property (S-DOP) if the following holds:
There are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 a-primary model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, that does not fork over $M_1 \cup M_2$.

S-DOP

The following theory in the language $\{P_0, P_1, P_2, f_0, f_1\}$ has the S-DOP.
Let P_0, P_1 and P_2 be disjoint unary relations such that for every a, $a \in P_0 \cup P_1 \cup P_2$.
The functions f_0 and f_1 satisfy:

- $f_i(a) = a$ for every $i < 2$ and $a \notin P_2$.
- $f_i(a) \in P_i$ for every $i < 2$ and $a \in P_2$.
- $\forall a \in P_0$, $\forall b \in P_1$, exists infinitely many $c \in P_2$ such that $f_0(c) = a$ and $f_1(c) = b$.

S-DOP

Lemma

If a theory T has the S-DOP, then T has the DOP.

Theorem

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, $\lambda \geq 2^\omega$, and κ an inaccessible cardinal. Then $\cong^\kappa_T \leqslant_B E_\lambda \leqslant_B \cong^\kappa_{T'}$

Sum up

- When $\kappa = \omega$ the classifications are different.

- It is consistent that there is a generalized Borel reducibility counterpart of the Main Gap Theorem in the space 2^κ.

- For κ inaccessible, the classifiable theories are at most as complex as the theories with OCP or S-DOP.

Questions

1. Denote by E^κ_λ and E^2_λ the relation E_λ in the spaces κ^κ and 2^κ respectively. For which λ holds $E^\kappa_\lambda \leqslant_B E^2_\lambda$?

2. For which λ holds $E_\lambda \leqslant_B \cong^\kappa_T$ in the space κ^κ for every theory T non-classifiable?

3. For which λ holds $E_\omega \leqslant_B \cong^\kappa_T$ in the space 2^κ for every theory T non-classifiable?

References

- S.D. Friedman, T. Hyttinen, and V. Kulikov, *Generalized descriptive set theory and classification theory*, Memoirs of the Amer. Math. Soc. Vol. 230/1081 (American Mathematical Society, 2014).
- T. Hyttinen, and M. Moreno, *On the reducibility of isomorphism relations*, Mathematical Logic Quarterly. To appear.
- T. Hyttinen, V. Kulikov, and M. Moreno, *A Generalized Borel-reducibility Counterpart of Shelah's Main Gap Theorem*, (arXiv:1602.00605).
- M. Laskowski, and S. Shelah, *Borel completeness of some $\aleph_0$-stable theories*, Fundamenta Mathematicae. 229(1) 1–46, 2015.
- S. Shelah, *Classification theory*, Stud. Logic Found. Math. Vol. 92, (North-Holland, Amsterdam, 1990).