A Generalisation of Closed Unbounded Sets and Square Sequences

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A sketch

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C is *1-club* in κ iff C is stationary in κ and stationary-closed.

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4. κ is *γ-reflecting* if for any γ-stationary $S, T \subseteq \kappa$ there is $\alpha < \kappa$ with S and T both γ-stationary below α.

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Notation

$d_\gamma(A) := \{\alpha : A \text{ is } \gamma\text{-stationary below } \alpha\}$

Restating the Definitions in Terms of d_γ

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Definition (restated)

1. $S \subseteq On$ is 0-stationary in κ if it is unbounded in κ.
2. $C \subseteq On$ is *γ-stationary closed* if $d_\gamma(C) \subseteq C$.
3. C is *γ-club* in κ if C is γ-stationary closed and γ-stationary below κ.
4. κ is *γ-reflecting* if for any γ-stationary $S, T \subseteq \kappa$, $d_\gamma(S) \cap d_\gamma(T) \cap \kappa \neq \emptyset$.
5. $S \subseteq \kappa$ is γ-stationary if for every $\gamma' < \gamma$ we have κ is γ'-reflecting and for any C γ'-club in κ we have $S \cap C \neq \emptyset$

how large is a subset of κ?

If κ is n-reflecting, then for a subset of κ we have these implications:

$$\begin{array}{ccc} n\text{-club} & \Longrightarrow & n+1\text{-stationary} \\ \Uparrow & & \Downarrow \\ n-1\text{-club} & & n\text{-stationary} \\ \Uparrow & & \Downarrow \\ \dots & & \dots \\ \Uparrow & & \Downarrow \\ 0\text{-club } (= \text{club}) & & \text{stationary} \\ & & \Downarrow \\ & & \text{unbounded} \end{array}$$

Origins

- In *"Reflection and indescribability in the constructible universe"*, Bagaria, Magidor, and Sakai (BMS) introduce a generalisation of stationary sets based on stationary reflection.
- Their definitions are the origins of those I give above, the definitions stated there aren't quite equivalent, and did not specify a notion of generalised club.
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Questions:

- What can we do with these notions?
- What properties do they have?
- What generalises and what doesn't?

Results in L

Theorem (Jensen) ($V = L$)

A regular cardinal reflects stationary sets iff it is weakly compact ($= \Pi^1_1$-indescribable).

Theorem (BMS) ($V = L$)

A regular cardinal reflects n-stationary sets iff it is Π^1_n-indescribable. ($1 < n < \omega$)

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- Bagaria, Magidor and Sakai use an induction with Jensen's result for the case $n = 1$, but no $\square$ analogue for $n > 1$.
- Can we define and construct some such generalised $\square$?

Generalising $\square$

Definition

A $\square^{\gamma}$ *sequence* below κ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ so that for all α:

1. C_α is an γ-club subset of α
2. for every $\beta \in d_\gamma(C_\alpha)$ we have $C_\beta = C_\alpha \cap \beta$

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Theorem (Brickhill) ($V = L$)

If κ is Π^1_n- but not Π^1_{n+1}-indescribable then for any $n+1$-stationary $A \subseteq \kappa$ there is an $n+1$-stationary set $A' \subseteq A$ and $\square^n$ sequence avoiding A'. Thus κ is not $n+1$-reflecting. $(0 < n < \omega)$

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- we need a definition of Π^1_γ-indescribable
- we also need a slightly different $\square$ sequence for the limit stages.

$\square^{<\omega}$

Definition

A $\square^{<\omega}$ *sequence* below κ is a sequence $\langle (n_\alpha, C_\alpha) : \alpha \in \kappa \rangle$ such that for each α:

1. $n \in \omega$ and C_α is an n_α-club subset of α
2. for every $\beta \in d_{n_\alpha}(C_\alpha)$ we have $n_\beta = n_\alpha$ and $C_\beta = C_\alpha \cap \beta$

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2. for every $\beta \in d_{n_\alpha}(C_\alpha)$ we have $n_\beta = n_\alpha$ and $C_\beta = C_\alpha \cap \beta$

Theorem (Brickhill) ($V = L$)

If κ is Π^1_n-indescribable for every $n < \omega$ but not Π^1_ω-indescribable then for any ω-stationary $A \subseteq \kappa$ there is an ω-stationary set $A' \subseteq A$ and $\square^{<\omega}$ sequence avoiding A'.
Thus κ is not ω-reflecting.

Π^1_γ-indescribability

Definition (Welch)

For a Δ_0 sentence $\varphi(v_0, v_1, v_2, v_3)$ and ordinal α with $A \subseteq \alpha$, we define the game $G_\alpha(\varphi, \gamma, A)$ with two players and finitely many moves such that in round n:

- Player 1 plays an ordinal $\gamma_n < \cdots < \gamma_0 = \gamma$ and $S_n \subseteq \alpha$
- Player 2 plays $P_n \subseteq \alpha$ such that $\varphi(\alpha, \bar{S}_n, \bar{P}_n, A)$
- The first player unable to play loses.

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Definition (Welch)

A cardinal κ is $\Pi^1_{2\gamma}$-indescribable if for every Δ_0 sentence $\varphi(v_0, v_1, v_2, v_3)$ and every $A \subseteq \kappa$ such that player II wins the game $G_\kappa(\varphi, \gamma, A)$ there is some $\alpha < \kappa$ such that player II wins $G_\alpha(\varphi, \gamma, A \cap \alpha)$.

$\square^\gamma$ and indescribability in L

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Theorem (Brickhill) ($V = L$)

For $\gamma < \kappa$, if κ is not Π^1_γ-indescribable but for all $\gamma' < \gamma$ we have $\Pi^1_{\gamma'}$-indescribable then for any γ-stationary $A \subseteq \kappa$ there is a γ-stationary set $A' \subseteq A$ and $\square^{\gamma-1}$ or $\square^{<\gamma}$ (if $\lim(\gamma)$) sequence avoiding A'. Thus κ is not γ-reflecting.

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Theorem

If the n-club filter at κ is κ-complete (which holds if κ is Π^1_n-indescribable) any $n+1$-stationary sets splits into κ many disjoint $n+1$-stationary sets.

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- Kunen showed that it is consistent relative to a weakly compact cardinal that a non-weakly compact cardinal reflects stationary sets. Using the same forcing:

Theorem

If κ is a Π^1_n-indescribable cardinal then there is a forcing extension of L in which κ is n-reflecting but not weakly compact.