# Reflection of Magidor-Malitz Quantifiers

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Let $\kappa < \lambda$ be two infinite cardinals, and let $\mathcal{A}$ be a structure of size $\lambda$. Can we always find a substructure $\mathcal{B} \subseteq \mathcal{A}$ of size $\kappa$ which is similar to $\mathcal{A}$?

The answer depends on the definition of "similar".

- If "similar" is "first-order elementary submodel" (with countable language), then by Löwenheim-Skolem-Tarski Theorem, the answer is always "Yes".
- If "similar" is "second-order elementary submodel", then the answer is "No" below the first supercompact cardinal (due to Magidor).

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- If "similar" is "first-order elementary submodel" (with countable language), then by Löwenheim-Skolem-Tarski Theorem, the answer is always "Yes".
- If "similar" is "second-order elementary submodel", then the answer is "No" below the first supercompact cardinal (due to Magidor).
- When "similar" is taken to be "elementary submodel relative to some intermediate logic", the answer is more complicated:
  - Probably independent of $\mathrm{ZFC}$.
  - Likely to require large cardinals.

We focus on a specific fragment of second order logic - the quantifiers of Jerome Malitz and Menachem Magidor.

**Definition**

Let $\mathcal{A}$ be a model and let $\vec{p} \in \mathcal{A}^{<\omega}$.
We define:

$$\mathcal{A} \models Q^n x_0, \ldots, x_{n-1} \varphi(x_0, \ldots, x_{n-1}, \vec{p})$$

iff

$$\exists I \subseteq \mathcal{A},\ |I| = |\mathcal{A}|,\ \forall a_0, \ldots, a_{n-1} \in I, \mathcal{A} \models \varphi(a_0, \ldots, a_{n-1}, \vec{p})$$

The quantifiers $Q^n$ (for $n < \omega$) are called Magidor-Malitz quantifiers.

## Definition

Let $\mathcal{B} \subseteq \mathcal{A}$. We say that $\mathcal{B} \prec_{Q^n} \mathcal{A}$ if for every $Q^n$-formula $\varphi$ and every $\vec{b} \in \mathcal{B}$,

$$\mathcal{B} \models \varphi(\vec{b}) \iff \mathcal{A} \models \varphi(\vec{b})$$

We denote by $\mathcal{B} \prec_{Q^{<\omega}} \mathcal{A}$ the statement $\forall n < \omega$, $\mathcal{B} \prec_{Q^n} \mathcal{A}$

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Note that the interpretation of the quantifiers $Q^n$ in both models is different, as we interpret the quantifiers according to the size of the model.

## Definition

Let $\kappa < \lambda$ be cardinals. We denote by $\lambda \underset{Q^n}{\twoheadrightarrow} \kappa$ the statement:

For every model $\mathcal{A}$ of cardinality $\lambda$ there is a submodel $\mathcal{B} \prec_{Q^n} \mathcal{A}$, $|\mathcal{B}| = \kappa$.

Recall:

**Definition (Chang's conjecture)**

For cardinals $\nu > \mu,\ \lambda > \kappa$ we say that

$$(\nu, \mu) \twoheadrightarrow (\lambda, \kappa)$$

if for every model $\mathcal{A}$ of cardinality $\nu$ with distinguished predicate $R$, $|R^{\mathcal{A}}| = \mu$, there is an elementary submodel $\mathcal{B}$, $|\mathcal{B}| = \lambda$, $|R^{\mathcal{B}}| = \kappa$.

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Chang's conjecture fails in $L$. The minimal instance $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ is equiconsistent with the existence of $\omega_1$-Erdős cardinal.
The exact consistency strength of $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$ is unknown.

### Fact

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### Proof:

$\Rightarrow$: Let $\mathcal{A}$ be a structure of cardinality $\lambda^+$, with distinguished predicate $R$, $|R^{\mathcal{A}}| = \lambda$. Enrich the model $\mathcal{A}$ with a well ordering of its elements, $<$, of order type $\lambda^+$, such that the first $\lambda$ elements in the order are $R^{\mathcal{A}}$. Add a function to $\mathcal{A}$, $F$, such that for every $a < b$, $F(a, b) \in R^{\mathcal{A}}$ and the function $x \to F(x, b)$ is one to one on $\{x \in \mathcal{A} \mid x < b\}$. Reflect the $Q^1$ sentence $\neg Q^1 x, R(x)$, together with the first order theory of the enriched model.

## Fact

$\lambda^+ \underset{Q^1}{\twoheadrightarrow} \kappa^+$ *iff* $(\lambda^+, \lambda) \twoheadrightarrow (\kappa^+, \kappa)$.

## Proof (Cont.):

$\Leftarrow$: For every formula $\varphi$ in the language $\mathcal{L}(Q^1)$ define a Skolem function $F_\varphi$ witnessing its truth value. Namely, if, for example, $\mathcal{A} \models Q^1 x\varphi(x)$ then $F_\varphi$ is one to one function from elements of the model and for every $y$, $\varphi(F_\varphi(y))$ holds. If $\neg Q^1\varphi(x)$ but $\exists x\varphi(x)$, then $F_\varphi$ is a surjection from the elements of $R$ to $\{x \mid \varphi(x)\}$.
Any substructure $\mathcal{B}$ of the enriched model that satisfies $|R^{\mathcal{B}}| = \kappa$, $|\mathcal{B}| = \kappa^+$, will satisfy $\mathcal{B} \prec_{Q^1} \mathcal{A}$ (in the original language). □

**Example**

If $\aleph_2 \underset{Q^2}{\twoheadrightarrow} \aleph_1$ then any $\omega_2$-Aronszajn tree has a subtree which is $\omega_1$-Aronszajn tree. Similarly, for Suslin trees.

This is true as the statements "There is a cofinal branch" or "There is an unbounded antichain" are $Q^2$-sentences.

Recall:

## Definition

Let $\kappa$ be a regular cardinal. A sequence $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\square(\kappa)$-sequence if:

1. $C_\alpha \subseteq \alpha$ is a club.
2. $\forall \beta \in \operatorname{acc} C_\alpha,\ C_\beta = C_\alpha \cap \beta$.
3. There is no club $D \subseteq \kappa$, such that $\forall \beta \in \operatorname{acc} D,\ D \cap \beta = C_\beta$.

# Magidor-Malitz Reflection is stronger than Chang's conjecture

## Theorem

*Assume that* $\lambda \underset{Q^2}{\twoheadrightarrow} \kappa$*, where* $\kappa < \lambda$ *are both regular. Then* $\neg\Box(\lambda)$*.*

Note that instances of Chang's Conjecture are consistent with this type of square.

## Proof.

Let $\mathcal{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ be a square sequence. Let $\mathcal{A}$ be a $Q^2$-elementary submodel of some structure that contains $\lambda$ and knows about $\mathcal{C}$, and assume that $|\mathcal{A}| = \kappa$. Let $\rho = \sup(\mathcal{A} \cap \lambda)$. Note that $\operatorname{otp} \mathcal{A} \cap \lambda = \kappa$ and therefore $\operatorname{cf} \rho = \kappa$.

## Proof (Cont.)

Let

$$\Delta = \operatorname{acc}(\mathcal{A} \cap \lambda) \cap \operatorname{acc} C_\rho.$$

Let $\delta < \delta' \in \Delta$ and let

$$\beta = \min \mathcal{A} \cap \lambda \setminus \delta, \ \beta' = \min \mathcal{A} \cap \lambda \setminus \delta'.$$

$\delta \in \operatorname{acc} C_\beta$, $\delta' \in \operatorname{acc} C_{\beta'}$ (as $\delta \in \operatorname{acc} \mathcal{A} \cap \lambda$ and by the minimality of $\beta$). Therefore:

1. $C_\delta \trianglelefteq C_\beta$,
2. $C_\delta \trianglelefteq C_{\delta'} \trianglelefteq C_{\beta'}$

## Proof (Cont.)

We claim that $C_\beta \trianglelefteq C_{\beta'}$. Otherwise, there was $\gamma \in (C_\beta \triangle C_{\beta'}) \cap \beta \in \mathcal{A}$ and if has to be strictly above $\delta$ and below $\beta$. By elementarity - there is such $\gamma$ in $\mathcal{A}$, which is impossible. Therefore,

$$\mathcal{A} \models Q^2\beta, \beta' \ \beta \geq \beta' \bigvee C_\beta \trianglelefteq C_{\beta'}$$

which is a contradiction to the $Q^2$-reflection. □

## Consistency of Magidor-Malitz Reflection

**Theorem**

*Assume that* $\kappa$ *is a Ramsey cardinal. Then for every regular cardinal* $\omega < \mu < \kappa$*,*

$$\kappa \xrightarrow[Q^{<\omega}]{} \!\!\!\!\rightarrow \mu.$$

**Proof.**

Let $\mathcal{A}$ be a model. Without loss of generality, the universe of $\mathcal{A}$ is $\kappa$, and let us enrich $\mathcal{A}$ with $Q^{<\omega}$-Skolem functions. Let $I = \{\rho_i \mid i < \kappa\}$ be a set of indiscernibles of order type $\kappa$. Let $\mathcal{B}$ be the Skolem closure of the first $\mu$ elements of $I$. Assume that

$$\mathcal{B} \models Q^n x_0, \ldots, x_{n-1}\, \varphi(x_0, \ldots, x_{n-1}, \vec{p})$$

## Proof (Cont.):

Let $J \subseteq \mathcal{B}$ be a witness. Let $\langle y_\xi \mid \xi < \mu \rangle$ be an enumeration of $J$.
$y_\xi = F_\psi(i_0^\xi, \ldots, i_{m-1}^\xi)$ where $i_0^\xi, \ldots, i_{m-1}^\xi \in I$.
Since cf $\mu$ is uncountable, we may assume that $\psi$ is fixed. Let $\gamma$ be the last indiscernible that appear in the definition of the parameter $\vec{p}$.

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Using the $\Delta$-system lemma, we may narrow $J$ down and assume (by reordering the entries of $F_\varphi$, is required) that there is $k < m - 1$ such that for all $\xi < \xi' < \mu$ if $j < k$, $i_j^\xi = i_j^{\xi'}$ and if $j \geq k$ then $\gamma < i_j^\xi < i_j^{\xi'}$.

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By indiscerniblity of the elements of $I$, the set

$$\{F_\psi(i_0^0, \ldots, i_{k-1}^0, \rho_{\omega\cdot\delta+k}, \rho_{\omega\cdot\delta+k+1}, \ldots, \rho_{\omega\cdot\delta+m-1}) \mid \mu \leq \delta < \kappa\}$$

Witnesses

$$\mathcal{A} \models Q^n x_0, \ldots, x_{n-1} \varphi(x_0, \ldots, x_{n-1}, \vec{p})$$

# Consistency of Magidor-Malitz Reflection

## Theorem

*Assume that $\kappa$ is a huge cardinal and let $\mu < \kappa$ be regular cardinal. There is a generic extension of the universe in which $\kappa = \mu^+$ and*

$$\mu^{++} \xrightarrow[Q^{<\omega}]{} \mu^+$$

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If we start with larger cardinals we can obtain larger gaps. For example, if there is an elementary embedding $j \colon V \to M$ with critical point $\kappa$, and $M$ is closed under sequences of length $j(\kappa)^{+\omega+1}$, then there is a generic extension in which

$$\aleph_{\omega\cdot 2+1} \xrightarrow[Q^{<\omega}]{} \!\!\!\!\rightarrow \aleph_{\omega+1}.$$

**Proof.**

Let $j\colon V \to M$, $\lambda = j(\kappa)$ and assume that ${}^{\lambda}M \subseteq M$. Let us force with two steps iteration of the Easton collapse - the first between $\mu$ and $\kappa$ and the second between $\kappa$ and $\lambda$. $j(\mathbb{P}) \cong \mathbb{P} * \mathbb{Q}$ where $\mathbb{Q}$ is subsumed by a product of $\lambda$-closed and $\lambda$-Knaster forcing notions.
By careful examination of the conditions of the form $j(p)$ for $p \in \mathbb{P}$, we can find a master condition.
Using Silver's criteria, it is possible to extend the elementary embedding by forcing with $\mathbb{Q}$. $\mathbb{Q}$ does not change the truth value of statements of the form $j\text{''}\,\mathcal{A} \models Q^n\varphi$. $\square$

# Consistency of Magidor-Malitz Reflection

**Theorem**

*Assume that there is an ideal* $\mathcal{I}$ *on* $\kappa$ *such that:*

1. $\mathcal{I}^+$ *is* $\omega+1$*-strategically closed.*
2. $\mathcal{I}$ *is* $\kappa$*-complete.*
3. $\forall\alpha<\kappa$, $\{\alpha\}\in\mathcal{I}$.

*Assume also that* $\diamondsuit(\omega_1)$ *holds. Then* $\kappa \underset{Q^{<\omega}}{\twoheadrightarrow} \aleph_1$.

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Let $\mathcal{A}$ be a model of cardinality $\kappa$. Without loss of generality, the universe of $\mathcal{A}$ is $\kappa$. Let us construct a continuous increasing sequence of countable models of length $\omega_1$, $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ such that for every $\alpha < \omega_1$, $M_\alpha \prec \langle H(\chi), \in \kappa, \mathcal{A}, \dots \rangle$ for some large $\chi$.

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For each $M_\alpha$, we define:

$$M_{\alpha+1} = \{f(\rho) \mid f \colon \kappa \to V,\ f \in M_\alpha\}$$

Where $\rho < \kappa$ is "$M_\alpha$-generic", namely, the ultrafilter

$$\{A \in \mathcal{P}(\kappa) \cap M_\alpha \mid \rho \in A\}$$

is $M_\alpha$-generic ultrafilter disjoint from the ideal $\mathcal{I}$. The exact choice of $\rho$ will play a crucial role.

For a formula $\varphi(x_0, \ldots, x_{n-1}, \vec{p})$, we say that $I$ is a $\varphi$-block if for every $a_0, \ldots, a_{n-1} \in I$, $\varphi(a_0, \ldots, a_{n-1}, p)$ holds.

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We want to show that $\rho$ can be chosen in a way that (some) maximal $\varphi$-blocks are not extended.
Let $\langle E_\alpha \mid \alpha < \omega_1 \rangle$ be a diamond sequence. By standard coding, we may assume that $E_\alpha \subseteq M_\alpha$.
Assume that the $E_\alpha$ is the set $A_\alpha \times \{\varphi, \vec{p}\}$, and $\vec{p}, A_\alpha \subseteq M_\alpha$ and $A_\alpha$ is a maximal $\varphi$ block (with parameters $\vec{p}$) and

$$\mathcal{A} \models \neg Q^n x_0, \ldots, x_{n-1}, \varphi(x_0, \ldots, x_{n-1}, \vec{p})$$

For simplicity, assume that $\varphi$ is symmetrical.

For a formula $\varphi(x_0, \dots, x_{n-1}, \vec{p})$, we say that $I$ is a $\varphi$-block if for every $a_0, \dots, a_{n-1} \in I$, $\varphi(a_0, \dots, a_{n-1}, p)$ holds.
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Assume that the $E_\alpha$ is the set $A_\alpha \times \{\varphi, \vec{p}\}$, and $\vec{p}, A_\alpha \subseteq M_\alpha$ and $A_\alpha$ is a maximal $\varphi$ block (with parameters $\vec{p}$) and

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For simplicity, assume that $\varphi$ is symmetrical. We want to show that there is a large set for choices for $\rho$ for which the block $A_\alpha$ cannot be extended. This is essentially a type-omission procedure, we want to omit the type:

$$\{x \neq a \mid a \in A_\alpha\} \cup \{\varphi(a_0, \dots, a_{n-2}, x) \mid a_0, \dots, a_{n-2} \in A_\alpha\}$$

In order to do this, we will omit much larger type, and by this we will restrict the growth of the $\varphi$-block in the next steps.
Assume that $A \subseteq \kappa$ is a positive set and let $\psi(x)$ be a formula. Let:

$$\partial_A \psi(x) \equiv x \text{ is a function from } \kappa \text{ and } \{\rho \in A \mid \psi(x(\rho))\} \in \mathcal{I}$$

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Assume that $A \subseteq \kappa$ is a positive set and let $\psi(x)$ be a formula. Let:

$$\partial_A \psi(x) \equiv x \text{ is a function from } \kappa \text{ and } \{\rho \in A \mid \psi(x(\rho))\} \in \mathcal{I}$$

This is a formula in the language of set theory. For a type $\Psi(x) \subseteq M_\alpha$ (not necessarily an element of $M_\alpha$) we define

$$\partial_A(\Psi) = \{\partial_A(\psi) \mid \psi \in \Psi\}$$

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$$\partial_A(\Psi) = \{\partial_A(\psi) \mid \psi \in \Psi\}$$

If $\partial_A(\Phi)$ is omitted in $M_\alpha$ and $A$ is in the generic filter for $M_\alpha$ then $\Phi$ is omitted in $M_{\alpha+1}$.

We iterate the definition. If $\dot{B}$ is a function from $\kappa$ to $V$ in $M_\alpha$ such that for every $\rho < \kappa$, $\dot{B}(\rho) \in \mathcal{I}^+$, and $A \in \mathcal{I}^+$, we define $\partial_A \partial_{\dot{B}} \psi(x)$ to be:

$$\forall^\star \rho \in A,\ \partial_{\dot{B}(\rho)} \psi(x(\rho))$$

## Lemma

*Let $Z \subseteq M$ be maximal $\varphi$-block and assume that $\mathcal{A} \models \neg Q^n \varphi$ let $\Psi$ be the type*

$$\{x \neq a \mid a \in Z\} \cup \{\psi(x, \vec{p}) \mid \forall a \in Z,\ \psi(a, \vec{p}),\ p \subseteq M\}$$

*Then $M$ omits all the derivatives of $\Psi$.*

## Proof.

$M$ omits $\Psi$ by the maximality of $Z$. Let us sketch the proof for the first derivative: $\partial_A \Psi$. Assume that $\partial_A \Psi(b)$ holds. Then

$$\forall^\star \alpha \in A,\ \varphi(a_0, \ldots, a_{n-2}, b(\alpha))$$

Taking $A$ and $b$ as parameters we get:

$$\forall^\star \alpha \in A \forall^\star \beta \in A,\ \varphi(a_0, \ldots, b(\beta), b(\alpha))$$

and so on. After $n$ applications of this process we get:

$$\forall^\star \alpha_0 \in A \forall^\star \alpha_1 \in A \ldots \forall^\star \alpha_{n-1} \in A\, \varphi(b(\alpha_0), \ldots, b(\alpha_{n-1}))$$

## Proof (Cont.):

This is enough in order to build a large $\varphi$-block. Construct inside of $M$ a new $\varphi$-block $D$, which has the following properties:

$$\forall b_0, \ldots, b_{k-1} \in D \forall^\star \alpha_k \in A \ldots \forall^\star \alpha_{n-1} \in A,$$
$$\varphi(b_0, \ldots, b_k, b(\alpha_k), \ldots, b(\alpha_{n-1}))$$

Using Zorn's lemma, one can construct a maximal set $D$ in of $M$. $M \models |D| = \kappa$, as otherwise, we can extend it using the "new" element that we get by applying $b$ on any element from some large set.

But by elementarity, $D$ we have size $\kappa$ in $V$, contradiction to the assumption that $\neg Q^n \varphi$.

# Open Questions

**Question**

If $\aleph_{\omega+1} \xrightarrow[Q^{<\omega}]{} \!\!\!\!\rightarrow \aleph_1$ consistent?

**Question**

What is the consistency strength of $\aleph_2 \xrightarrow[Q^{<\omega}]{} \!\!\!\!\rightarrow \aleph_1$?

In an unpublished work, Shelah showed that $\aleph_2 \xrightarrow[Q^{<\omega}]{} \!\!\!\!\rightarrow \aleph_1$ is consistent relative to a Ramsey cardinal.

Thank You!