Hanf numbers for properties of AEC's

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September 28, 2016

Background

First order logic is largely impervious to extensions of ZFC.

There is a deep entanglement between infinitary logic and axiomatic set theory.

Spectra of AEC

Underlying question

Do the connections of large cardinals with fundamental properties of AEC's represent *algebraic* or *geometric* rather than *combinatorial* phenomena?

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Related Issues

What is significant about *complete* sentences of $L_{\omega_1,\omega}$?
A slightly different method of establishing completeness.

Independence in Boolean Algebras

κ-free implies κ^+-free

Hanf's principle

If a certain property can hold for only set-many objects then it is eventually false.

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If a certain property can hold for only set-many objects then it is eventually false.
Hanf refines this twice.

1. If $\mathcal{K}$ a *set* of collections of structures $\boldsymbol{K}$ and $\phi_P(X, y)$ is a formula of set theory such $\phi(\boldsymbol{K}, \lambda)$ means some member of $\boldsymbol{K}$ with cardinality λ satisfies P.

$$\mu_{\boldsymbol{K}} = \sup\{\lambda : P(\boldsymbol{K}, \lambda) \text{holds if there is such a sup } \}$$

   Hanf number of P = $\sup_{\boldsymbol{K}} \mu_{\boldsymbol{K}}$

2. If the property P is closed down for sufficiently large members of each $\boldsymbol{K}$, then 'arbitrarily large' can be replaced by 'on a tail' (i.e. eventually).

Examples

Boney- Unger -Shelah

The Hanf number for 'all aec's are tame' (with various decorations) is a compact cardinal with various decorations.

JEP

B-Koerwien-Souldatos

If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals whose models satisfy JEP($< \lambda_0$), there is an $L_{\omega_1,\omega}$-sentence ψ such that

1. The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
2. There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
3. ψ has arbitrarily large models.

Maximal models

B-Souldatos

1. There is a complete sentences of $L_{\omega_1,\omega}$, which has a (κ^+, κ) model for every κ.
2. Assume for simplicity that $2^{\aleph_0} > \aleph_\omega$. For each $n \in \omega$, there is a complete $L_{\omega_1,\omega}$-sentence ϕ'_n with maximal models in cardinalities $2^{\aleph_0}, 2^{\aleph_1}, \ldots, 2^{\aleph_n}$.
3. Assume κ is a homogeneously characterizable cardinal and for simplicity let $2^{\aleph_0} \geq \kappa$. Then there is a complete $L_{\omega_1,\omega}$-sentence ϕ_κ with maximal models in cardinalities 2^λ, for all $\lambda \leq \kappa$.

All of this is taking place below $\beth_{\omega_1}$.

The big gap

Theorem. B-Boney

The Hanf number for Amalgamation is at most the first strongly compact cardinal

The best known lower bound is $\beth_{\omega_{\omega_1}}$.

Disjoint Amalgamation

1. (B, Kolesnikov, and Shelah): There are $L_{\omega_1,\omega}$-definable classes with disjoint amalgamation up to $\aleph_\alpha$ for every countable α (but did not have arbitrarily large models).
2. (Kolesnikov and Lambie-Hanson): The Hanf number for amalgamation (equivalently, disjoint amalgamation) in Coloring Classes with κ predicates is precisely $\beth_{\kappa^+}$ (and many of the classes have arbitrarily large models).

Maximality, JEP, AP, Arbitarily Large

A maximal model plus (global) JEP or AP implies a bound on cardinality of models.

Test question: non-maximality

Let $\boldsymbol{K}_0$ be the collection of models of a complete sentence in $L_{\omega_1,\omega}$ in a countable vocabulary.

to avoid negatives:
$\boldsymbol{K}_0$ is *universally extendible in* λ if every model in λ is extendible – has a proper $L_{\omega,\omega}$ extension.

Theorem. B-Shelah

The Hanf number for universal extendibility is the first measurable cardinal μ if it exists.

Clearly, every model with cardinality at least μ has a proper extension.

A schema

Notation

1. Let $(\boldsymbol{K}_0, \leq)$ be a collection of countably many countable structures.
2. Let $(\boldsymbol{K}_1, \leq)$ (often $\hat{\boldsymbol{K}}$) the collection of direct limits of structures in $\boldsymbol{K}_0$.

If $(\boldsymbol{K}_0, \leq)$ has the amalgamation property and joint embedding then it has generic model M – universal and homogenous with respect to $(\boldsymbol{K}_0, \leq)$.

Let $(\boldsymbol{K}_2, \leq)$ (often $\boldsymbol{K}^R$, **R** rich) be the collection of models of the Scott sentence of M.

Plan

Ingredients

1. P_1 will the domain of a Boolean algebra
2. In each model there will a (non-definable) homomorphism from P_1 into the BA of subsets of P_0. G_1 is a bijection between $P_{4,1}$ (atoms of P_1) and P_0. $R(u, b)$ iff $H_1(u) \leq b$.
3. P_2 is a set with no structure but for each n, there is a function such that $\{F_n(c) : c \in P_2\}$ is a set of elements of P_1. Cofinitely many of them are independent.
4. $P_{4,n}$ is the set of elements of P_1 that are a join of n-atoms; $P_4 = \bigcup_n P_{4,n}$

Relative Independence

Definition

1. For $X \subseteq B$ and B a Boolean algebra, $\overline{X} = X_B = \langle X \rangle_B$ be the subalgebra of B generated by X.
2. A set Y is independent from X over an ideal I in a Boolean algebra B if and only if for any Boolean-polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0), and any $a \in \overline{X} - I$,

$$p(y_0, \ldots, y_k) \wedge a \notin I.$$

Facts

Observations

1. If I is the 0 ideal, (read Y is independent from X), the condition becomes any $a \in \overline{X} - \{0\}$, $B \models p(y_0, \ldots, y_k) \wedge a > 0$.
2. It is easy to check that 'Y is independent from X over I' implies the image of Y is free from the image of X in B/I.

Free Amalgamation

We can amalgamate Boolean algebras B and A over C .

notation

Let $C \subseteq A, B$ be Boolean algebras.
The disjoint amalgamation $D = A \otimes_C B$ is the Boolean algebra obtained as the pushout of A and B over C.

It is characterized internally by the following condition. For $a \in A - C, b \in B - C$: $a \leq b$ in D if and only if there is a $c \in C$ with $a < c < b$ (and symmetrically). D is generated as a Boolean algebra by $A \cup B$.

Some Notation

Notation

For any Boolean algebra B, $\mathrm{At}(B)$ denotes the set of atoms of B.

Definition

Following Vaught we say a Boolean algebra is *atomistic* if every element is above an atom.

Following general practice in the study of Boolean algebras, we say B is *atomic* it every element is the join of atoms.

Amalgamation preserving atoms

Theorem

Let $A_0 \subseteq A_1, A_2$ be atomistic Boolean algebras. There is a Boolean algebra amalgamating A_1 and A_2 such that $\mathrm{At}(A_3) = \mathrm{At}(A_1) \cup \mathrm{At}(A_2)$.

The proof uses the notion of independence defined above.

The Complete Sentence

Basic Framework

Definition

$M \in \boldsymbol{K}_{-1}$ is the class of structures M satisfying.

1. P_0^M, P_1^M, P_2^M partition M.
2. $(P_1^M, 0, 1, \wedge, \vee, <, ^-)$ is a Boolean algebra ($^-$ is complement).
3. G_1^M is a bijection from P_0^M onto $P_{4,1}^M$ such that $R(M, G_1^M(a)) = \{a\}$. H_1^M is defined on $P_{4,1}^M$ and is the inverse of G_1^M.
4. $R \subset P_0^M \times P_1^M$ with $R(M, b) = \{a : R^M(a, b)\}$ and the set of $\{R(M, b) : b \in P_1^M\}$ is a Boolean algebra. $f^M : P_1^M \mapsto \mathcal{P}(P_0^M)$ by $f^M(b) = R(M, b)$ is a Boolean algebra homomorphism into $\mathbb{P}(P_0^M)$.

Basic Framework II

K_{-1} continued

5. $P^M_{4,n} = \{b \in P^M_1 : |\{c \in P^M_1 : c \leq b\}| = n\}$ and P^M_4 is the union of the $P^M_{4,n}$.
6. If $b_1 \neq b_2$ are in P^M_4 then $R(M, b_1) \neq R(M, b_2)$.
7. If $c \in P^M_2$, the $F_n(c)$ for $n < \omega$ are pairwise distinct.

Note that f is not 1-1 [1] in τ; it is simply a convenient abbreviation for the relation between the Boolean algebra P^M_1 and the set algebra on P_0 by the map $b \mapsto R(M, b)$.

[1] The subsets of P^M_0 are *not* elements of M.

Finitely generated

$\boldsymbol{K}_0$

M is in the class of structures $\boldsymbol{K}_0$ if $M \in \boldsymbol{K}_{-1}$ and there is a *witness* $\langle n_*, \boldsymbol{B}, b_* \rangle$ such that:

1. $b_* \in P_1^M$ is a finite union of atoms. Further, for some k, $P_{4,k}^M = \{c : c \leq b_*\}$ and for all $n > k$, $P_{4,k}^M = \emptyset$.
2. $\boldsymbol{B} = \langle B_n : n \geq n_* \rangle$ is an increasing sequence of finite Boolean subalgebras of P_1^M.
3. B_{n_*} contains the $\{c \in P_1^M : c \leq b_*\}$
4. $\bigcup_{n \geq n_*} B_n = P_1^M$.
5. P_2^M is finite and not empty.

Finitely Generated II

K_0 continued

6. The set $\{F_m(c) : m \geq n_*, c \in P_2^M\}$ is free from B_n over P_4^M.
7. For every $n \geq n_*$, B_n, is generated by $B_{n_*} \cup \{F_m(c) : n > m \geq n_*, c \in P_2^M\}$. Thus P_1^M and so M is generated by $B_{n_*} \cup P_2^M$.
8. If $n < n_*$ and $c \in P_2^M$, $F_n^M(c) \in B_{n_*}$
9. If $a \in P_0^M$ and $c \in P_2^M$ then for every large enough n $a \notin R(M, F_n(c))$. Equivalently $\bigcap_n F_n^M(c) = \emptyset$.

B_{n_*} could be taken to be generated by $P_4^M \cup \{F_m(c) : c < n_*, c \in P_2^M\}$.

$\boldsymbol{K}_0$ is countable

Lemma

$\boldsymbol{K}_0$ is countable.

Proof. Let $M \in \boldsymbol{K}_0$, witnessed by $\langle n_*, \boldsymbol{B}, b_* \rangle$.

The isomorphism type of M is determined by the structure on P_4^M induced by the $F_n(c_i)$ and $c_i \in P_2^M$, as the tail, is just an atomless boolean algebra in the sense of P_1^M.

And there can be only countably many structures induced on the finite P_4^M by the countable set $F_n(c_i)$ through the formulas $x < F_n(c_i)$ which determine the values of R on P_4^M.

Generic Model

Theorem

$(\boldsymbol{K}_0, \subseteq)$ has the disjoint amalgamation property.

Proof sketch:
Suppose M^0 is extended by M^1 and M^2. Let $\mathcal{B}^i$ be the Boolean algebra with domain $P_1^{M_i}$.

1. Find a 'freeish' amalgamation $\mathcal{B}^3$ of $\mathcal{B}^1$ $\mathcal{B}^2$ over $\mathcal{B}^0$ with no new atoms.
2. Extend $\mathcal{B}^3$ to $M^3 \in \boldsymbol{K}_{-1}$.
3. Prove $M^3 \in \boldsymbol{K}_0$.

Complete Sentence

Corollary

There is a countable generic model M for $\boldsymbol{K}_0$.

Lemma

If M is the generic model and $b_1 \neq b_2 \in P_1^M - P_4^M$ then P_1^M to $\mathcal{P}(P_)$ is injective.

Further, if $b \in P_1^M - P_4^M$, $R^M(b, M)$ is infinite and b is not an atom.

So P_1^M / P_4^M is an atomless boolean algebra, hence free.

prototypical model.
$P_4^M = \mathcal{P}_\omega(N)$
$P_1^M = \mathcal{P}(N)$ – or rather, a countable elementary submodel.

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The actual example

Definition

M in $\boldsymbol{K}_1$ is rich if for any $N_1, N_2 \in \boldsymbol{K}_0$ with $N_1 \subseteq N_2$ and $N_1 \subseteq M$, there is an embedding of N_2 into M over N_1. We denote the class of rich models in $\boldsymbol{K}_1$ by $\boldsymbol{K}_2 = \boldsymbol{K}^R$.

It is easy to construct a back-forth-showing:

Lemma

The class $\boldsymbol{K}_2$ is the collection of models of the Scott sentence of the generic (rich) model.

Free Extensions

Free Extensions

Definition

M_2 is free over M_1 written $M_1 \subseteq_{fr} M_2$ if

1. There is an $\boldsymbol{I}$ with $\boldsymbol{I} \subset (P_1^{M_2} - P_1^{M_2}) \cup P_4^{M_2}$ such that:
   - i $\boldsymbol{I} \cup P_1^{M_1} \cup P_4^{M_2}$ generates $P_1^{M_2}$
   - ii $\boldsymbol{I}$ is independent from $P_1^{M_1}$ over $P_4^{M_2}$ in $P_1^{M_2}$.
2. There is a function H from $P_2^{M_2} \setminus P_2^{M_1}$ to $\mathbb{N}$ such that the $F_n(c)$ for $n \geq H(c)$ are distinct and

$$\{F_n^M(c) : c \in P_2^{M_2} \setminus P_2^{M_1} \text{ and } n \geq H(c)\} \subset \boldsymbol{I}.$$

M is free if it is free over the empty model i.e., P_1^M has a free basis over P_4^M.

Properties of Free Extensions

Transitivity

1. If $M_1 \subseteq_{fr} M_2$ by $\boldsymbol{I}_1$ and $M_2 \subseteq_{fr} M_3$ by $\boldsymbol{I}_2$ then $M_1 \subseteq_{fr} M_3$ by $\boldsymbol{I}_1 \cup \boldsymbol{I}_2$. Thus, $\subseteq_{fr}$ is a partial order.
2. More generally if M_α with $\alpha < \delta$ is continuous $\subseteq_{fr}$ increasing then $M = \bigcup M_\alpha$ satisfies $M_\alpha \subseteq_{fr} M$ witnessed by $\bigcup_{\alpha<\beta<\delta} I_\beta$.

Proof is immediate from analogous result for Boolean algebras. This proof depends heavily on the notion of free -from -over.

Amalgamation of finitely generated with free

Theorem

Suppose $M_1 \in \boldsymbol{K}_1$ is free and $N_1 \subset M_1$. Let $N_1 \subset N_2$ with both in $\boldsymbol{K}_0$.
Then there are an M_2 and an f such that:

1. $M_2 \in \boldsymbol{K}_1$, $M_1 \subseteq_{fr} M_2$ and so M_2 is free. Similarly $N_2 \subseteq_{fr} M_2$.
2. f maps N_2 into M_2 over N_1. Moreover, the image of N_2 is free in M_2.

Amalgamation depends on free extension and finite base and uses our atom-preserving amalgamation.

Existence of Free Extensions

Theorem

Let M_1 be free in $\boldsymbol{K}_1$.

1. There exists an M_2 which is a free extension of M_1.
2. We can choose $M_2 \in \boldsymbol{K}_2$.

Proceeding inductively we get:

Corollary

For every μ there is a free $M \in \boldsymbol{K}_1$, ($\boldsymbol{K}_2$) of cardinality μ.

P_0-maximality

Definition: Maximal Models

1. A model $M \in \boldsymbol{K}_2 = \boldsymbol{K}^R$ is *P_0-maximal* (for $\boldsymbol{K}^R$) if $M \subseteq N$ and $N \in \boldsymbol{K}_2 = \boldsymbol{K}^R$ implies $P_0^M = P_0^N$.
2. A model $M \in \boldsymbol{K}_2 = \boldsymbol{K}^R$ is *maximal* (for $\boldsymbol{K}^R$) if $M \subseteq N$ and $N \in \boldsymbol{K}_2 = \boldsymbol{K}^R$ implies $M = N$.

Background Set theory

Definition

[$\diamond_S$] Given a cardinal κ and a stationary set $S \subseteq \kappa$, $\diamond_S$ is the statement that there is a sequence $\langle A_\alpha : \alpha \in S \rangle$ such that

1. each $A_\alpha \subseteq \alpha$
2. for every $A \subseteq \kappa$, $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary in κ

Definition

[S reflects] Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. If $\alpha < \kappa$ has uncountable cofinality, S reflects at α if $S \cap \alpha$ is stationary in α . S reflects if it reflects at some $\alpha < \kappa$.

Let $S^\lambda_{\aleph_0}$ denote the stationary set $\{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0, \delta \text{ is divisible by } |\delta|\}$.

Main Theorem for Maximality

Theorem

If

1. there is no measurable cardinal ρ with $\rho \leq \lambda$, $\lambda = \lambda^{<\lambda}$,
2. and there is an $S \subseteq S^{\lambda}_{\aleph_0}$, that is stationary non-reflecting, and $\diamond_S$ holds.

Then there is a P_0-maximal model $M \in \boldsymbol{K}^R$ of card λ

We give this argument first; then sketch a black box to remove the set theoretic hypotheses.

Easy to go from P_0-max to max.

Goals of construction

We will choose M_α for $\alpha < \lambda$ by induction to satisfy the following conditions.

Construction: Goal

Let $\langle U_\alpha : \alpha < \lambda \rangle$ list $[\lambda]^{<\lambda}$ so that each subset is enumerated λ times and $U_\alpha \subseteq \alpha$.
Let $\overline{A}^* = \langle A^*_\delta : \delta \in S \rangle$ be a $\diamond_S$-sequence.

1. $M_\alpha \in \boldsymbol{K}^R$ has universe an ordinal between α and λ and M_0 is empty.
2. $\langle M_\beta : \beta < \alpha \rangle$ is $\subseteq$- continuous.
3. If $\beta \in \alpha - S$ then M_α is free over M_β.

Goals of construction II

4. If $\alpha = \beta + 2$ and $U_\beta \subseteq P_0^{M_\beta}$ then there is a $b_\beta \in P_1^{M_\alpha}$ such that $R(M_\alpha, b_\beta) \cap M_{\beta+1} = U_\beta$ and in the Boolean algebra $P_1^{M_\alpha}$, $\{b_\beta\}$ is free over $P_1^{M_\beta+1} \cup P_4^{M_\alpha}$.
5. If $\delta \in S$ and $\alpha = \delta + 1$ then a) implies b), where:
   a) there is an increasing sequence $\overline{\gamma} = \langle \gamma_{\delta,n}, b_{\delta,n} : n < \omega \rangle$, where the $\gamma_{\delta,n}$ are not in S and increasing with n satisfying:
      i) $\gamma_{\delta,n} < \gamma_{\delta,n+1} < \delta$ with $\sup_n \gamma_{\delta,n} = \delta$;
      ii) $b_{\delta,n} \in P_1^{M_{\gamma_{\delta,n+1}}} \cap A^*_\delta$ and so $b_{\delta,n} \in P_1^{M_\delta}$;
      iii) $\{b_{\delta,n} : n < \omega\}$ is independent over $P_1^{M_{\gamma n}} \cup P_4^{M_\delta}$;
      iv) if $a \in P_0^{M_\delta}$ then for all but finitely many n, $\neg R(a, b_{\delta,n})$.

   b) For some $\overline{\gamma} = \langle \gamma_{\delta,n}, b^\delta_n : n < \omega \rangle$, there is a $c_\delta \in P_2^{M_{\delta+1}}$ such that for each n, $F_n^{M_{\delta+1}}(c_\delta) = b_{\delta,n}$.

The actual Construction

1-4a are easy

Case 1: $\alpha = 0$.

Case 2: $\alpha = \beta + 1$ and $\beta \notin S$.

Case 3: $\alpha = \delta$, a limit ordinal that is not in S. Use the fact that S does not reflect to show M_δ is free.

Case 4a: $\alpha = \delta + 1$, $\delta \in S$, and clause 5a fails. This is just as in case 2.

Case 4b: $\alpha = \delta + 1$, $\delta \in S$, but clause 5a holds. Use the following lemma.

Key Lemma

Claim

Suppose that for $n < \omega$, $M_n \subset_{fr} M_{n+1}$ are in $\widehat{\boldsymbol{K}}$. If Condition A) holds then so does condition B).

A
1. $P_2^{M_{n+1}} - P_2^{M_n}$ is infinite
2. there is a $b_n \in P_1^{M_{n+1}}$ so that $\{b_n\}$ is free over $P_1^{M_n}$.
3. if $a \in P_1^{M_i}$, then for all but finitely many $n \geq i$, $a \notin R(M_{n+1}, b_n)$.

B) then there is a pair (M, c)
1. $M = \bigcup M_n \cup \{c\}$, $c \in P_2^M$, c is not in any M_n,
2. $M_n \subset_{fr} M$ for each n,
3. $F_n^M(c) = b_n$.

The Construction Suffices

M is P_0-maximal for $\boldsymbol{K}^R$

Suppose for contradiction there exists N extending M in $\widehat{\boldsymbol{K}}$ such that $P_0^N \supsetneq P_0^M$. Choose $a^* \in P_0^N - P_0^M$. Let

$$A = \{b \in P_1^M : R^N(a^*, b)\}.$$

Then A is a non-principal ultrafilter on P_1^M.

We will derive a contradiction using the choice of the stationary set $S_A = \{\delta \in S : M_\delta \text{ has universe } \delta \ \& \ A^*_\delta = A \cap \delta\}$.

2 cub's

C_1

There is a closed unbounded set C_1 such that if $\delta \in C_1$, for every sequence $\overline{\gamma} \in M_\delta^\omega$ satisfying condition 5a), there is a $c_\delta \in P_2^{M_{\delta+1}}$ such that for each n, $F_n^{M_{\delta+1}}(c_\delta) = b_{\delta,n}$.

Note that if $\delta^* \in C_1$, the key lemma implies there can be no sequence $\langle \gamma_n^{\delta^*} : n < \omega \rangle$ with limit δ^* so that $\gamma_n \notin S$ and $b_{\gamma_n^{\delta^*}} < \gamma_{n+1}^{\delta^*}$.

C_2

$C = \{\delta < \lambda : \delta \text{ limit } \ \& \ \alpha < \lambda \rightarrow b_\alpha < \delta\}$ is a club of λ.

Case 1

We will show case 1 is outright impossible and case 2 is impossible unless λ is measurable.

Case 1

For every $\alpha < \lambda$ there is a $b_\alpha \in P_1^M \cap A$ such that $R(M, b_\alpha)$ is disjoint from α and $\{b_\alpha\}$ is independent over $P_1^{M_\alpha} \cup P_4^M$.

Fix a $\delta^* \in S_A \cap C \cap C_1$.
Since $N \in \widehat{\boldsymbol{K}}$, by Definition, $N \models \neg(\exists x) \bigwedge_n R(x, F_n(c^*_\delta))$.
This contradicts that we chose $b_{\gamma_n^{\delta^*}} \in A$, so by the definition of A, for each $n < \omega$, $R^N(a, b_{\gamma_n^{\delta^*}})$ holds.

Case 2

Case 2

For some α^*, there is no such b_{α^*}.
That is, if $b \in P_1^M$ is independent from $P_1^{M_\alpha}$ and $R(M, b)$ is disjoint from α^* then $\neg R(a^*, b)$.

More complicated argument - invoking the non-measurability

$\boldsymbol{K}_2$-maximal

We can construct M_λ to be in $\boldsymbol{K}_2$ as well as P_0-maximal.

If M is not maximal build an increasing sequence of proper extensions (freely extend if possible) in $\boldsymbol{K}_2$ with the same P_0, we find an actual maximal model M' in $\boldsymbol{K}_2$.
This must happen before $(2^\lambda)^+$ steps.
We know M itself satisfies every subset of size less than λ is embedded in a free

The role of Set theory

1. Non-reflecting is used in the construction to obtain freeness at limits in S.
2. $\diamondsuit$ is used to verify the construction works.

Main Construction: ZFC version

Main Theorem

Theorem

Suppose $\lambda = (2^\chi)^+$ and there is no measurable cardinal less than or equal λ, then there is a P_0-maximal model of $\boldsymbol{K}_1$. (Hence a maximal model of $\boldsymbol{K}_1$ of cardinality at most 2^λ.

Reduction

Lemma

If $\mu = 2^\chi$ and χ is not measurable there is a Boolean algebra *B* of cardinality μ contained in $\mathbb{P}(\mu)$ which has no $\aleph_1$-complete non-principal ultrafilter.

Straightforward diagonalization.

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Blackbox

Göbel-Shelah blackbox

Assume $\lambda = \mu^+$ and $\lambda = \mu^\theta$ and $S \subseteq \{\delta : \delta < \lambda, \mathrm{cf}(\delta) = \aleph_0\}$ is a stationary subset of λ and $\langle C_\delta : \delta \in S \rangle$ guess clubs (and C_δ is an unbounded subset of δ of order type ω, of course). Then, we can find $\langle \overline{N}_\eta : \eta \in \Gamma \rangle$ such that:

(a) $\Gamma = \cup\{\Gamma_\delta : \delta \in S\}$ where $\Gamma_\delta \subseteq \{\eta : \eta$ in an increasing ω-sequence of ordinals $< \delta$ with limit $\delta\}$ and $\delta(\eta) = \delta$ when $\eta \in \Gamma_\delta, \delta \in S$

(b) $\overline{N}_\eta$ is $\langle N_{\eta,n} : n \leq \omega \rangle$ in $\prec$-increasing, and we let $N_\eta = N_{\eta,\omega}$

(c) each N_η is a model of cardinality κ with vocabulary $\subseteq H(\kappa^+)$ for notational simplicity, and universe $\subseteq \delta := \delta(\eta)$ and $N_{\eta,n} = N_\eta \restriction \gamma^\delta_n$ where γ^δ_n is the n-th member of C_δ

Blackbox continued

(*d*) for every distinct $\eta, \nu \in \Gamma_\delta$ where $\delta \in S$, for some $n < \omega$ we have $N_\eta \cap N_\nu = N_{\eta,n} = N_{\nu,n}$

(*e*) for every $\eta, \nu \in \Gamma_\delta$ the models N_η, N_ν are isomorphic, moreover there is such isomorphism f which preserve the order of the ordinals and maps $N_{\eta,n}$ onto $N_{\nu,n}$

(*f*) if $\mathcal{A}$ is a model with universe λ and vocabulary $\subseteq \mathcal{A}(\kappa^+)$ then for stationarily many $\delta \in S$ for some $\eta \in \Gamma_\delta \subseteq \Gamma$ we have $N_\eta \prec \mathcal{A}$. Moreover, if $\kappa^{<\kappa} = \kappa$ and h is a one to one function from ${}^{\aleph_0}\lambda$ into λ then, we can add: if $\rho \in {}^{\aleph_0}(N_{\eta,n})$ then $h(\rho) \in N_{\eta,n}$.

The role of Set theory

In the ZFC+ case:

1. Non-reflecting is used in the construction to obtain freeness at limits in S.
2. $\diamondsuit$ is used to verify the construction works.

Here we give up the freeness and blackbox guesses the clubs from a collection of possibilities.

We must give up freeness since by Magidor-Shelah, assuming the consistency of countably many supercompacts), one cannot prove in set theory that there are almost free nonfree Abelian groups (Boolean algebras) whose cardinality is above the first cardinal fixed point.

Proof Sketch

Let $\mu = 2^{\chi}$. Then $\mu = \mu^{\aleph_0}$ and there is no measurable cardinal $\leq \mu$.

By the reduction, there is a Boolean algebra *B* of cardinality μ contained in $\mathbb{P}(\mu)$ which has no $\aleph_1$-complete non-principal ultrafilter.

We will now construct a model *M* on λ and apply the fact taking *M* as the $\mathcal{A}$ in Fact45. In particular, we include a unary predicate *Q* which will approximate the diamond sequence in the ZFC+ proof.

Construction

1. Define M^1_γ by induction for $\gamma \leq \lambda$ as $\{\mathbf{m} = \langle \mathbf{M}_\alpha : \alpha < \gamma \rangle\}$ so that **m** mimics an initial stage of the construction in the ZFC+ argument.
2. Replace the diamond sequence by requiring: if $\alpha < \gamma$ and $\alpha \notin S$ and $A \in \boldsymbol{B}_{\leq\alpha}$ there is a $b \in P^{M_{\beta+1}}$ that is free from P^{M_β} over $P_4^{M_{\beta+1}}$ such that $R(M_\alpha, b) = A$ and even $R(M_\beta, b) = A$.
3. Expand each N_λ by a predicate Q with gives a non-principal ultrafilter on P_1^N.
4. Construct sequences $c_{\delta,\eta}, c_{\delta,\eta,n}$; the black box ensures that one η works as required.

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What (if anything) is special about measurable cardinals?