

## Local distributions and rate fluctuations in a unified scaling law for earthquakes

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A recently proposed unified scaling law for interoccurrence times of earthquakes is analyzed, both theoretically and with data from Southern California. We decompose the corresponding probability density into local-instantaneous distributions, which scale with the rate of earthquake occurrence. The fluctuations of the rate, characterizing the nonstationarity of the process, show a double power-law distribution and are fundamental to determine the overall behavior, described by a double power law as well.

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Earthquakes constitute an extremely complex phenomenon in nature, with the deformation and sudden rupture of some parts of the Earth's crust driven by convective motion in the mantle, and the radiation of energy in the form of seismic waves. Only a part of this complexity is collected by earthquake catalogs, where magnitude, epicenter spatial coordinates, and starting time of events, among other measurements, are recorded. This information, which converts the phenomenon in a spatiotemporal point process marked by the magnitude, nevertheless reveals some important scale-invariant properties.

First, the Gutenberg-Richter law states that the number of earthquakes in some region with magnitude larger than some threshold value decreases exponentially with the threshold. Taking into account that (to a first approximation) the released energy increases exponentially with the magnitude, the probability distribution of the released energy turns out to be a power law, precisely the hallmark of scale-free behavior [1–3]. Second, the introduction of fractal geometry soon led to the recognition that the spatial distribution of epicenters (or hypocenters) draws a fractal object over the Earth's surface [2,3]. And third, the Omori law, proposed more than 100 years ago, accounts for the number of events (called aftershocks) that follow a large shock after some time. This number is another power law, with exponent close to minus one [4].

This lack of characteristic scales suggests that the crust is in a critical state, like the well-known critical points studied in equilibrium systems, but without external adjustments of control parameters. Therefore, one may talk about a self-organized critical (SOC) state for the seismic system [5]. This concept has important implications for the issue of earthquake prediction; indeed, a critical crust implies that a fracture process may or may not develop to provoke a large earthquake depending on minor microscopic details that are intrinsically out of control [6].

An important quantity characterizing earthquake occurrence is the time interval between successive earthquakes. This time (that can be referred to as interoccurrence time, recurrence time, or waiting time) although related to the Omori law, has a distribution that is not clearly known. In fact, all the possibilities have been proposed, from periodic

behavior for large earthquakes to totally random occurrence. The most extended view is to consider the existence of two separated processes, one for the main shocks, which should occur randomly following a Poisson distribution, and another process for the aftershocks; but this should not hold for large events, for which clustering has been reported [7]. In general, the usual studies proceed by fixing a limited area of observation where aftershocks are skillfully identified and removed from data. On the opposite side, other works concentrate only on series of aftershocks.

Bak *et al.* [8] have followed an alternative approach, which is to consider the problem in its complete spatiotemporal complexity. They divide the area of South California into regions of size  $L$  degrees in the north-south (meridian) direction and  $L$  degrees as well in the east-west (parallel) direction [9,10]. Only earthquakes with magnitude  $m$  larger than a threshold value  $m_c$  are taken into account (but no other events are eliminated, all shocks are equally treated). For each  $L \times L$  region the time interval  $\tau$  between consecutive earthquakes is obtained for the period from 1984 to 2000 as  $\tau_i = t_i - t_{i-1}$ , where  $t_i$  is the time coordinate of the  $i$ th earthquake within the region with  $m > m_c$ . The probability density for this interoccurrence time,  $D(\tau, m_c, L)$ , is computed and the results give  $D(\tau, m_c, L) \propto 1/\tau$  for short times and a faster decay for long times, with a dependence also on  $L$  and  $m_c$ .

Remarkably, when a scaling analysis is performed, all the distribution functions corresponding to different values of  $L$  and  $m_c$  collapse into a single curve if the axes are rescaled by  $S^b/L^{d_f}$ , with  $d_f \approx 1.2$ ,  $b \approx 1$ , and  $S \equiv 10^{m_c}$  (related to the energy roughly as  $S \propto E^{2/3}$ ). In mathematical words,

$$D(\tau, m_c, L) \approx \frac{L^{d_f}}{S^b} F\left(\frac{L^{d_f}}{S^b} \tau\right) = \frac{1}{\tau} G\left(\frac{L^{d_f}}{S^b} \tau\right); \quad (1)$$

this scaling law constitutes the Bak-Christensen-Danon-Scanlon (BCDS) proposal [8]. For short times, the function  $G$  shows a slow variation not affecting the power-law ( $1/\tau$ ) behavior; for long times a fast decay is obtained, which could be consistent with an exponential distribution and therefore with a Poisson process, according to Ref. [8]. [From now on, to simplify the notation, we will omit the dependence of  $D$  on  $L$  and  $m_c$  and just write  $D(\tau)$ .]

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This result is relevant for several reasons, among them: it shows scaling in the spatiotemporal occurrence of earthquakes, a key element to consider earthquakes as a critical phenomenon. Second, it is the first law that relates interoccurrence times, the Gutenberg-Richter law (factor  $1/S^b$ ), and the fractal dimension of the spatial distribution of events ( $d_f$ ), allowing a unified description. Third, the law is valid for all earthquakes, no matter their size or location, and no matter also if they are considered as aftershocks, foreshocks, or main shocks. Fourth, the power law tells us that immediately after any earthquake there is a high probability of return, and this probability decreases in time with no characteristic scale up to  $S^b/L^{d_f}$ ; that is, there is a correlation time that depends on the region size and magnitude under consideration, and therefore for any event one may find clusters of aftershocks in all time scales up to an appropriate length scale  $L$ .

The importance of this law deserves further study. Here we are interested in a general understanding of the BCDS law and its origins. We will analyze the same catalogs as Bak *et al.* [11] and will show that the fast decay for long times is not exponential, but another power law.  $D(\tau)$  is related to its local and instant components and to the rate of earthquake occurrence  $r$ ; this quantity, which counts the number of events per unit time in a given region, displays large fluctuations across several orders of magnitude, doubly power-law distributed. This is in contrast to simple SOC models.

Let us pay more attention to the obtaining of the distribution  $D(\tau)$  by Bak *et al.* As we have mentioned, this distribution accounts for the time difference between successive earthquakes with magnitude larger than  $m_c$  in every  $L \times L$  region. Times from different regions are counted together in  $D(\tau)$ . But the total number of earthquakes differs from region to region (as it is well known, due to the fractal spatial distribution), with a high variability [12]. Therefore, the local distributions  $D_{xy}(\tau)$  accounting for the time difference in a given  $L \times L$  region (of spatial coordinates  $x, y$ ) are clearly different. This means that  $D(\tau)$  is a mixed distribution constructed from all the different  $D_{xy}(\tau)$ .

But further, looking into a single  $L \times L$  region one can see a high variability in the rate along time  $t$  [12] (see Fig. 1). In fact, the rate typically exhibits a quite stable behavior for some periods of time, with small fluctuations, but for other periods develops sudden burst of activity where its value increases sharply and then decreases to become stationary again, or not. This intermittency, of course related to the occurrence of larger earthquakes in the region, recalls the punctuated-equilibrium behavior of SOC systems [5,13], but note however that the variable that displays punctuated equilibrium is not only the signal  $m(t)$ , but also the rate.

Again, the local distribution  $D_{xy}(\tau)$  is obtained as a mixture of distributions, the densities of interoccurrence times in a given region at a certain time  $t$ ,  $D_{xyt}(\tau)$ . From this we can write

$$D(\tau) \propto \sum_{x,y} \int D_{xyt}(\tau) r(x,y,t) dt, \quad (2)$$

where the rate  $r(x,y,t)$  is the number of earthquakes per unit

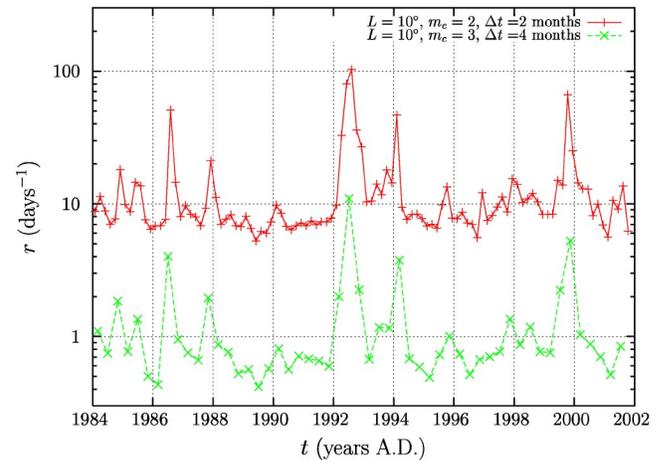


FIG. 1. Rate of earthquake occurrence as a function of time in the  $L=10^\circ$  region of South California, for  $m_c=2$  with  $\Delta t=2$  months and for  $m_c=3$  with  $\Delta t=4$  months. The vertical log scale should not make one under-rate the large variations in  $r$ . Note, for example, the almost constant rate in 1991 in contrast to 1992.

time in the region  $(x,y)$  at time  $t$ . (All functions here depend as well on  $L$  and  $m_c$ , though the dependence is not explicitly written.) The rate  $r$  in the integral acts as a weight factor, due to the fact that the higher the rate in a given region and time, the larger the number of earthquakes that are produced and contribute to the distribution.

We now make the hypothesis that the dependence on space and time enters into the distribution  $D_{xyt}$  only through the rate  $r(x,y,t)$ . That is, we assume that different regions at different times but with the same rate of occurrence will have the same distribution of interoccurrence times (if the rate is stationary), i.e.,

$$D_{xyt}(\tau) = D(\tau|r(x,y,t)), \quad (3)$$

which is a conditional density. Therefore,

$$D(\tau) = \int_0^\infty D(\tau|r) \frac{r\rho(r)}{\mu} dr, \quad (4)$$

with  $\rho(r)$  being the probability density of the rate and  $\mu = \langle r \rangle$  just a normalization factor.

Since  $D_{xyt}(\tau)$  is an instantaneous quantity (and we have a single realization of the process), it would be impossible to measure if we did not have the periods of stationarity in  $r$ . Figure 2 shows these distributions for several periods of stationarity and several regions of different  $L$  and spatial coordinates. Indeed, the distributions  $D_{xyt}$  do not only depend exclusively on  $r$ , but they scale with it, i.e.,

$$D(\tau|r) \approx r f(r\tau) \quad (5)$$

(a behavior that could have been derived by dimensional analysis), with the scaling function  $f$  being a power law for short times and presenting a fast decay for large ones. In fact, the distributions can be fit by a function of the type

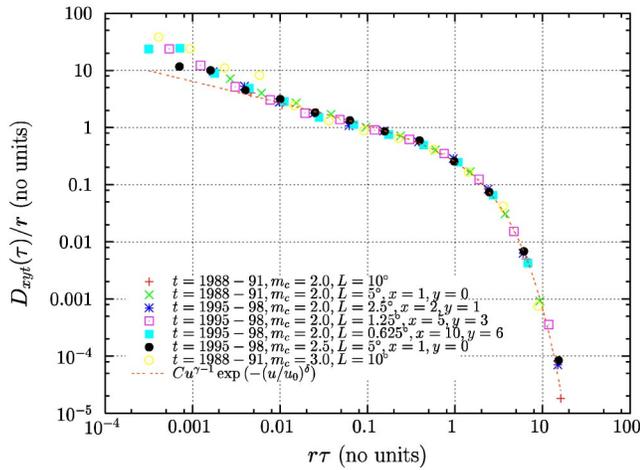


FIG. 2. Local distributions of interoccurrence times for several stationary periods and different regions, after scaling by the rate. The regions are labeled from  $x, y = 0$  to  $10^\circ/L - 1$  from west to east ( $x$ ) and from south to north ( $y$ ). The fit is explained in the text; deviations at small times are due to short-scale disturbances of the stationarity. As in Bak *et al.*'s paper, times smaller than 38 s are not considered.

$$f(u) = C \frac{1}{u^{1-\gamma}} e^{-(u/u_0)^\delta}, \quad (6)$$

with  $\gamma = 0.63 \pm 0.05$ ,  $\delta = 0.92 \pm 0.05$ ,  $u_0 = 1.50 \pm 0.15$ , and  $C = 0.5 \pm 0.1$ . We could approximate then  $D(\tau|r)$  to a gamma distribution ( $\delta = 1$ ), which ensures that the large-scale cutoff is close to exponential [14].

Next step is to look at the distribution of rates  $\rho(r)$ . To be precise,  $r(x, y, t)$  is defined by counting the number of events above the threshold  $m_c$  into the  $L \times L$  region of coordinates  $x, y$  during a time interval  $(t, t + \Delta t)$  and dividing the result by the duration of the interval,  $\Delta t$ . The corresponding probability density, which is calculated from 1984 to 2001 only for the regions in which there is earthquake activity, depends on  $L$  and  $\Delta t$  (as well as on  $m_c$ ) and is shown in Fig. 3. In fact, the value of  $\Delta t$  should be small enough to ensure  $r \approx$  constant (but large enough for statistical significance), but there is no characteristic scale for constant  $r$  and therefore no typical value for  $\Delta t$ . Also, the two-power-law behavior is noteworthy in the figure, one power law for low rates and another one for high rates, which can be modeled as

$$\rho(r) = C' \theta \frac{(\theta r)^{\alpha-1}}{[1 + (\theta r)^c]^{(\alpha+\beta)/c}}, \quad (7)$$

which gives  $\rho \propto 1/r^{1-\alpha}$  for  $r \leq \theta^{-1}$  and  $\rho \propto 1/r^{1+\beta}$  for  $r \gg \theta^{-1}$ . We obtain exponents for  $r$  about  $1.0 \pm 0.1$  and  $2.2 \pm 0.1$ , so  $\alpha \approx 0$  and  $\beta \approx 1.2$ . Parameter  $c$  just controls the sharpness of the transition from one regime to the other and  $\theta^{-1}$  is a scaling parameter. The double power law implies that there is no characteristic occurrence rate up to the value  $\theta^{-1}$ , but for values in the tail of the distribution there is also

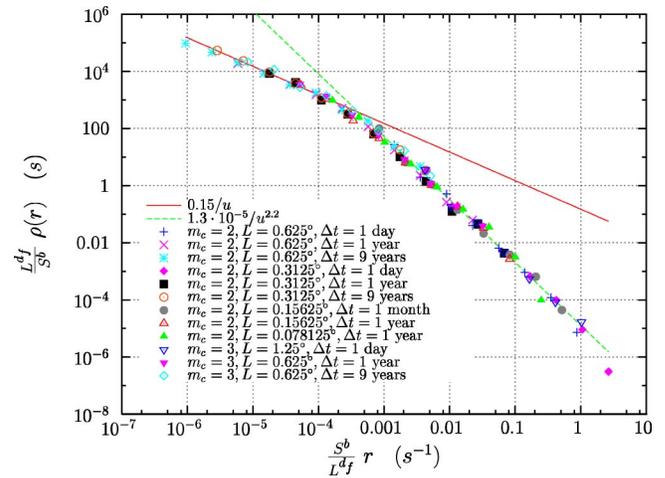


FIG. 3. Scaled distributions of rates, for several  $\Delta t$ ,  $L$ , and  $m_c$ , using  $d_f = 1.6$  and  $b = 0.95$ . Two power laws with exponents 1 and 2.2 fit the data.

scale invariance. This could be understood as criticality, not only in the time domain as we knew, but also in the rate domain.

Additionally, it is easy to obtain the form of the scaling factor  $\theta^{-1}$ . The mean rate  $\mu = \langle r \rangle$  is given by the total number of events divided by the total time and by the number of regions with activity; the former, because of the Gutenberg-Richter law, scales as  $1/S^b$  and the latter as  $1/L^{d_f}$ , which gives  $\langle r \rangle \propto L^{d_f}/S^b$ . Since the distribution turns out to scale in the same way,  $\theta^{-1} \propto L^{d_f}/S^b$ . For the scaling plot in Fig. 3 we have used  $b = 0.95$  [8] and  $d_f = 1.6$ , which was obtained from a box-counting method for spatial distributions of epicenters with  $m \geq 2$ .

Now that we know the form of the functions  $D(\tau|r)$  and  $\rho(r)$  we can answer the question about how the large variations of the rate influence the distribution  $D(\tau)$ , just by integrating Eq. (4) with the use of Eqs. (5)–(7). The limit  $\tau \rightarrow \infty$  is obtained directly with the use of Laplace's method to

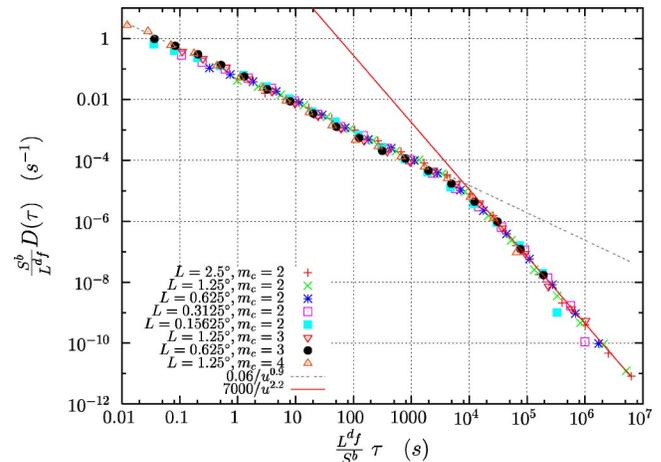


FIG. 4. Scaled distributions of interoccurrence times,  $D(\tau)$ , for different  $L$  and  $m_c$  with  $d_f = 1.6$  and  $b = 0.95$ ;  $\tau \geq 38$  s again. The straight lines illustrate the double power-law behavior, with exponents 0.9 and 2.2.

evaluate asymptotic integrals [15]. We get

$$D(\tau) \propto \frac{\theta^\alpha}{\mu} \tau^{\gamma-1} \int_0^\infty r^{\alpha+\gamma} e^{-(\tau r/u_0)^\delta} dr \propto \frac{\theta^\alpha}{\delta \mu \tau^{2+\alpha}}; \quad (8)$$

this is in fact independent of the tail of  $\rho(r)$ , it does not matter if it is a power law or not. But there is another limit to be studied. Indeed, the behavior of the integral depends on the relation between  $\tau$  and  $\theta$ . We have just calculated what happens for  $\tau \gg \theta$ ; the opposite case,  $\tau \ll \theta$ , can be obtained for long times if we first perform the limit  $\theta \rightarrow \infty$  and then apply Laplace's method for  $\tau$ . So,  $\rho(r) \sim C' \theta^{-\beta} / r^{1+\beta}$ , and the integral gives

$$D(\tau) \propto \frac{\tau^{\gamma-1}}{\mu \theta^\beta} \int_0^\infty r^{\gamma-\beta} e^{-(\tau r/u_0)^\delta} dr \propto \frac{1}{\delta \mu \theta^\beta \tau^{2-\beta}}. \quad (9)$$

Since  $\mu$  is proportional to  $\theta^{-1}$  the last results can be summarized as follows:

$$D(\tau) \propto \frac{\theta^{1-\beta}}{\tau^{2-\beta}} \quad \text{for } \tau \ll \theta$$

and

$$D(\tau) \propto \frac{\theta^{1+\alpha}}{\tau^{2+\alpha}} \quad \text{for } \tau \gg \theta, \quad (10)$$

and they are displayed in Fig. 4, where the two exponents for  $\tau$  turn out to be  $0.9 \pm 0.1$  and  $2.2 \pm 0.1$ , yielding  $\beta \approx 1.1$  and  $\alpha \approx 0.2$  in good agreement with our calculations. Both results for  $D(\tau)$  do not depend on the form given by Eq. (7) for  $\rho(r)$  as long as it exhibits the two-power-law behavior. In addition, the exponent  $\gamma$  of  $D(\tau|r)$  does not affect the value of the two exponents of  $D(\tau)$ . Notice that the scaling factor for  $\tau$  in  $D(\tau)$  is  $\theta$ , that is, the inverse of the scaling factor for  $r$ , so  $\theta \propto S^b / L^{d_f}$ .

Finally, we would like to point out that  $D(\tau)$  is described by the same function (including the same values of the exponents) than the one that characterizes the trapping time distribution in a rice-pile model, (see Fig. 1 of Ref. [16]). Also the coincidence between our Fig. 3 and Fig. 3 in Ref. [17] is notable, although no probability density is measured there. Accordingly, the exponents about 0.9 and 2.2 should be quite universal.

In conclusion, we have performed a ‘‘microscopic’’ analysis of the BCDS law for earthquakes, which provides a way to deal with the heterogeneity and nonstationarity of seismic occurrence.

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- [1] B. Gutenberg and C. F. Richter, *Seismicity of the Earth* (Hafner, New York, 1965).
- [2] D. L. Turcotte, *Fractals and Chaos in Geology and Geophysics* (Cambridge University Press, Cambridge, 1997).
- [3] Y. Y. Kagan, *Physica D* **77**, 162 (1994).
- [4] T. Utsu, Y. Ogata, and R. S. Matsu'ura, *J. Phys. Earth* **43**, 1 (1995).
- [5] P. Bak, *How Nature Works: The Science of Self-Organized Criticality* (Copernicus, New York, 1996); H. J. Jensen, *Self-Organized Criticality* (Cambridge University Press, Cambridge, 1998); D. L. Turcotte, *Rep. Prog. Phys.* **62**, 1377 (1999); D. Sornette, *Critical Phenomena in Natural Sciences* (Springer, Heidelberg, 2000).
- [6] I. Main, *Nature (London)* **385**, 19 (1997); R. J. Geller, D. D. Jackson, Y. Y. Kagan, and F. Mulargia, *Science* **275**, 1616 (1997).
- [7] For references see J.-H. Wang and C.-H. Kuo, *J. Seismol.* **2**, 351 (1998); D. Sornette and L. Knopoff, *Bull. Seismol. Soc. Am.* **87**, 789 (1997).
- [8] P. Bak, K. Christensen, L. Danon, and T. Scanlon, *Phys. Rev. Lett.* **88**, 178501 (2002); *ibid.* **90**, 109901 (2003); K. Christensen, L. Danon, T. Scanlon, and P. Bak, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 2509 (2002).
- [9] K. Ito, *Phys. Rev. E* **52**, 3232 (1995).
- [10] Note that, due to the use of the longitude, the sides of a region do not have the same length and the regions are not equally sized. Nevertheless this does not influence the results.
- [11] Southern California Seismographic Network catalogs: <http://www.scecdc.scec.org/ftp/catalogs/SCSN>. We will use data from 1984 to 2001; almost all events with  $m \geq 2$  are in the region  $(30^\circ N, 40^\circ N) \times (123^\circ W, 113^\circ W)$ , so  $L = 10^\circ$  will be the size of the area under study.
- [12] In fact, it is an infinite variability, due to its power-law distribution, as we will see.
- [13] S. J. Gould and N. Eldredge, *Nature (London)* **366**, 223 (1993).
- [14] The universality of this scaling behavior and their counterintuitive consequences will be analyzed elsewhere.
- [15] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, New York, 1999).
- [16] M. Bogañá and A. Corral, *Phys. Rev. Lett.* **78**, 4950 (1997).
- [17] A. Corral and M. Paczuski, *Phys. Rev. Lett.* **83**, 572 (1999).