

Tracer Dispersion in a Self-Organized Critical System

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We have studied experimentally transport properties in a slowly driven granular system which recently was shown to display self-organized criticality [Frette *et al.*, *Nature (London)* **379**, 49 (1996)]. Tracer particles were added to a pile and their transit times measured. The distribution of transit times is a constant with a crossover to a decaying power law. The average transport velocity decreases with system size. This is due to an increase in the active zone depth with system size. The relaxation processes generate coherently moving regions of grains mixed with convection. This picture is supported by considering transport in a 1D cellular automaton modeling the experiment. [S0031-9007(96)00529-7]

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The avalanches that occur when grains are dropped onto a pile illustrate the spontaneous generation of complexity in simple dynamical systems [1]. When grains are dropped onto a finite base, a pile builds up. However, it cannot become infinitely high, and, eventually, the system settles in a stationary state where the outflux over the edge of the base on average equals the influx. Intermittent flow of grains down the slope of the pile (small and large avalanches) maintains the system in this state. Bak, Tang, and Wiesenfeld constructed a 2D cellular automaton of a slowly driven dynamical system. They showed that the “pile” spontaneously evolves, or self-organizes, into a state with avalanches of all sizes distributed according to a power law; that is, there is no internal system-specific scale. Because of the lack of any characteristic avalanche size, the system is referred to as critical [1].

It has been a long-standing question whether real granular systems display self-organized criticality (SOC) when slowly driven. Recently, however, an experiment on a quasi-one-dimensional pile of rice has shown that the occurrence of SOC depends on details in the grain-level dissipation mechanisms [2]. Only with sufficiently elongated grains, avalanches with a power-law distribution occurred. The elongated rice grains could pack in a variety of ways, and each avalanche replaced, locally or globally, one surface configuration with another. Thus a dynamically varying medium disorder (coupled to the relaxation processes) was generated. This is conceptually different from transport in media with a quenched disorder; see, e.g., Refs. [3,4]. Furthermore, in SOC systems, a small perturbation may lead to arbitrarily large avalanches, and it is not clear at all how this affects the transport properties. Thus it is quite surprising that there are no experiments and only a few theoretical and numerical studies on transport in systems displaying SOC [5–7].

We have measured the transit times of colored tracer particles, that is, the time tracer particles spent in the pile. Experimentally, we find that the distribution of transit

times is essentially constant for small transit times T and decays as a power law for large T . The average transit time is $\langle T \rangle \propto L^\nu$, where L is the system size and $\nu = 1.5 \pm 0.2$. Thus the average velocity of tracer particles $\langle V \rangle \propto L/\langle T \rangle$ decreases with system size. This implies the existence of long-range correlations in the rice pile. Correlations in the sequence of transit times decay exponentially with a correlation time that increases with system size but have a crossover to a power-law decay. This indicates that the system has coherently moving regions, the size of which increases with system size. A 1D cellular automaton model, related to the experiment, displays SOC behavior and the distribution of transit times agrees qualitatively with the experimental findings. The scaling of the average transit time with system size is reproduced and hence the decrease of the average velocity with increasing system size. This phenomenon is related to an increase in the active zone depth as the system size is increased. Finally, the numerical and experimental results for the correlations in the transit times are very similar.

The experimental system consisted of a rice pile confined between two vertical 5 mm thick glass plates supported by 15 mm thick 100 cm \times 120 cm polymethylmethacrylate plates. Aluminum rods were inserted between the glass plates to form a vertical wall at one side and a variable base with length \mathcal{L} of the quasi-one-dimensional pile. The other side was open, allowing grains to fall off the pile. Grains of “Geisha Naturris” from Stabburet a.s. (Trollåsen, Norway) with length $\delta = 7.6 \pm 0.9$ mm and width 2.0 ± 0.1 mm were slowly fed into the gap between the plates close to the vertical wall using an Övrum Nibex 500 (Övrum, Sweden) single seed machine. We used a plate separation of $d = 6$ mm and system sizes $\mathcal{L} = 15, 30, 60,$ and 85.7 cm, which, expressed in units of the grain length δ , are $\mathcal{L}/\delta = L = 20, 39, 79,$ and 113 . The injection rate was 2–3 uncolored grains every 7.7 s or, on average, 20 grains/min. When the pile had reached the stationary

state, color coded particles were added to the pile by hand and the transit times measured. A total number of 400 tracer particles for the two smaller systems and 800 for the two larger systems were used. Except for the smallest system, where a color coded tracer particle was injected every second minute, the rate was one tracer particle every fourth minute. The injection of uncolored grains was continued until all tracers had left the system.

Figure 1 shows a part of the $L = 113$ system at a late stage of the experiment. Some of the tracer particles are close to the surface layer while others are buried quite deep in the pile. These tracers had to be released by large avalanches in order to move on. However, large avalanches occurred with a very small probability, and hence deeply buried particles tended to stay in the system for a very long time. Figure 2 is a full record of the experimental findings for the $L = 113$ rice pile. The projections of the horizontal lines onto the x axis represent the time interval each tracer particle spent in the pile, that is, the transit time of the i th tracer $T(i) = T_{\text{out}}(i) - T_{\text{in}}(i)$, where $T_{\text{in}}(i)$ and $T_{\text{out}}(i)$ denote the input and output time measured in units of additions of uncolored grains (one addition every 7.7 s), respectively. There is a huge variability in the transit times. The distribution functions of transit times $P(T, L)$ for all system sizes are shown in Fig. 3. A data collapse for different system sizes L is obtained when plotting $L^\beta P(T, L)$ against the rescaled variable T/L^ν when using $\nu = 1.5 \pm 0.2$ and $\beta = 1.4 \pm 0.2$; see Fig. 3 inset. Thus we can write

$$P(T, L) = L^{-\beta} F(T/L^\nu), \quad (1)$$

where F is a scaling function and ν a critical exponent expressing how the crossover transit time T_c scales with

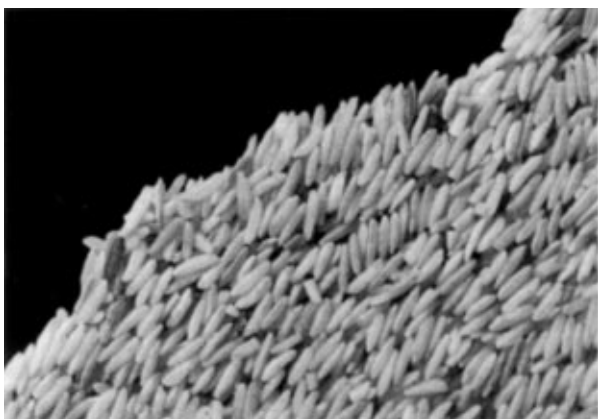


FIG. 1. A close up photograph of the rice pile with size $L = 113$. The plate separation/grain length ratio $d/\delta = 0.79$ and most of the grains are aligned along the flow direction. Each tracer particle is uniquely color coded; thus it is possible to measure individual transit times. This picture was taken after 638 out of 800 tracer particles had been added. The region shown contains 5 of a total of 73 tracer particles that were inside the pile at this time.

system size. The scaling function F is of the form $F(x) = \text{const}$ for $x < 1$ and $F(x) \propto x^{-\alpha}$ for $x > 1$, where $\alpha = 2.4 \pm 0.2$. Since $\alpha > 1$, it follows from the form of the scaling function F and the normalization constraint that $\beta = \nu$. The power-law tail does not affect the scaling of the mean transit time $\langle T \rangle$ if $\alpha > 2$, and $\langle T \rangle \propto T_c \propto L^\nu$. In the experiments, the angle of repose was independent of system size. Thus the average velocity of tracer particles scales like $\langle V \rangle \propto L/\langle T \rangle \propto L^{1-\nu} = L^{-0.5 \pm 0.2}$. It is quite surprising that the average velocity is not a constant but decreases with increasing system size. The (tracer) particles have information on the system size. This can happen only if correlations exist throughout the system. In the statistically stationary state, the steady influx of particles is balanced with a steady outflux. During a time Δt , each particle moves, on average, a distance $\Delta t \langle V \rangle$. Since the feeding rate was the same for all system sizes, the number of particles that crossed any given section in this time interval $\Delta t \langle V \rangle \lambda_L = \text{const}$ where λ_L denotes the active zone depth, that is, the depth of the zone of the grains that participated in the dynamics. This implies $\langle V \rangle \propto 1/\lambda_L$ and an increase in λ_L with system size would be consistent with the experimental finding of a decreasing average velocity.

Further insight into the correlations that exist in this dynamical state can be extracted from the data. We noticed that, once in a while, many tracers dropped out

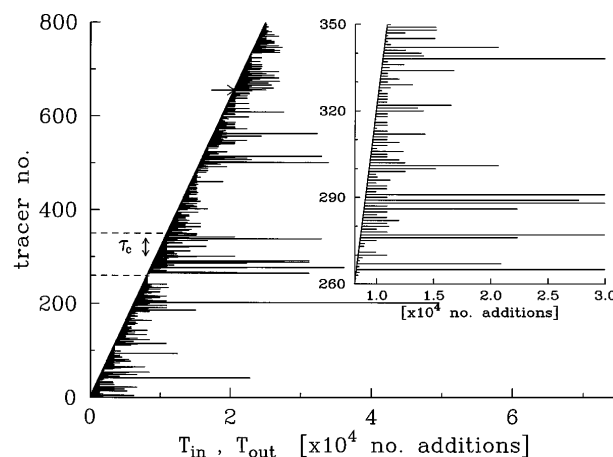


FIG. 2. A record of the tracer experiment in a pile of size $L = 113$ where a total number of 800 tracer particles were added, one every fourth minute. The tilted line connects the injection times for all the tracers. The transit time for each tracer particle is represented by the length of a horizontal line whose projection onto the x axis of the left (right) end point equals the time the particle entered T_{in} (left T_{out}) the system. The transit time is measured in units of the number of injections of uncolored grains (number of additions), one addition every 7.7 s. Note the large variation in the transit times $T = T_{\text{out}} - T_{\text{in}}$ and that, repeatedly, many tracers left the system at the same time. The horizontal arrow indicates the time at which the photograph in Fig. 1 was taken. Furthermore, the correlation time τ_c [see text related to Eq. (2)] is indicated. The inset is the part of the full record marked by dashed lines.

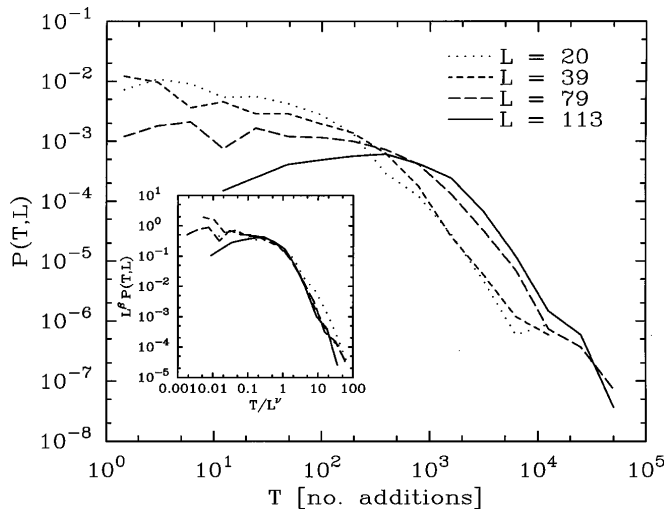


FIG. 3. The experimental results for the normalized distribution of transit times in piles with sizes $L = 20, 39, 79,$ and 113 . The data have been averaged over exponentially increasing bins with base 2 in order to reduce the fluctuations in the statistics due to the relatively small number of tracer particles. The functions are essentially constant for small T and have a decaying power-law tail with a slope of $\alpha = 2.4 \pm 0.2$. These large transit times correspond to tracer particles which, during the transport through the systems, become deeply embedded in the pile. Disregarding the smallest system and using Eq. (1), a reasonable data collapse of the three largest systems is obtained with $\nu = 1.5 \pm 0.2$ and $\beta = 1.4 \pm 0.2$, as shown in the inset.

of the system at the same time since they were part of a large avalanche that reached the rim of the pile. The corresponding “steps” in the diagrams, like the one shown in Fig. 2, have many different sizes and are interwoven in a complex way. The correlation function

$$c(\tau) = \langle I(i, i + \tau) \rangle, \quad (2)$$

where

$$I(i, i + \tau) = \begin{cases} 1 & \text{if } |T_{\text{out}}(i) - T_{\text{out}}(i + \tau)| < \delta t, \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

is the indicator function of simultaneous dropout and $\langle \cdot \rangle$ denotes average over all i , gives the probability that tracers i and $i + \tau$ left the system simultaneously. Using $\delta t = 10$, this quantity is found to behave as $c(\tau) \propto \exp[-\tau/\tau_c(L)]$ for small τ , where τ is the time difference in units of 4 min and $\tau_c(L) = 5.8 \pm 1.3, 8.7 \pm 2, 19 \pm 4,$ and 34 ± 4 is the correlation time for system sizes $L = 20, 39, 79,$ and 113 , respectively. The results do not depend on the value of δt . In Fig. 2, we see that $\tau_c(L = 113) = 34$ corresponds to the average size of the steps. Since the injection rate was 20 grains/min, the characteristic number of grains spanned by these correlated sequences of tracers were 476, 705, 1539, and 2754. The solidlike motion of domains described

in Ref. [2] is probably one aspect of this coherent motion. These domains can be characterized by their volume. Using $\lambda_L \propto \text{volume}/L$ and disregarding the smallest system, we obtain $\lambda_L \propto L^{0.25 \pm 0.2}$, which should be compared to the result obtained in a more direct way above. However, there is significant dispersion. Grains are being transferred between different coherently moving areas; see Fig. 2. The average domain size increases with system size; however, in large systems large domains are broken up more often, which might explain the lower exponent.

Inspired by the experiments, we considered a refined version of a simple 1D cellular automaton studied in Ref. [8]. In a system of size L , an integer variable h_x gives the height of the pile at site x . The local slope z_x at site x is given by $z_x = h_x - h_{x+1}$, and we impose $h_{L+1} = 0$. The addition of a grain at the wall increases the slope by 1 at $x = 1$, that is, $z_1 \rightarrow z_1 + 1$. We proceed by dropping grains at the wall until the slope z_1 exceeds a critical value, $z_1 > z_1^c$, then the site topples by transferring one grain to its neighboring site on the right, $x = 2$. If $z_x > z_x^c$, this site topples in turn according to

$$z_x \rightarrow z_x - 2, \quad (4)$$

$$z_{x+1} \rightarrow z_{x+1} + 1$$

(unless at the rightmost site where the grains fall off the pile) generating an avalanche. During the avalanche, no grains are added to the pile. The avalanche stops when the system reaches a stable state with $z_x \leq z_x^c \forall x$ and grains are added at the wall until a new avalanche is initiated and so on. The critical slopes z_x^c are dynamical variables chosen randomly to be 1 or 2 every time site x has toppled. This is a simple way to model the changes in the local slopes observed in the rice pile experiment. Thus the model differs from the trivial 1D Bak-Tang-Wiesenfeld (BTW) model where $z_x^c = 1$ is a constant and where grains are added on randomly chosen sites [1]. The randomness in the BTW model is external. In our model, the randomness is internal and inherent in the dynamics. For arbitrary initial conditions, the system reaches a stationary state where the avalanche sizes are power-law distributed with an exponent $\tau_s = -1.55 \pm 0.10$ and a cutoff in the power-law distribution that scales with system size as $L^{D_s} = L^{2.25 \pm 0.10}$. Since the average avalanche size scales linearly with system size, these exponents must fulfill the scaling relation $\tau_s = 2 - 1/D_s$ [9].

We have measured the transit times (in units of added grains) of the added particles after the pile has reached the stationary critical state. Using Eq. (1), a reasonable data collapse is obtained with $\nu = 1.30 \pm 0.10$ and $\beta = 1.35 \pm 0.10$ as displayed in Fig. 4. We find $\alpha = 2.22 \pm 0.10$; that is, the results agree well with the experimental findings. The average depth of tracer particles during the transport through the system as a function of the average transit time is displayed as an

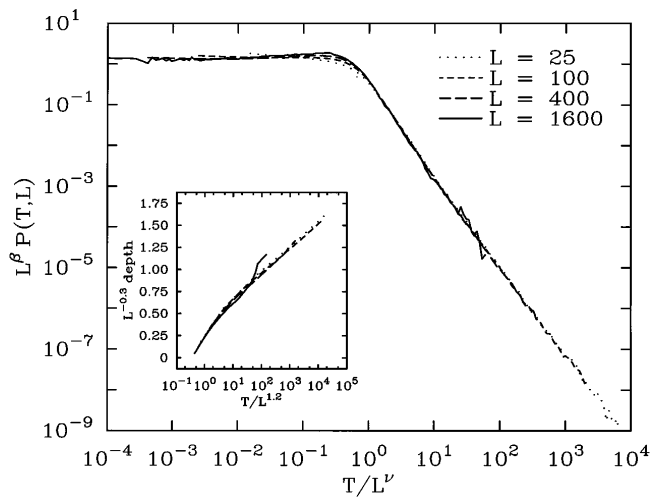


FIG. 4. A finite-size scaling plot using Eq. (1) with $\nu = 1.30 \pm 0.10$ and $\beta = 1.35 \pm 0.10$ of the normalized distribution of transit times in the numerical model with system sizes $L = 25, 100, 400,$ and 1600 . The statistics shown correspond to 10^7 tracer particles (10^6 for $L = 1600$), and the data have been averaged over exponentially increasing bins with base 1.1. The functions are constant for small transit times and decay as power laws with a slope of $\alpha = 2.22 \pm 0.10$. The inset shows that a data collapse of the correlation between the transit times and depth of the tracer particles can be obtained using a finite-size scaling plot analogous to Eq. (1), with $\nu' = 1.20 \pm 0.10$ and $\beta' = -0.30 \pm 0.10$.

inset of Fig. 4, and it shows that the power-law tail in the distribution of transit times arises from tracer particles that become deeply buried in the pile, while tracer particles that stayed close to the surface of the pile during the transport through the system mainly contributed to the constant part. The average velocity $\langle V \rangle \propto L/\langle T \rangle \propto L^{1-\nu} = L^{-0.30 \pm 0.10}$, in fairly good agreement with the experimental result. Furthermore, in the model, we are able to measure directly the scaling of the active zone depth with system size. Defining

$$\lambda_L = \sqrt{\frac{1}{L} \sum_{x=1}^L \langle [h_x(t) - \langle h_x(t) \rangle_t]^2 \rangle_t}, \quad (5)$$

$h_x(t)$ being the height at position x at time t and $\langle \cdot \rangle_t$ denotes the temporal average, we find $\lambda_L \propto L^{0.25 \pm 0.10}$ in agreement with the relation $\langle V \rangle \propto 1/\lambda_L$. Paczuski and Boettcher have mapped the model to a depinning transition of an interface dragged at one end through a random medium, and it is argued that the scaling of the active zone depth with system size is given by the exponent $\nu - 1 = D_s - 2 = 0.25$ as well [9]. Alternatively, a third measure for the active zone depth is available from the inset in Fig. 4, giving $\lambda_L \propto \text{depth} \propto L^{0.3 \pm 0.10}$. Moreover, correlation functions calculated from Eqs. (3) and (4) give an exponential decay for short times

with a crossover to a decaying power law for large times. The scaling of the correlation time with system size leads to $\lambda_L \propto \text{volume}/L = L^{0.20 \pm 0.1}$. Adding grains at the wall and allowing for dynamical critical slopes, we have seen similar results for the nonlocal limited model in Ref. [10]. Thus the behavior seems to be universal.

In conclusion, this new direction of research sheds light upon the dynamics of SOC systems in general and granular systems in particular. We find that transport in a SOC granular medium is characterized by an average grain velocity that approaches zero when the system size increases. The dynamics of the system displays comprehensive correlations. These experimental findings agree well with the behavior seen in simple one-dimensional computer models of the self-organized critical pile.

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