On model-theoretic characterizations of large cardinals

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Set-theoretical Aspects of the Model Theory of Strong Logics
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Goal

Find characterizations of large cardinals in terms of compactness $\mathbb{L}_{\kappa,\omega}$ and $\mathbb{L}_{\kappa,\kappa}$ (and other logics).

Motivation

- Certain large cardinals have standard characterizations in terms of $\mathbb{L}_{\kappa,\omega}$ and $\mathbb{L}_{\kappa,\kappa}$. This makes it very easy for me as a model-theorist/Abstract Elementary Classist to understand and compare them

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- Can we extend these characterizations to other cardinals?
- As this progresses, it would be nice to see how these characterizations interact with AECs.

What is defined

Definition

Let τ be a language. $\mathbb{L}_{\kappa,\lambda}(\tau)$ is the logic that allows (negation and)

- *conjunctions of size $< \kappa$*
- *quantification over $< \lambda$-many variables at once*

(We assume that the functions and relations of τ are $< \lambda$-ary).
$\mathbb{L}^n(\tau)$ allows nth order quantification (along with membership).
$\mathbb{L}^n_{\kappa,\kappa}(\tau)$ has the obvious meaning.
A theory $T \subset \mathbb{L}_{\kappa,\lambda}(\tau)$ is satisfiable iff it has a model.

We call $\mathbb{L}_{\omega,\omega}$ first-order, even if it's not quite the best terminology. I also might say consistent instead of satisfiable, even if it's wrong.

What is known

Theorem (Goedel, Mal'cev)

ω is the compactness cardinal for $\mathbb{L}_{\omega,\omega}$; that is, for every $T \subset \mathbb{L}_{\omega,\omega}(\tau)$, if every $< \omega$-sized subset of T is satisfiable, then T is satisfiable.

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What about compactness in other logics?

What is known

Theorem

$\kappa > \omega$ is a weak compactness cardinal for $\mathbb{L}_{\kappa,\kappa}$ iff κ is weakly compact.

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We can do a parameterized version of this:

Theorem

κ is (δ, λ)-strong compact iff every $T \subset \mathbb{L}_{\delta,\delta}$ of size λ such that every $< \kappa$-sized subset is satisfiable is satisfiable.

κ is (δ, λ)-strong compact iff every κ-complete filter generated from λ-many sets extends to a δ-complete ultrafilter.

What is known

We can even go to second-order and above. Note that the property of having an *LST* cardinal is simply true in $\mathbb{L}_{\lambda,\kappa}$ (Skolem functions).

Theorem (Magidor)

1. κ *is the first supercompact iff for every* τ *of size* $< \kappa$ *and* τ*-structure* M*, there is a* τ*-structure* N *such that* $N \prec_{\mathbb{L}^2} M$ *and* $\|N\| < \kappa$.
2. κ *is extendible iff* κ *is a compactness cardinal for* $\mathbb{L}^2_{\kappa,\kappa}$.

What is known, but not well-known

Theorem

κ is measurable iff every $T \subset \mathbb{L}_{\kappa,\kappa}(\tau)$ that can be written as $T = \cup^{\uparrow}_{i<\kappa} T_i$ with each T_i satisfiable is itself satisfiable.

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Proof: In one direction, we want to build a κ-complete uniform ultrafilter on κ.

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Proof: In one direction, we want to build a κ-complete uniform ultrafilter on κ.Set

$$\begin{aligned}
\tau_0 &= \{U, V, E, c_X\}_{X \in \mathcal{P}(\kappa)} \\
M &= (\kappa, \mathcal{P}(\kappa), \in, X)_{X \in \mathcal{P}(\kappa)} \\
T &= Th_{\mathbb{L}_{\kappa,\kappa}}(M) \cup \{U(d), dEc_{(\alpha,\kappa)} \mid \alpha < \kappa\} \\
T_i &= Th_{\mathbb{L}_{\kappa,\kappa}}(M) \cup \{U(d), dEc_{(\alpha,\kappa)} \mid \alpha < i\}
\end{aligned}$$

Each T_i is satisfiable by (M, i). Given $N \vDash T$, we can define an ultrafilter on κ by $X \in U$ iff $N \vDash dEc_X$.

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Proof: In the other, take $j : V \to \mathcal{M}$ with crit $j = \kappa$ and ${}^{\kappa}\mathcal{M} \subset \mathcal{M}$ and $T = \cup_{i<\kappa} T_i$ with T_i satisfiable. Let $\bar{M} = \langle M_i \mid i < \kappa\rangle$ so $M_i \vDash T_i$ and $\bar{T} = \langle T_i \mid i < \kappa\rangle$. Then

$$j(\bar{M}) = \langle j(M_i) \mid i < \kappa\rangle^\frown \langle M_i^* \mid \kappa \leq i < j(\kappa)\rangle$$

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Claim 1: $\mathcal{M} \vDash$ "$M^*_\kappa \vDash j(\bar{T})_\kappa$"

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Claim 1: $\mathcal{M} \models$ "$M_\kappa^* \models j(\bar{T})_\kappa''$

Claim 2*: $j''T \subset j(\bar{T})_\kappa \cap \mathbb{L}_{\kappa,\kappa}(j''\tau)$ is T after some renaming according to j.

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Claim 2*: $j''T \subset j(\bar{T})_\kappa \cap \mathbb{L}_{\kappa,\kappa}(j''\tau)$ is T after some renaming according to j.
Claim 3*: $\mathcal{M}$ is right about satisfaction in $\mathbb{L}_{\kappa,\kappa}$.

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Claim 2*: $j''T \subset j(\bar{T})_\kappa \cap \mathbb{L}_{\kappa,\kappa}(j''\tau)$ is T after some renaming according to j.
Claim 3*: $\mathcal{M}$ is right about satisfaction in $\mathbb{L}_{\kappa,\kappa}$.
Thus (the renaming of) M^*_κ is our model of T. †

Type Omission

Definition

$p(x) \subset \mathbb{L}_{\kappa,\lambda}(\tau)$ iff all formulas in p have free variable x (or no free variables). We will only consider types of single elements, but could be longer.
A τ-structure M omits p iff for every $m \in M$, there is $\phi(x) \in p$ such that $M \models \neg\phi(m)$.

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- Type omission can be expressed via the $\mathbb{L}_{\lambda+|p|^+,\kappa}(\tau)$ sentence $\forall x \bigvee_{\phi \in p} \neg\phi(x)$.
- A great theorem of Chang says that $\mathbb{L}_{\lambda^+,\omega}$ is equivalent to omitting types in $\mathbb{L}_{\omega,\omega}$ (for some definition of equivalent).
- A not well-known model theorist once said: "Any fool can realize a type, but it takes a model theorist to omit one."

Type Omission

Theorem (Omitting Types Theorem)

Let $T \subset \mathbb{L}_{\omega,\omega}(\tau)$ be countable, and $\{p_n(x) \mid n < \omega\}$ be a sequence of nonisolated types. Then there is a countable model of T omitting each p_n.

- Going beyond countable is harder, either the size of the model or the logic or the number of types
  - ($\mathbb{L}_{\omega_1,\omega}$ is “equivalent” to omitting a countable set of types, so omitting types there is not much harder.)

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- Going beyond countable is harder, either the size of the model or the logic or the number of types
  - ($\mathbb{L}_{\omega_1,\omega}$ is "equivalent" to omitting a countable set of types, so omitting types there is not much harder.)
- There's an omitting types theorem for κ-sized theories where the types are not κ-isolated
  - (Jouko used this to prove that $\mathbb{L}(Q^{cof}_{\omega})$ is compact last week)

What is known, but not well-known

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Theorem (Benda)

κ is supercompact is equivalent to the following: Let T be a $\mathbb{L}_{\kappa,\kappa}(\tau)$ that can be written as an increasing union $T = \cup_{s \in \mathcal{P}_\kappa \lambda} T_s$ and $p(x) = \{\phi_i(x) \mid i < \lambda\}$ (possibly with repetitions). If every T_s has a model omitting $p_s(x) := \{\phi_i(x) \mid i \in s\}$, then T has a model omitting p.

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- This gives us a way to omit types
- Important to note that if $p(x) \subset q(x)$, then it is *easier* to omit q than p
- For $s \in \mathcal{P}_\kappa \lambda$, omitting p_s is a sentence in $\mathbb{L}_{\kappa,\kappa}(\tau)$, so T should *not* be a complete theory

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- Replacing κ with ω makes this theorem *false*!

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Proof: First, suppose $j : V \to \mathcal{M}$ with crit $j = \kappa$, $j''\lambda \in \mathcal{M}$. Let $\bar{M} = \{M_s \mid s \in \mathcal{P}_\kappa \lambda\}$ be the sequence of witnesses. Following the measurable case $\mathcal{M}$ thinks that $j(\bar{M})(j''\lambda)$ is a model of $j''T$. Furthermore, $\mathcal{M}$ is right about this and and $j''T$ is just some renaming of T.

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Also, $j(\bar{M})(j''\lambda)$ omits $j(p)_{j''\lambda} = \{j(\phi_i) \mid i < \lambda\} = j''p$. Thus, we have a model of T that omits p.

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Proof: Second, suppose $\mathbb{L}_{\kappa,\kappa}$ satisfies this type omitting compactness. Again, we write down the language for ultrafilters on $\mathcal{P}_\kappa \lambda$.

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M &= (\mathcal{P}_\kappa\lambda, \mathcal{P}(\mathcal{P}_\kappa(\lambda)), \in, X)_{X \in \mathcal{P}(\mathcal{P}_\kappa\lambda)} \\
T &= Th_{\mathbb{L}_{\kappa,\kappa}}(M) \cup \{U(d), dEc_{[\alpha]} \mid \alpha < \kappa\} \\
T_s &= Th_{\mathbb{L}_{\kappa,\kappa}}(M) \cup \{U(d), dEc_{[\alpha]} \mid \alpha \in s\}
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Type Omitting Compactness

We write down the language for ultrafilters on $\mathcal{P}_\kappa\lambda$.

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T &= Th_{\mathbb{L}_{\kappa,\kappa}}(M) \cup \{U(d), dEc_{[\alpha]} \mid \alpha < \kappa\} \\
T_s &= Th_{\mathbb{L}_{\kappa,\kappa}}(M) \cup \{U(d), dEc_{[\alpha]} \mid \alpha \in s\} \\
p_F(x) &= \{x = d \wedge xEc_{\{s \in \mathcal{P}_\kappa\lambda \mid F(s) \in s\}} \wedge \neg(xEc_{\{s \in \mathcal{P}_\kappa\lambda \mid F(s) = \alpha\}}) \mid \alpha < \lambda\} \\
p_{F,s}(x) &= \{x = d \wedge xEc_{\{s \in \mathcal{P}_\kappa\lambda \mid F(s) \in s\}} \wedge \neg(xEc_{\{s \in \mathcal{P}_\kappa\lambda \mid F(s) = \alpha\}}) \mid \alpha \in s\}
\end{aligned}$$

As F varies over functions from $\mathcal{P}_\kappa\lambda$ to λ.

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Claim 1: A model of T omitting p gives a κ-complete, fine, normal ultrafilter on $\mathcal{P}_\kappa\lambda$.

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Getting the ultrafilter is the same as before: $X \in U$ iff $N \vDash dEc_X$.
For normality, suppose that $F : \mathcal{P}_\kappa\lambda \to \lambda$ is regressive on $Y \in U$.
Then $N \vDash dEc_Y$.

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For normality, suppose that $F : \mathcal{P}_\kappa\lambda \to \lambda$ is regressive on $Y \in U$.
Then $N \vDash dEc_Y$.
Since N omits p_F, there is some $\alpha_0 < \lambda$ such that
$Y' := \{s \in \mathcal{P}_\kappa\lambda \mid F(s) = \alpha_0\} \in U$ because $N \vDash dEc_{Y'}$.

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We can take the principal filters here as we did before. (M, s) works. If F is regressive on s, then it's actually just equal to some element of s.

Type Omitting Compactness

Theorem (Benda)

κ is supercompact is equivalent to the following: Let T be a $\mathbb{L}_{\kappa,\kappa}(\tau)$ that can be written as an increasing union $T = \cup_{s \in \mathcal{P}_\kappa \lambda} T_s$ and a set of types $\Gamma = \{p^j(x) \mid j \in J\}$ such that $p^j = \{\phi_i^j \mid i < \lambda\}$. If every T_s has a model simultaneously omitting each p_s^j, then T has a model omitting each p.

This is a great theorem. This is the sort of template I want for more characterizations of large cardinals.

Lessons Learned

Lesson

Suppose that $j : V \to \mathcal{M}$ *with crit* $j = \kappa$ *and* ${}^{<\lambda}\mathcal{M} \subset \mathcal{M}$. *Then* $\mathcal{M}$ *is right about* $\mathbb{L}_{\kappa,\lambda}$ *satisfaction.*

Lessons Learned

Lesson

Suppose that $j : V \to \mathcal{M}$ with crit $j = \kappa$ and ${}^{<\lambda}\mathcal{M} \subset \mathcal{M}$. Then $\mathcal{M}$ is right about $\mathbb{L}_{\kappa,\lambda}$ satisfaction.

- This comes from the coding of formulas in $\mathbb{L}_{\kappa,\kappa}$ as sets.
- The only non-absolute (given the language) parts are the infinite conjuctions or quantification, which are coded by an ordinal below κ. Thus, individual sentences aren't changed below the critical point.
- This gives the $\mathbb{L}_{\kappa,\omega}$-correctness
- For the infinite sequences, $\mathcal{M}$ thinks that it satisfies them, but without being closed under $< \lambda$ sequences it might be wrong.

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Above weakly compacts, the size of the theory doesn't matter, just the filtration of it.

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- The characterization of λ-strong compact typically states that the entire theory is of size λ, but we really have compactness of any theory filtrated as $\mathcal{P}_\kappa \lambda$.

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- The fact that the theories are increasing is enough
- The characterization of λ-strong compact typically states that the entire theory is of size λ, but we really have compactness of any theory filtrated as $\mathcal{P}_\kappa\lambda$.
- This even holds true for definable class theories in definable class languages.

Lessons Learned

- We can go directly between compactness and elementary embedding characterizations by using the following theories for the listed large cardinals.

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- We can go directly between compactness and elementary embedding characterizations by using the following theories for the listed large cardinals.
- Due to undefinability of truth, there's a technical issue with getting these classes, but we sweep this under the rug as we do with saying there's elementary $j : V \to \mathcal{M}$.
- In each of the following, the basic class language is $(E, c_x)_{x \in V}$. We include the elementary diagram of the "standard model" V. This means that $x \mapsto c_x^{\mathcal{M}}$ will define an elementary embedding.
- Then some extra pieces are added to capture whatever extra conditions we put on $\mathcal{M}$.

Lessons Learned

1. κ is measurable

$$
\begin{aligned}
T &= Th_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x\in V} \cup \{c_i < c < c_\kappa \mid i < \kappa\} \\
T_\alpha &= Th_{\mathbb{L}_{\kappa,\kappa}}(V, \in, x)_{x\in (V_{\geq\kappa} \cup V_\alpha)} \cup \{c_i < c < c_\kappa \mid i < \alpha\}
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2. κ is λ-strongly compact

$$\begin{aligned} T &= Th_{\mathbb{L}_{\kappa,\kappa}}(V,\in,x)_{x\in V} \cup \{c_i < c < c_\kappa \mid i < \kappa\} \\ &\quad \cup\{c_\alpha \in d \mid \alpha < \lambda\} \cup \{|d| < c_\kappa\} \\ T_s &= Th_{\mathbb{L}_{\kappa,\kappa}}(V,\in,x)_{x\in(V_{\geq\kappa}\cup V_{otp(s)})} \cup \{c_i < c < c_\kappa \mid i \in otp(s)\} \\ &\quad \cup\{c_\alpha \in d \mid \alpha \in s\} \cup \{|d| < c_\kappa\} \end{aligned}$$

Lessons Learned

1. κ is λ-supercomapct

$$
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&\quad \cup\{c_\alpha \in d \mid \alpha<\lambda\} \\
p(x) &= \{xEd \wedge \neg(x=c_\alpha) \mid \alpha<\lambda\} \\
T_s &= Th_{\mathbb{L}_{\kappa,\kappa}}(V,\in,x)_{x\in(V_{\geq\kappa}\cup V_{otp(s)})} \\
&\quad \cup\{c_i < c < c_\kappa \mid i < otp(s)\} \cup \{c_\alpha \in d \mid \alpha\in s\} \\
&\quad \cup\{|d| < c_\kappa\} \\
p_s(x) &= \{xEd \wedge \neg(x=c_\alpha) \mid \alpha\in s\}
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Lessons Learned

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Normality $=$ *closure under sequences* $=$ *type omission.*

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- This already showed up in supercompactness
- Gives a nice characterization of measurability with type omission
- Can also see this looking directly at the ultraproduct: given $\langle M_i \mid i < \kappa \rangle$ and a *normal* ultrafilter on κ, $\Pi M_i / U$ omits *every* type $\{\phi_\alpha(x) \mid \alpha < \kappa\}$ such that M_i omits $\{\phi_\alpha(x) \mid \alpha < i\}$.

Lessons Learned

Lesson

Normality = closure under sequences = type omission.

- Pins down the difference between strong compact and supercompact: both realize the type of an element that contains every c_α and has no surjection to c_κ. However, supercompact omits the type of an element in that set that is not some $j(\alpha)$. This is formally an $\mathbb{L}_{\lambda^+,\kappa}$-statement, but we can do it with type omission directly (i. e., not up to equivalence).

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- The hope/goal is to characterize all large cardinals via compactness or type omitting compactness properties
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- There might be other X compactness properties besides type omission that we find later
- As an Abstract Elementary Classist, type omission feels very natural to me. The fact that we can characterize large cardinals with type compactness helps argue that this is the case.
- To move toward this goal, we can do two things:
  - Apply existing type omitting compactness to other logics
  - Find new templates for type omitting compactness

Huge cardinals

- For a new template, let's turn to huge cardinals
- Huge cardinals have a characterization in terms of a normal ultrafilter/elementary embedding to a target model that is very closed under sequences, so this is a good place to look.

Definition

κ is huge at λ iff

- *there is $j : V \to \mathcal{M}$ with crit $j = \kappa$ and ${}^{j(\kappa)}\mathcal{M} \subset \mathcal{M}$ and $\lambda = j(\kappa)$*
- *there is a normal, fine, κ-complete ultrafilter on $[\lambda]^\kappa$*

Huge Type Omission

We can characterize huge cardinals based on type omission patterned on $[\lambda]^\kappa$.

Theorem

κ is huge at λ iff for every language τ and $\mathbb{L}_{\kappa,\kappa}$-theory $T = \cup_{s \in [\lambda]^\kappa} T_s$ that behaves according to the index and $\mathbb{L}_{\kappa,\kappa}$-type $p(x) = \{\phi_i(x) \mid i < \lambda\}$, if every T_s has a model omitting p_s, then T has a model omitting p

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"Behaving according to the index" is some condition that's necessary because the ultrafilter is not closed under intersection of size $|s|$. Enough to have that there are T^α for $\alpha < \lambda$ and $T_s = \cup_{\alpha \in s} T^\alpha$.

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Proof: To get this compactness, consider a function f that maps s to a model of T_s omitting p_s. Then (up to suppressing the renaming that sends τ to $j''\tau$) $j(f)(j''\lambda)$ is a model of $T \subset j(T)_{j''\lambda}$.

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$$[s] = \{t \in [\lambda]^\kappa \mid s \subset t\} \in U$$

So we need this behavior condition to imply that $\{t \in [\lambda]^\kappa \mid \phi \in T_t\} \in U$.

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Proof: For the type omission, $j(f)(j''\lambda)$ omits $j(p)_{j''\lambda}$. Since $\mathcal{M}$ is closed under λ sequences, this is exactly $\cup_{s \in [\lambda]^\kappa} j(p)_{j''s} = j''p = p$. This all happened in $\mathcal{M}$, but the closure under sequences and critical point mean that $\mathcal{M}$ is correct about satisfaction in $\mathbb{L}_{\kappa,\kappa}$.

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Proof: For the other direction, use the ultrafilter characterization and write down the various parts. Given an $s \in [\lambda]^\kappa$, the principle ultrafilter on s gives a model of T_s omitting p_s.

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- A soft answer is that it doesn't seem like it; the monotonicity of omission works in the opposite direction of satisfaction, so it seems like there's something more here.
- The answer can be made definite with a theorem of Stavi.

The characterization of huge is huge

Theorem (Stavi)

The following are equivalent:

- *Vopenka's Principle*
- *Every logic has a strong compactness cardinal*

- With apologies to Kanamori, this says that Vopenka's Principle "rallies at last to form a veritable Goetterdammerung" for compactness cardinals. (Also, apologies to Wagner and the German language)

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- If κ is the first huge, then V_κ models Vopenka's Principle and thus has strong compactness for every logic. However, no $\mathbb{L}_{\lambda,\lambda}$ satisfies type omitting compactness-for-huge.
- So even your strongest logic (sort logic) cannot represent type omitting compactness-for-huge as a compactness property.

Supercompact-for-$\mathbb{L}^2_{\kappa,\kappa}$

- We can apply the supercompact type omitting template to second-order logic $\mathbb{L}^2_{\kappa,\kappa}$

Definition

κ is supercompact-for-$\mathbb{L}^2_{\kappa,\kappa}$ iff for every $T \subset \mathbb{L}^2_{\kappa,\kappa}$ that is an increasing union $\cup_{s \in \mathcal{P}_\kappa \lambda} T_s$ and $p(x) = \{\phi_i(x) \mid i < \lambda\}$ such that every T_s that has a model omitting p_s, then T has a model omitting p.

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Theorem

κ is supercompact-for-$\mathbb{L}^2_{\kappa,\kappa}$ iff κ is extendible (which is strong compact-for-$\mathbb{L}^2_{\kappa,\kappa}$.

- It seems surprising that there's no extra strength here.
- One direction is clear ($p(x)$ is $x \neq x$ repeated)

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We use a nice characterization of extendible by Tsaprounis.

Theorem (Tsaprounis)

κ is extendible iff for every $\lambda > \kappa$, there is $j : V \to \mathcal{M}$ with crit $j = \kappa$, $V_{j(\lambda)} \subset \mathcal{M}$, ${}^{\lambda}\mathcal{M} \subset \mathcal{M}$, and $\lambda < j(\kappa)$.

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Find the appropriate $j : V \to \mathcal{M}$ with μ. Then $\mathcal{M}$ thinks $j(f)(j''\lambda)$ models $j''T$ and omits $j''p$ as is normal.

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Find the appropriate $j : V \to \mathcal{M}$ with μ. Then $\mathcal{M}$ thinks $j(f)(j''\lambda)$ models $j''T$ and omits $j''p$ as is normal. In addition, $j(f)(j''\lambda) \in V_{j(\mu)}$ since each $f(s) \in V_\mu$. Thus, $\mathcal{M}$ has all the proper power sets for $j(f)(j''\lambda)$, so it computes satisfaction in $\mathbb{L}^2_{\kappa,\kappa}$ correctly *for* $j(f)(j''\lambda)$.

The curious case of supercompacts

- In measurability, the existence of a κ-complete uniform ultrafilter on κ is equivalent to the existence of a κ-complete normal ultrafilter on κ. This means that chain compactness and chain compactness for type omission are equivalent.
- At the level of strong compact and supercompact, the normality of the fine ultrafilter is
- But the identity criss between compact-for-$\mathbb{L}^2_{\kappa,\kappa}$ and supercompact-for-$\mathbb{L}^2_{\kappa,\kappa}$ doesn't exist! They are one and the same.
- What is it about $\mathbb{L}_{\kappa,\kappa}$ compared with $\mathbb{L}^2_{\kappa,\kappa}$ that causes this difference? Does it propagate to other logics?

Is second order logic necessary here?

- Looking more at some of the results, it seems closure under sequences is like type omission in first-order (in the broad sense) and strong-like reflection $V_\lambda \subset \mathcal{M}$ uses second order

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Is second order logic necessary here?

- Looking more at some of the results, it seems closure under sequences is like type omission in first-order (in the broad sense) and strong-like reflection $V_\lambda \subset \mathcal{M}$ uses second order
- However (we will see), that even these properties can be written as type omission in first-order
- So do we need second order to characterize these other logics? E. g., is there some way to characterize extendible cardinals based solely on $\mathbb{L}_{\kappa,\kappa}$?

Strong cardinals

Definition

κ is λ-strong iff there is some $j : V \to \mathcal{M}$, crit $j = \kappa$, $V_\lambda \subset \mathcal{M}$.

κ is λ-strong iff there is a (class) model of T omitting $\{p_y \mid y \in V_\lambda\}$, where

$$\begin{aligned} \tau &= \{E, c_x, d_y\}_{x \in V, y \in V_\lambda} \\ T &= Th_{\mathbb{L}_{\kappa,\omega}}(V, \in, x)_{x \in V} \cup \{E \text{ is a well-founded relation}\} \\ &\quad \cup \{c_\alpha < c < c_\kappa \mid \alpha < \kappa\} \cup \{d_z \in d_y \mid z \in y \in V_\lambda\} \end{aligned}$$

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- Note the ω in $\mathbb{L}_{\kappa,\omega}$. This is because of the fact that $\mathcal{M}$ might not be closed under ω sequences and because extender powers are directed colimits but not ω_1-directed colimits
- What's missing here is a locality property that reflects this type omission.

And now for something a little different

Whirlwind introduction to AECs for strong logicists

- Abstract Elementary Classes are a great framework for nonelementary model theory
- More precise definitions are forthcoming, but a fine way to think about them (especially at this conference) is the following: Fix a theory T in a logic $\mathbb{L}(\tau)$ along with $\prec^*_{\mathbb{L}}$ ($\mathbb{L}$-elementary substructure with a possible twist). Then $\mathcal{K} = (\text{Mod } T, \prec^*_{\mathbb{L}})$ is an AEC iff
  1. $\mathbb{L}$ has an LST number $LS(\mathcal{K})$
  2. $\prec^*_{\mathbb{L}}$ is smooth for increasing unions of chains (and thus for colimits of all directed systems)
- Shelah has introduced the amazing project of classification theory into model theory, and AECs are a framework to do this outside of $\mathbb{L}_{\omega,\omega}$
- Typically, $\mathbb{L}$ has the form of $\mathbb{L}_{\lambda^+,\omega}$ with extra quantifiers.
- A key feature is that well-order is undefinable in AECs.

Whirlwind introduction to tame AECs

- Lots of things go wrong when you move outside of compactness
- Large cardinals give you some of this back

Theorem

If $\mathcal{K}$ is an AEC and $\kappa > LS(\mathcal{K})$ is strongly compact, then $\mathcal{K}$ is $< \kappa$-tame.

- Tameness will also go undefined until tomorrow
- Types are very important in classification theory, but just looking at sets of formulas won't work
- Galois types are the proper semantic replacement (roughly orbits of automorphisms fixing a particular model)
- However, distinct Galois types don't necessarily have distinct restrictions over any smaller submodel
- Tameness says they do

Large cardinals from AECs

- The amazing thing is this is enough to characterize large cardinals

Theorem (B.-Unger)

If every AEC $\mathcal{K}$ with $LS(\mathcal{K}) < \kappa$ is $< \kappa$-tame (and $\sigma < \kappa \to \sigma^\omega < \kappa$), then κ is almost strongly compact.

- So there are questions about classes that don't define well-orders that have large cardinal strength (and are more subtle than writing down the theory of a κ-complete ultrafilter)
- Is there such a characterization for supercompacts? (or huge or extendible or...)
- My suspicion is yes, but not as naturally as strong compact

Thanks!

Any questions?