Pure-injective modules, H-subgroups and duality

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Introduction

This is a series of four lectures I gave at the Universitat Autònoma de Barcelona at the end of February 2003. My object was on one side to sketch some fundamentals of the theory of purity and pure-injective modules and on the other side to present variants of duality results of Ánh, Herbera and Menini. The first three lectures emphazised the characterization of pure-injective modules via matrix subgroups and H-subgroups. This material also plays a decisive role in the last lecture on duality where I formulated two theorems saying that certain weak duality assumptions concerning the socle together with AB5*-conditions for sets of H-subgroups imply stronger duality conditions. Due to the lack of time I confined myself to recording the definitions, some main results and many examples. For the benefit of the reader I have slightly enlarged these notes by presenting complete proofs of some important characterizations of pure-injective modules. Concerning other aspects and applications of pure-injectivity, for instance in the representation theory of artin algebras, the reader is referred to [9, 13, 14, 21].

Throughout these notes modules will mostly be right modules over a ring R. Given a module M_R, $\mathcal{L}(M)$ will denote the set of all submodules of M, $\mathrm{Fac}(M)$ the class of all epimorphic images of M, $\mathrm{Cogen}(M)$ the class of modules A such that there is an embedding $A \to M^I$ into a direct product of copies of M, $\mathrm{Gen}(M)$ the class of modules B such there is an epi $M^{(I)} \to B$ from a direct sum of copies of M onto B, and M^+ the left R-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. For modules M_R, N_R we shall use the abbreviation (M_R, N_R) or (M, N) instead of the lengthy $\mathrm{Hom}_R(M, N)$. Furthermore $J(S)$ will denote the Jacobson radical of a ring S.

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Lecture 1

Pure-exact sequences, pure-projective and pure-injective modules.

We start with an exposition of some important notions and results needed later. We emphazise that most of it is based on the pioneering article "Purity and algebraic compactness for modules" by R. B. Warfield Jr. [25].

Definition and Proposition 1.1. An exact sequence $\mathcal{E}\colon 0 \to M'_R \xrightarrow{f} M_R \xrightarrow{g} M''_R \to 0$ is called *pure* if it satisfies the following equivalent conditions:

1) For all (finitely presented) modules ${}_RN$ the induced sequence $0 \to M' \otimes_R N \xrightarrow{f\otimes 1} M \otimes_R N \xrightarrow{g\otimes 1} M'' \otimes_R N \to 0$ is exact.

2) For all finitely presented modules L_R the sequence $0 \to (L, M') \xrightarrow{(1,g)} (L, M) \xrightarrow{(1,f)} (L, M'') \to 0$ is exact.

3) [8] Suppose that A is a matrix with coefficients in R having m rows and n columns and that $(y_1, \ldots, y_n)$ is a row of elements in $f(M')$. Then the system of equations $(x_1, \ldots, x_m)A = (y_1, \ldots, y_n)$ is solvable in $f(M')$ provided it is solvable in M.

4) The induced sequence of left R-modules $0 \to M''^{+} \xrightarrow{(g,1)} M^{+} \xrightarrow{(f,1)} M'^{+} \to 0$ splits.

5) $\mathcal{E}$ is isomorphic to the direct limit of a direct system of split exact sequences.

Obviously, condition 1) is satisfied if all maps $f \otimes 1_N$ are injective, and 2) is satisfied if all maps $(1_L, f)$ are surjective. Two further definitions: A mono $f\colon M' \to M$ (an epi $g\colon M \to M''$) is called *pure*, if the exact sequence $0 \to M' \xrightarrow{f} M \to \mathrm{Coker} f \to 0$ $(0 \to \mathrm{Ker} g \to M \xrightarrow{g} M'' \to 0)$ is pure-exact, and a submodule

M' of M is called pure, if the inclusion $M' \to M$ is pure.

The equivalence of 1), 2), 3) and 5) is shown for instance in Lam's book [17]. To show the equivalence of 2) and 4) we note that the canonical map $\phi\colon F \otimes_R G^+ \to (F_R, G_R)^+$ where $\phi(x \otimes \alpha)(h) = \alpha(h(x))$ for $x \in F$, $\alpha \in G^+$ and $h \in (F, G)$ is an isomorphism in case G_R is arbitrary and F_R is finitely presented. Using this isomorphism it is easy to see that $\mathcal{E}$ is pure-exact iff the induced sequence $\mathcal{E}^+\colon 0 \to M''^+ \xrightarrow{(g,1)} M^+ \xrightarrow{(f,1)} M'^+ \to 0$ is pure-exact. We shall soon see that M''^+ is pure-injective, hence pure-exactness of $\mathcal{E}^+$ is equivalent with splitting.

Examples 1.2. 1) Split exact sequences are pure.

2) Given a module M_R and an upward directed family $(M_i)_{i \in I}$ of pure submodules (e.g. direct summands) of M, the union $\bigcup_{i \in I} M_i$ is pure in M. In particular: If $(M_i)_{i \in I}$ is a family of modules, then the direct sum $\coprod_{i \in I} M_i$ is a pure submodule of the product $\prod_{i \in I} M_i$. More generally, if $\mathcal{F}$ is a filter on I and if $M_{I \setminus F} = \prod_{i \in I \setminus F} M_i$ for $F \in \mathcal{F}$, then $\bigcup_{F \in \mathcal{F}} M_{I \setminus F}$ is pure in $\prod_{i \in I} M_i$.

3) Let $(I, \leq)$ be an upward directed ordered set, let $\varphi_{ji}\colon M_i \to M_j$, $i \leq j$ in I, be a direct system of modules with limit $M = \varinjlim_{i \in I} M_i$ and let $\varphi_i\colon M_i \to M$, $i \in I$, be the canonical maps. Then the exact sequence $0 \to \mathrm{Ker}\varphi \to \coprod_{i \in I} M_i \xrightarrow{\varphi} M \to 0$ is pure, where φ is the sum maps of the φ_i. In fact every pure-exact sequence is a push-out of an exact sequence of this type, where in addition the M_i are finitely presented [15].

This is seen as follows. Given a pure-exact sequence $\mathcal{E}\colon 0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ it is well-known [7] that there exists a directed set $(I, \leq)$ and a direct system $\varphi_{ji}\colon M_i'' \to M_j''$, $i \leq j$ in I, of finitely presented modules, such that $M'' = \varinjlim_{i \in I} M_i''$; let $\varphi_i\colon M_i'' \to M''$, $i \in I$, be the canonical maps. By condition 2) in 1.1 there exist $\beta_i \in (M_i'', M)$ such that $g\beta_i = \varphi_i$ for all $i \in I$. We obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\varphi \arrow[r, "j"] \arrow[d, "\gamma"'] & \coprod_{i \in I} M_i'' \arrow[r, "\varphi"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

where φ and β are the sum maps of the φ_i and β_i resp. and γ is the kernel map. It follows that f is the push-out of the pure inclusion j.

4) [26] Let ${}_SM_R$ and ${}_SU_T$ be bimodules such that $M \otimes_R X \in \mathrm{Cogen}({}_SU)$ for all finitely presented modules ${}_RX$, let V_T be a module which is $({}_SM, {}_SV)_T$-injective and contains U_T. Then the canonical map $\gamma\colon M_R \to (({}_SM, {}_SU)_T, V_T)_R$ where $\gamma(m)(\varphi) = \varphi(m)$ for $m \in M$, $\varphi \in ({}_SM, {}_SU)$, is a pure monomorphism. In particular, the canonical monomorphism $\gamma_M\colon M \to M^{++} = ((M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ is pure for all modules M_R.

5) Let R be a principal ideal domain. A submodule M' of an R-module M is pure iff $M'r = M' \cap Mr$ for all $r \in R$. In case $R = \mathbb{Z}$ this condition is due to Prüfer and in fact constitutes the origin of the theory of purity. More generally, if R is a Dedekind domain, a submodule M' of an R-module M is pure iff $M'\mathfrak{a} = M\mathfrak{a} \cap M'$ for all ideals $\mathfrak{a}$ of R.

Remarks 1.3. 1) The class of pure monomorphisms is a proper class which is closed under limits of direct systems. (See [18] for the definition of a proper class.).

2) The notion of purity is well-suited to characterize certain well-known classes of modules. We give two examples.

a) A module M'_R is called absolutely pure if it satisfies the following equivalent conditions:

i) Given an exact sequence $0 \to A_R \to B_R \to C_R \to 0$ where A is finitely generated, B is finitely generated and free, the induced sequence $0 \to (C, M') \to (B, M') \to (A, M') \to 0$ is exact.

ii) Every mono $M'_R \to M_R$ is pure.

iii) There exists a pure mono $M' \to M$ into an injective module M.

b) A module M''_R is called flat if it satisfies the following equivalent conditions:

i) Given an exact sequence $0 \to_R A \to_R B \to_R C \to 0$ the induced sequence $0 \to M'' \otimes_R A \to M'' \otimes_R B \to M'' \otimes_R C \to 0$ is exact.

ii) Every epi $M_R \to M''_R$ is pure.

iii) There exists a pure epi $M \to M''$ where M is projective.

3) R is von Neumann regular (i.e. all finitely generated left and right ideals of R are direct summands of ${}_RR$ and R_R resp.) if and only if all exact sequences of R-modules are pure. This in turn is equivalent to the fact that all R-modules are absolutely pure resp. flat.

Next we consider the modules that are injective (projective) with respect to the class of pure monos (pure epis).

Definition 1.4. A module N_R is called pure-injective, p.i. for short, if for every pure-exact sequence $0 \to M'_R \xrightarrow{f} M_R \xrightarrow{g} M''_R \to 0$ the induced sequence $0 \to (M'', N) \xrightarrow{(g,1)} (M, N) \xrightarrow{(f,1)} (M', N) \to 0$ is exact. Dually, a module P_R is called pure-projective, p.p. for short, if for every pure-exact sequence $0 \to M'_R \xrightarrow{f} M_R \xrightarrow{g} M''_R \to 0$ the sequence $0 \to (P, M') \xrightarrow{(1,f)} (P, M) \xrightarrow{(1,g)} (P, M'') \to 0$ is exact.

Obviously, N_R is p.i. iff for every pure mono $f\colon M'_R \to M_R$ the map $(f,1)\colon (M,N) \to (M',N)$ is surjective, and P_R is p.p. iff for every pure epi $g\colon M_R \to M''_R$ the map $(1,g)\colon (P,M) \to (P,M'')$ is surjective. As in the case of ordinary injective and projective modules the following observations are obvious.

Proposition 1.5. *1) N_R is p.i. if and only if every pure mono $N_R \to N'_R$ splits. Given a family of modules $(N_k)_{k\in K}$, the product $\prod_{k\in K} N_k$ is p.i. iff all N_k, $k \in K$, are so.*
2) P_R is p.p. if and only if every pure epi $P'_R \to P_R$ splits.
Given a family $(P_k)_{k\in K}$ of modules, the direct sum $\coprod_{k\in K} P_k$ is p.p. iff all P_k, $k \in K$, are p.p..

Examples 1.6. 1) Every injective module is p.i.. More precisely, a module is injective if and only if it is absolutely pure and p.i..
If R is von Neumann regular, a module is p.i. iff it is injective. (In fact this characterizes v.N. regular rings.) But in general p.i. modules need not be injective. For instance the $\mathbb{Z}$-modules $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{Z}\setminus\{0,1,-1\}$ are p.i. but not injective.

2) Given a bimodule ${}_RA_S$ and a p.i. module B_S, the module $(A_S, B_S)_R$ is p.i.. The proof is easy: Supposing that $\mathcal{E}_R$ is a short pure-exact sequence of right R-modules, we have an isomorphism of complexes $(\mathcal{E}_R, (A_S, B_S)_R) \cong (\mathcal{E} \otimes_R A_S, B_S)$. The right hand complex is exact, because $\mathcal{E} \otimes_R A_S$ is pure-exact and B_S is p.i., hence the left hand complex is exact as well.
We mention some special cases. Let R be an algebra over a commutative ring k and B a p.i. k-module. Then for every module ${}_RA$ the right R-module $({}_kA, {}_kB)_R$ is p.i.. Taking $k = \mathbb{Z}$ we see that for every module ${}_RA$ over an arbitrary ring R the module $A^+_R = ({}_{\mathbb{Z}}A, {}_{\mathbb{Z}}\mathbb{Q}/\mathbb{Z})_R$ is p.i.. In case k is a field the k-dual $(A,k)_R$ of A is p.i., because k is p.i. over itself. If, in addition, the module A is of finite k-dimension, it is isomorphic to its double k-dual and therefore p.i..

3) It follows from condition 2) in 1.1 that every finitely presented module P_R is p.p.. More generally, given a family $(P_k)_{k\in K}$ of finitely presented modules, the direct sum $\coprod_{k\in K} P_k$ and all of its direct summands are p.p..

Using these examples we see that there exist sufficient p.i. and p.p. modules.

Proposition 1.7. *Let M_R be a module.*
1) There exists a pure epi $P \to M$ with a p.p. module P.
2) There exists a pure mono $M \to N$ with a p.i. module N.

The proof is easy. 1) As mentioned there exists a limit presentation $M = \varinjlim_{i\in I} M_i$ by finitely presented modules M_i. Letting $\varphi_i\colon M_i \to M$, $i \in I$, be the canonical maps, the sum map $\varphi = (\varphi_i)\colon \coprod_{i\in I} M_i \to M$ is a pure epi starting in a p.p. module.
2) We know that the canonical map $\gamma_M\colon M \to M^{++}$ is a pure mono and M^{++} is p.i..

Corollary 1.8. *1) M is p.p. if and only if there exists a family $(M_i)_{i\in I}$ of finitely presented modules such that M is isomorphic to a direct summand of $\coprod_{i\in I} M_i$.*
2) M is p.i. if and only if there exists a module ${}_RN$ such that M is isomorphic to a direct summand of N_R^+.

The characterization of p.p. modules in this corollary is quite satisfying, since there are many results concerning direct summands of direct sums of finitely generated modules, e.g. the Crawley-Jønnson-Warfield theorem [1]. Contrary to this the description of p.i. modules is not at all satisfactory. Hence it is desirable to look for "inner" properties of p.i. modules. This is achieved by the socalled "matrix subgroups" to be introduced next.

Lecture 2

Matrix and H-subgroups

Definition 2.1. [11, 31] Let $A = (a_{ij})_{(i,j)\in I\times J}$ be a column-finite matrix with coefficients in R and $\alpha \in I$ (I labels the rows, J the columns of A). For a module M_R we put

$$\begin{aligned} M[A;\alpha] &= \{x = x_\alpha \in M; \exists (x_i)_{i\in I\setminus\{\alpha\}} \in M^{I\setminus\{\alpha\}} : (x_i)_{i\in I} \cdot A = 0\} \\ &= \mathrm{pr}_\alpha\{(x_i)_{i\in I} \in M^I; (x_i)_{i\in I}\cdot A = 0\}, \end{aligned}$$

where $\mathrm{pr}_\alpha \colon M^I \to M$ is the α-th projection. This defines a subfunctor $[A;\alpha]$ of the forgetful functor $V\colon \mathrm{Mod}R \to \mathrm{Mod}\mathbb{Z}$ that commutes with arbitrary direct sums and products. In particular $M[A;\alpha]$ is a submodule of M, considered as a left module over $S = \mathrm{End}_R(M)$. $M[A;\alpha]$ is a submodule of M_R if R is commutative, but not in general. Subgroups of the kind $M[A;\alpha]$ are called matrix subgroups. If I and J are finite, say $I = \{1,\dots,m\}$, $J = \{1,\dots,n\}$, then A is called finite and $M[A;\alpha]$ a finite matrix subgroup. In this case we may assume that $\alpha = 1$ and we will write $M[A]$ for $M[A;1]$. We shall use the notations $\mathcal{M}(M_R)$ resp. $\mathcal{M}_f(M_R)$ for the set of all resp. all finite matrix subgroups of M.

Examples 2.2. 1) Let $\mathfrak{a} = \sum_{i=1}^m Ra_i$ be a finitely generated left ideal of R. Then $M\mathfrak{a} = \sum_{i=1}^n Ma_i$ is finite matrix subgroup, since $M\mathfrak{a} = M[A;\alpha]$ with $A = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ an \end{pmatrix}$, where $a_0 = 1$ and $\alpha = 0$.
2) Let $\mathfrak{b} = \sum_{j\in J} b_j R$ be a right ideal. Then $\mathrm{Ann}_M(\mathfrak{b}) = M[B;\beta]$ where B is the row matrix $(b_{1j})_{j\in J}$ with $b_{1j} = b_j$ for all $j \in J$ and $\beta = 1$. For finite J we have $\mathrm{Ann}_M(\mathfrak{b}) \in \mathcal{M}_f(M_R)$.

In order to get further examples it is useful to give a “basis-free” description of the functors $[A;\alpha]$.

Definition 2.3. [32] Let (X,x) be a pointed module, i.e. a module X_R and a “point” $x \in X$. We let $H_{X,x}(M)$ be the image of the map $\varepsilon_x \colon (X_R, M_R) \to M$, $h \mapsto h(x)$, for short $H_{X,x}(M) = (X, M)(x)$. This defines a subfunctor $H_{X,x} \subset V$ commuting with direct sums and products.

Proposition 2.4. *1) Given a column-finite matrix* $A = (a_{ij})_{(i,j)\in I\times J}$ *over* R *and* $\alpha \in I$ *there exists a pointed module* (X_R, x) *such that* $M[A; \alpha] = H_{X,x}(M)$ *for all modules* M_R*. If* A *is finite, then* X *can be chosen finitely presented.*
2) Given a pointed module (X_R, x)*, there exists a column-finite matrix* $A = (a_{ij})_{(i,j)\in I\times J}$ *over* R *and* $\alpha \in I$ *such that* $H_{X,x}(M) = M[A; \alpha]$ *for all* M_R*. If* X_R *is finitely presented, then* A *can be chosen finite.*

We give a hint of the proof of 1). Let $(f_j)_{j\in J}$ and $(e_i)_{i\in I}$ be the canonical bases of $R^{(J)}$ and $R^{(I)}$ resp., let $g\colon R^{(J)} \to R^{(I)}$ be defined by $g(f_j) = \sum_{i\in I} e_i a_{ij}$ for $j \in J$, X the cokernel of g and x the image of e_α in X. Then (X, x) is as wanted.

Examples and Remarks 2.5. 1) Let $S = \mathrm{End}_R(M)$ and $x = (x_1, \dots, x_n) \in M^n$. Then $\sum_{i=1}^n Sx_i = H_{M^n,x}(M) \in \mathcal{M}(M_R)$.
2) Let $\mathfrak{b}$ be a right ideal of R. Then $\mathrm{Ann}_M(\mathfrak{b}) = H_{R/\mathfrak{b},\overline{1}}(M)$, where $\overline{1} = 1+\mathfrak{b} \in R/\mathfrak{b}$.
3) We have $H_{X,x}(M) = \mathrm{Ann}_M\mathrm{Ann}_R(x)$ if and only if M is injective with respect to the inclusion $xR \hookrightarrow X$. In particular, if M_R is injective, then $H_{X,x}(M) = \mathrm{Ann}_M\mathrm{Ann}_R(x)$ for all (X, x).

Dually, $H_{X,x}(M) = M\cdot H_{X,x}(R_R)$ if and only if the following condition holds: Let $p\colon P \to M$ be an epi with a projective module P. Then for all $h \in (X, M)$ there is $h' \in (X, P)$ such that $h(x) = ph'(x)$. In particular, if M is projective then $H_{X,x}(M) = M \cdot H_{X,x}(R_R)$ for all (X, x).
4) $\mathcal{M}(M_R)$ is closed under finite sums and arbitrary intersections. $\mathcal{M}_f(M_R)$ is closed under finite sums and finite intersections.
5) Let ${}_SM_R$ be a bimodule. If ${}_SM$ is a self-cogenerator (i.e. $M/U \in \mathrm{Cogen}({}_SM)$ for all $U \in \mathcal{L}({}_SM)$) and $R = \mathrm{End}_S(M)$, then $\mathcal{L}({}_SM) = \mathcal{M}(M_R)$, because $U = \mathrm{Ann}_M\mathrm{Ann}_R(U)$ for all $U \in \mathcal{L}({}_SM)$. On the other hand, if M_R is a self-generator (i.e. $W \in \mathrm{Gen}(M_R)$ for all $W \in \mathcal{L}(M_R)$ and $S = \mathrm{End}_R(M)$, then every finitely generated submodule W of M_R is a finite matrix subgroup of ${}_SM$.

Another type of subfunctor of the forgetful functor $V\colon \mathrm{Mod}R \to \mathrm{Mod}\mathbb{Z}$ can be constructed via the tensor functor.

Definition 2.6. Let $({}_RY, y)$ be a pointed module and M_R a module. Then $T_{Y,y}(M)$ is to be the kernel of the map $\tau_y\colon M \to M \otimes_R Y$, $m \mapsto m \otimes y$. This defines a subfunctor $T_{Y,y} \subset V$ commuting with direct limits.

In analogy with Prop 2.4 we have:

Proposition 2.7. *1) Given a finite matrix* A *with coefficients in* R *there exists a finitely presented pointed module* $({}_RY, y)$ *such that* $M[A] = T_{Y,y}(M)$ *for all* M_R*.*
2) If $({}_RY, y)$ *is finitely presented, then there exists a finite matrix* A *such that* $T_{Y,y}(M) = M[A]$ *for all* M*.*

We note that in this latter way finite matrix subgroups have been introduced by Gruson and Jensen under the name "sousgroupe de définition fini" [11]. Due to the right exactness of the tensor product this description of finite matrix subgroups is most suitable when dealing with purity, flatness, Π-flatness and coherence. For instance, it can be used to show the following characterization of purity which generalizes Prüfer's condition (1.2, Example 5): A submodule M' of M_R is pure iff $M'[A] = M' \cap M[A]$ for all finite matrices A. This has been shown, among others, by Rothmaler [24]. Furthermore, we note that for arbitrary (Y, y) the group $T_{Y,y}(M)$ is a directed union of finite matrix subgroups.

The precise connection between the subfunctors $H_{X,x}$ and $T_{Y,y}$ is as follows.

Proposition 2.8. *Given a finitely presented pointed module* (X_R, x) *there exists a finitely presented module* $({}_RY, y)$ *such that* $H_{X,x} = T_{Y,y}$ *and* $T_{X,x} = H_{Y,y}$. *(Similarly one can start with a pair* $({}_RY, y)$ *and construct* (X_R, x)*).*

We hint at the proof. If $A = (a_{ij})_{a\leq i\leq n, 1\leq j\leq m}$ is a finite matrix such that X_R is the cokernel of $\beta\colon R^m_R \to R^n_R$, $(r_1, \ldots, r_m)^t \mapsto A(r_1, \ldots, r_m)^t$, and x is the image of $(1, 0, \ldots, 0)^t \in R^n$ in X, then ${}_RY$ is to be the cokernel of the map $\gamma\colon {}_RR^n \to_R R^{m+1}, (r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_n)\begin{pmatrix} 1 & \\ 0 & \\ \vdots & \mathrm{A} \\ 0 & \end{pmatrix}$, and y the image of $(1, 0, \ldots, 0) \in R^{m+1}$ in Y. Hence Y is very similar to the Auslander-Bridger transpose of X.

The correspondence in the last Proposition is called "elementary duality". It has been intensely studied by Herzog, Prest and Rothmaler among others. There is also an extremely useful correspondence between the finite matrix subgroups of a module and of its dual.

Proposition 2.9. *[22, 30] Let* ${}_SM_R$ *be a bimodule,* ${}_SU$ *a module such that* $M \otimes_R X \in$ *Cogen(*${}_SU$*) for all finitely presented modules* ${}_RX$ *and that* ${}_SU$ *is* ${}_SM$*-injective. Furthermore, let* $M^* =_R({}_SM,_S U)$. *Then the map* $\mathcal{M}_f(M_R) \to \mathcal{M}_f({}_RM^*)$, $H \mapsto Ann_{M^*}(H)$, *is bijective with inverse* $L \mapsto Ann_M(L)$. *More precisely, we have* $Ann_{M^*}(H_{X,x}(M)) = T_{X,x}(M^*)$ *and* $Ann_M(T_{X,x}(M^*)) = H_{X,x}(M)$ *for all finitely presented* (X, x).

An important special case of this situation is $M^* = M^+ = (M, \mathbb{Q}/\mathbb{Z})$.

We shall see in the last lecture that it is useful to further generalize the notion of a matrix subgroup.

Definition 2.10. Let A, M, X be right R-modules and $\varphi \in (A, X)$. Let $H_\varphi(M)$ be the image of the map $(\varphi, 1)\colon (X, M) \to (A, M)$, $f \mapsto f \circ \varphi$, for short $H_\varphi(M) =$

$(X, M) \circ \varphi$. This defines a subfunctor H_φ of the representable functor $(A, -)\colon \mathrm{Mod}R \to \mathrm{Mod}\mathbb{Z}$, which commutes with direct products; in case A is finitely generated, it also commutes with direct sums. The subgroups of the type $H_\varphi(M)$ of (A, M) are called H-subgroups. What is more they are S-submodules of ${}_S(A, M)$ where $S = \mathrm{End}_R(M)$. It is useful to know that the module X in the definition of the $H_\varphi(M)$ can always be choosen in $\mathrm{Cogen}(M)$. This is seen as follows: Let $(\alpha_i)_{i \in I}$ be a generating system of the S-module (X, M) and $\alpha\colon X \to M^I$ the product map of the α_i. It is easy to see that $H_\varphi(M) = H_{\alpha \circ \varphi}(M)$. We shall denote the set of all H-subgroups of (A, M) by $\mathcal{H}(A, M)$.

Examples 2.11. 1) First we show that H-subgroups embody matrix subgroups. Let $A = R_R$, $m \in M_R$ and $\varphi\colon R \to M$, $r \mapsto mr$. Then the image of the map $(X, M) \xrightarrow{(\varphi,1)} (R, M) \cong M$ is $H_{X,x}(M)$. Hence we have $H_\varphi(M) = H_{X,x}(M)$ if we identify (R, M) with M. More generally: Let ${}_TA_R$ be a bimodule (e.g. with $T = \mathrm{End}_R(A)$). It is then easy to see that every matrix subgroup of the T-module $(A_R, M_R)_T$ is an H-subgroup.
2) Every finitely generated submodule of the S-module ${}_S(A, M)$, where $S = \mathrm{End}_R(M)$, is an H-subgroup.
3) Let A' be a submodule of A and $\varphi\colon A \to A/A'$ the canonical map. Then $H_\varphi(M) = \mathrm{Ann}_{(A,M)}(A')$. Every H-subgroup of (A, M) is of this kind if and only if M_R is injective with respect to all monos $U \to W$ where $W \in \mathrm{Cogen}(M)$ and $U \in \mathrm{Fac}(A)$.
4) Let $\varphi\colon A \to I$ be a mono into an injective module I. Then $H_\varphi(M)$ is the set of all $h \in (A, M)$ factorizing through an injective module. H-subgroups of this type play a role in the representation theory of artin algebras.
5) For $U = (X, M) \circ \varphi \in \mathcal{H}(A, M)$ and $a_1, \dots, a_n \in A$ we have $\sum_{i=1}^n U(a_i) = \sum_{i=1}^n H_{X,\varphi(a_i)}(M) \in \mathcal{M}(M_R)$. Specifically: Let $S = \mathrm{End}_R(M)$. If ${}_SM$ contains isomorphic copies of all simple left S-modules, then the Jacobson radical $J(S)$ is an H-subgroup of $S = (M_R, M_R)$, since $J(S) = \mathrm{Ann}_S(\mathrm{Soc}_S M)$. Furthermore, if $V = \sum_{i=1}^n Sx_i$ is a finitely generated submodule of ${}_SM$, then $J(S)V = \sum_{i=1}^n J(S)x_i$ is a matrix subgroup of M_R.
6) $\mathcal{H}(A, M)$ is closed under finite sums and arbitrary intersections.

The next proposition contains two further constructions of H-subgroups.

Proposition 2.12. *Let ${}_TA_R$ be a bimodule, $U \subset (A, M)$ an H-subgroup which is also a T-submodule.*
1) Every matrix subgroup of U_T is an H-subgroup of (A, M).
2) If W is a matrix subgroup of the T-module $(A, M)/U$, then its inverse image in (A, M) is an H-subgroup.

This proposition has the following consequence for matrix subgroups. Suppose that U is a matrix subgroup of M_R and R' is a subring of R such that U is also an R'-submodule of M. Then every matrix subgroup of $U_{R'}$ is a matrix subgroup of M_R and if W is a matrix subgroup of the R'-module M/U, then its inverse image in M is a matrix subgroup of M_R.

Lecture 3

Pure-injective modules

Now we dispose of the technical tools for "inner" characterizations of p.i. modules.

Theorem 3.1. *The following conditions are equivalent for a module M_R.*
1) M is pure-injective.
2) Given a column-finite $I \times J$-matrix A with coefficients in R and a family $(m_j)_{j\in J}$ of elements in M, the system of equations $(x_i)_{i\in I} \cdot A = (m_j)_{j\in J}$ is solvable in M in case it is finitely solvable in M (i.e. every finite subsystem is solvable).
3) The canonical map $\gamma_M \colon M \to M^{++}$ splits.
4) Given a downward directed family of cosets $(m_i + U_i)_{i\in I}$ with $m_i \in M$ and $U_i \in \mathcal{M}(M_R)$ (resp. $U_i \in \mathcal{M}_f(M_R)$) we have $\bigcap_{i\in I}(m_i + U_i) \neq \emptyset$.
5) For all modules A_R and all downward directed families of cosets $(f_i + H_i)_{i\in I}$ with $f_i \in (A, M)$ and $H_i \in \mathcal{H}(A, M)$ we have $\bigcap_{i\in I}(f_i + H_i) \neq \emptyset$.
6) Let $f \in (X_R, Y_R)$, $g \in (X_R, M_R)$ and let $(X_i)_{i\in I}$ be an upward directed family of submodules of X with $X = \bigcup_{i\in I} X_i$. If $g|_{X_i}$ factors through $f|_{X_i}$ for all $i \in I$, then g factors through f.
7) Given a pure mono $f \colon M_R \to N_R$ and $y \in N$ there exists $x \in M$ such that $f(M) \cap (f(x) + y)R = 0$ and the composed map $M \xrightarrow{f} N \xrightarrow{can} N/(f(x) + y)R$ is a pure mono, too.

We note that some further important characterizations of p.i. modules can be found in the book of Jensen and Lenzing [14]. The equivalence of 1), 2), 3) has been proven in Warfield's paper [25]; condition 4) first appeared in the articles [11, 31]; condition 6) in essence is due to Azumaya [6]; condition 7) appeared in [35] and is a kind of Baer criterion; and 5) seems to be new.

We now prove some of the equivalences. We disregard conditions 2) and 6) and show the equivalence of the remaining ones.
1)⇔ 3) This has been shown in Lecture 1.
3)⇒ 5) We assume that $\gamma_M \colon M \to M^{++}$ splits; let d be a left inverse of γ_M, i.e., $d \circ \gamma_M = 1_M$. First we show that there exist homomorphisms $\delta_A \colon (A, M)^{++} \to$

(A, M) that are natural in A, such that the composition of $\gamma = \gamma_{(A,M)}\colon (A, M) \to (A, M)^{++}$ with δ_A is equal to the identity on (A, M). To this aim let $\phi\colon A \otimes_R M^+ \to (A, M)^+$ be the map considered in Lecture 1, namely $\phi(a \otimes \alpha)(h) = \alpha(h(a))$ for $a \in A$, $\alpha \in M^+$ and $h \in (A, M)$. Then δ_A is to be the composition $(A, M)^{++} \xrightarrow{\phi^+} (A \otimes_R M^+)^+ \xrightarrow{adj} (A, M^{++}) \xrightarrow{(1,d)} (A, M)$ where the middle arrow is the Hom-$\otimes$-adjunction. One checks that $\delta_A(F)(a) = d(F \circ \phi \circ \tau_a)$ for $F \in (A, M)^{++}$ and $a \in A$, where $\tau_a\colon M^+ \to A \otimes M^+$, $\alpha \mapsto a \otimes \alpha$. A calculation shows that $\delta_A \circ \gamma$ is the identity on (A, M). In order to show 5) we follow an idea of B. J. Mueller [19]. Let $(f_i + H_i)_{i \in I}$ be a downward directed family as in 5) and put $H_i^0 = \mathrm{Ann}_{(A,M)^+}(H_i)$. It is easy to see that $\Gamma\colon \bigcup_{i \in I} H_i^0 \to \mathbb{Q}/\mathbb{Z}$ where $\Gamma(h) = h(f_i)$ for $h \in H_i^0$ defines a group homomorphism. Letting $\widetilde{\Gamma}$ be an extension of Γ to $(A, M)^+$ we have $\widetilde{\Gamma}\ (h) = \Gamma(h) = h(f_i) = \gamma(f_i)(h)$ for all $h \in H_i^0$, hence $\widetilde{\Gamma}\ -\gamma(f_i) \in H_i^{00} = \mathrm{Ann}_{(A,M)^{++}}(H_i^0)$. It is easy to check that $\delta_A(H_i^{00}) \subset H_i$, hence $\delta_A(\widetilde{\Gamma}) - f_i = \delta_A(\widetilde{\Gamma}\ -\gamma(f_i)) \in H_i$, for all $i \in I$. This shows that $\delta_A(\widetilde{\Gamma}) \in \bigcap_{i \in I}(f_i + H_i)$.

5)⇒ 4) This is trivial.

4)⇒ 7) We assume that 4) holds for finite matrix subgroups. Let M be a pure submodule of N, $Z = N/M$ and let $\pi\colon N \to Z$ be the canonical epi. Furthermore, let $y \in N$ and $z = \pi(y)$. It is not difficult to see that there exists a direct system $\varphi_{ji}\colon Z_i \to Z_j$, $i \le j$ in I, of finitely presented modules with points $z_i \in Z_i$ satisfying $\varphi_{ji}(z_i) = z_j$ for all $i \le j$, such that $Z = \varinjlim Z_i$ and $\varphi_i(z_i) = z$ for all $i \in I$, where $\varphi_i\colon Z_i \to Z$, $i \in I$, denote the canonical maps. Due to purity of π there exists $\psi_i \in (Z_i, M)$ such that $\pi\psi_i = \varphi_i$, $i \in I$. It follows that $\pi(\psi_i(z_i) - y) = \varphi_i(z_i) - z = 0$, hence $\psi_i(z_i) - y \in M$ for all $i \in I$. If $i \le j$ in I, we have $(\psi_i(z_i) - y) - (\psi_j(z_j) - y) = (\psi_i - \psi_j\varphi_{ji})(z_i) \in M \cap H_{Z_i,z_i}(N) = H_{Z_i,z_i}(M)$, hence $\psi_i(z_i) - y + H_{Z_i,z_i}(M) \subset \psi_j(z_j) - y + H_{Z_j,z_j}(M)$. By assumption we have $\bigcap_{i \in I}(\psi_i(z_i) - y + H_{Z_i,z_i}(M)) \neq \emptyset$. Letting x be in this intersection we obtain $x + y \in \psi_i(z_i) + H_{Z_i,z_i}(M)$, hence there exists $\lambda_i \in (Z_i, M)$ such that $x + y = (\psi_i + \lambda_i)(z_i)$, for all $i \in I$. Thus, replacing ψ_i by $\psi_i + \lambda_i$, we can assume from the beginning that $\psi_i(z_i) = x + y$ for some $x \in M$ and all $i \in I$. It is mere routine now to verify that $x + y$ satisfies 7).

7) ⇒ 1) Let $f\colon M \to N$ be a pure mono. By Zorn's lemma there exists a submodule L of N that is maximal such that $f(M) \cap L = 0$ and the composed map $M \xrightarrow{f} N \xrightarrow{can} N/L$ is a pure mono. Now it is an easy consequence of 7) that $can \circ f$ is an isomorphism. It follows that $N = f(M) \oplus L$.

We add some consequences of Theorem 3.1.

A) Transfer of pure-injectivity.

The following Corollary makes use of 2.12.

Corollary 3.2. *Let $_TA_R$ be a bimodule, M_R a module and U an H-subgroup of (A,M) that is also a T-submodule. If M is p.i., then U and $(A,M)/U$ are p.i. as T-modules.*

In particular: If M_R is p.i., $U \in \mathcal{M}(M_R)$, R' a subring of R such that U is an R'-submodule of M, then U and M/U are p.i. as R'-modules.

Theorem 3.3. *Let A, M be right R-modules and $T = End_R(A)$. If M is p.i., then the T-modules $Ext^n_R(A,M)$, $n \geq 0$, are p.i..*

This result seems to be new. It generalizes a result of Warfield [25], saying that the R-modules $\operatorname{Ext}^n_R(A,M)$, $n \geq 0$, are p.i. in case R is commutative and M is p.i..

B) Linear compactness implies pure-injectivity; AB5* for H-subgroups.

Corollary 3.4. *Let $_SM_R$ be a bimodule. If $_SM$ is linearly compact, then M_R is p.i..*

This result first appeared in [31]. It was rediscovered by Ringel at the end of the nineties. A special case can be found in Warfield's paper [25]. He showed that a linearly compact module over a commutative ring is p.i.. We add a proof of the corollary: Let $(m_i + U_i)_{i \in I}$ be a downward directed family of residue classes of matrix subgroups. Since the U_i are S-submodules we have $\bigcap_{i \in I}(m_i + U_i) \neq \emptyset$. By condition 4) of Theorem 3.1 M is p.i.. A different proof has been given by Onodera [20].

Corollary 3.5. *Given an arbitrary module A_R and an p.i. module M_R, the set $\mathcal{H}(A,M)$ satisfies the AB5* condition. In particular the set of matrix subgroups of M_R satisfies AB5*.*

Recall that a set $\mathcal{G}$ of subgroups of an abelian group G satisfies the AB5* condition if the equality $\bigcap_{i \in I}(G_i + H) = \bigcap_{i \in I} G_i + H$ holds for every downward directed family $(G_i)_{i \in I}$ of subgroups in $\mathcal{G}$ and every $H \in \mathcal{G}$. The corollary is derived from conditions 4) and 5) of Theorem 3.1 just as AB5* for all submodules is shown for a linearly compact module.

In the last lecture we shall deal with the question under which additional conditions the converse holds in these corollaries.

C) Pure-injective hulls.

Here we present a new module-theoretic proof of the existence of pure-injective hulls. We start with a proposition which generalizes a result due to Krause and Saorin [16].

Proposition 3.6. *Let M_R be a module with the exchange property such that the set of all H-subgroups of $S = End_R(M)$ satisfies AB5*. Given a module A and*

$f \in (A, M)$ there exists a direct summand M' of M containing Imf such that the map $f\colon A \to M'$ is left minimal, i.e., every $t \in End_R(M')$ with $f = tf$ is invertible.

We note that the assumptions on M are valid if M is p.i. (by [29] and B).

Proof. We consider the H-subgroup $U = \mathrm{Ann}_S(\mathrm{Im} f)$ of S. As $\mathcal{H}(M_R, M_R)$ has AB5* there exists an H-subgroup V of S that is minimal with respect to $U + V = S$. Since cyclic left ideals of S are H-subgroups, V is even minimal among all left ideals W of S with $V + W = S$, hence $U \cap V \subset J(S)$. Because M has the exchange property there exists an idempotent $e \in S$ such that $S(1-e) \subset U$ and $Se \subset V$ (see [29]). It follows that $V = Se$ and $(1-e)f(M) = 0$, in particular $f(M) \subset M' = eM$. Let $t \in \mathrm{End}_R(M')$ with $f = tf$ and let $\tilde{t}$ be the trivial extension of t to M. Then $t = e\tilde{t}e|_{M'}$, $e - e\tilde{t}e \in U \cap Se$, hence $e - e\tilde{t}e \in eJ(S)e$ and t is invertible.

As for every module A there exists a pure mono $A \to N$ into a p.i. module N, the following is a consequence of this proposition.

Theorem 3.7. *For every module A there exists a p.i. hull $f\colon A \to M$, i.e., a pure and left minimal mono f into a p.i. module M.*

D) Chain conditions for finite matrix subgroups.

Theorem 3.8. *The following are equivalent for a module M_R.*
1) The direct sum $M^{(I)}$ is p.i. for all sets I.
1') $M^{(\mathbb{N})}$ is p.i.
2) The inclusion $M^{(I)} \subset M^I$ splits for all sets I.
2') The inclusion $M^{(\mathbb{N})} \subset M^{\mathbb{N}}$ splits.
3) $\mathcal{M}_f(M_R)$ (resp. $\mathcal{M}(M_R)$) satisfies the minimum condition.
4) For every set I the product M^I is a direct sum of modules with local endomorphism rings.

We note that in 3) we even have $\mathcal{M}_f(M_R) = \mathcal{M}(M_R)$. The equivalence of 1)-3) has been shown in [12, 31]; condition 4) is due to [27].

Definition 3.9. A module M_R is called Σ-pure-injective if conditions 1)-4) hold.

Examples and Remarks 3.10. 1) Let ${}_SM_R$ be a bimodule. If ${}_SM$ is artinian, then M_R is Σ-p.i..

In particular: Artinian modules over a commutative ring are Σ-p.i.. Every finitely generated module over an artin algebra is Σ-p.i..
2) [28] Let k be a field, $R = k[X_i]_{i \in I}/(X_i, i \in I)^n$ where $n \geq 1$. Then R is Σ-p.i..
3) [31] Let R be a Dedekind domain. A module M_R is Σ-p.i. if and only if $M/D(M)$ is restricted ($D(M)$ denotes the divisible part of M).
4) [10] Let R be serial (i.e R_R and ${}_RR$ are direct sums of uniserial right and left ideals resp.). Then M_R is Σ-p.i. iff M is artinian over $\mathrm{End}_R(M)$.

5) [12, 31] Let M_R be Σ-p.i.. Then $M^{(I)}$, M^I are Σ-p.i. for all sets I. Furthermore, every pure mono $M' \to M$ splits and M' is Σ-p.i..

The next result is an immediate consequence of Prop. 2.9.

Proposition 3.11. *Let $_SM_R$ be a bimodule and $_SU$ an injective cogenerator. Then $\mathcal{M}_f(M_R)$ satisfies the minimum (maximum) condition if and only if $\mathcal{M}_f({}_R({}_SM, {}_SU))$ satisfies the maximum (minimum) condition.*

We conclude this lecture with some results on modules with maximum conditions for finite matrix subgroups. The first is a reinterpretation of the Krull-Akizuki theorem, the second a generalization of the Hilbert basis theorem.

Proposition 3.12. *Let R be a noetherian integral domain of Krull dimension 1. If M is a torsion-free R-module of finite rank then $\mathcal{M}_f(M_R)$ satisfies the maximum condition.*

Proposition 3.13. *[34] Let M_R be a module over an arbitrary ring R. If $\mathcal{M}_f(M_R)$ has maximum condition, then the polynomial module $M[X]_{R[X]}$ has so, too.*

This in turn yields a means to construct Σ-p.i. modules.

Corollary 3.14. *[34] Let $R[X_1, \dots, X_n] \to W$ be a ring homomorphism such that W is finitely generated as a right $R[X_1, \dots, X_n]$-module. If M_R is Σ-p.i., then $(W_R, M_R)_W$ is Σ-p.i. as well.*

Lecture 4

Pure-injectivity, linear compactness, duality

This part is focused on the problem of specifying assumptions under which pure-injectivity of a module on one side implies linear compactness on the opposite side (compare 3.4 and 3.5). We begin with recalling some known results.

Examples and Remarks 4.1.
1) [31] Let M_R be p.i. and $S = \mathrm{End}_R(M)$. If ${}_SM$ is noetherian or uniserial, then ${}_SM$ is linearly compact.

The proofs are easy. Let $(m_i + U_i)_{i\in I}$ be a downward directed family of cosets with $m_i \in M$ and $U_i \in \mathcal{L}({}_SM)$. If ${}_SM$ is noetherian, then the U_i are finitely generated, hence matrix subgroups and we have $\bigcap_{i\in I}(m_i + U_i) \neq \emptyset$ by Theorem 3.1. Now let ${}_SM$ be uniserial. In this case let $\overline{U}_i$ be the intersection of all cyclic submodules of ${}_SM$ comprising U_i; then $\overline{U}_i \in \mathcal{M}(M_R)$. As the family $(m_i + \overline{U}_i)_{i\in I}$ is also upward directed, we have $\bigcap_{i\in I}(m_i + \overline{U}_i) \neq \emptyset$. Obviously we may assume that the family $(m_i + U_i)_{i\in I}$ has no smallest member. In this case we can show as follows that $\bigcap_{i\in I}(m_i + U_i) = \bigcap_{i\in I}(m_i + \overline{U}_i)$. Letting $i \in I$ there is $j \in I$ such that $m_j + U_j \underset{\neq}{\subsetneq} m_i + U_i$. Then $U_j \underset{\neq}{\subsetneq} U_i$, hence $U_j \subset \overline{U}_j \subset U_i$ and $m_j + \overline{U}_j \subset m_i + U_i$. This implies the asserted equality.
2) [23] Let R be uniserial. If M_R is indecomposable and p.i., then M is linearly compact as an $\mathrm{End}_R(M)$-module.
3) Let ${}_SM_R$ be a bimodule with $R = \mathrm{End}_S(M)$ and let ${}_SM$ be a selfcogenerator. If M_R is p.i., then ${}_SM$ is linearly compact. This follows from the fact that $U = \mathrm{Ann}_M\mathrm{Ann}_R(U) \in \mathcal{M}(M_R)$ for all $U \in \mathcal{L}({}_SM)$. A special case is due to B. J. Mueller [19] : Let ${}_SM$ be a cogenerator and $R = \mathrm{End}_S(M)$. Then M_R is injective iff ${}_SM$ is linearly compact. To see this, note that M_R is absolutely pure because ${}_SM$ is a cogenerator [26]. Hence M_R is injective iff M_R is p.i..

Now we present generalizations of results of Ánh, Herbera and Menini. We will show that under certain assumptions concerning the socle of a module pure-injectivity implies stronger conditions like linear compactness or injectivity. Our procedure closely follows the articles [2, 3] of Ánh and the subsequent articles [4, 5] of Ánh, Herbera and Menini in which Ánh's ingenious results have further been extended. As we want to formulate the results in a rather general frame we have to introduce some notation and a number of definitions.

Let A, M be right R-modules, $S = \mathrm{End}_R(M)$ and $\mathcal{H} = \mathcal{H}(A, M)$. For subsets $B \subset A$ and $U \subset (A, M)$ we let $B^0 = \mathrm{Ann}_{(A,M)}(B)$ and $U^0 = \mathrm{Ann}_A(U)$ resp.; note that $B^0 \in \mathcal{H}$. If $\mathcal{T}$ is a set of subsets of A or (A, M) we let $\mathcal{T}^0 = \{T^0 : T \in \mathcal{T}\}$. For $a \in A$ let $\mathcal{H}(a) = \{H(a) : H \in \mathcal{H}\}$; this is a subset of $\mathcal{M}(M_R)$. We shall say that $\mathcal{H}(a)$ has the intersection property, I.P. for short, if for every downward directed family $(H_i)_{i\in I}$ in $\mathcal{H}$ with $\bigcap_{i\in I} H_i(a) = 0$ there is $k \in I$ such that $H_k(a) = 0$. To give examples this happens if ${}_SM$ is finitely cogenerated (i.e. $\mathrm{Soc}_S M$ is finitely generated and essential in ${}_SM$) or if M_R is Σ-p.i.. We will denote the set of all finitely generated submodules of ${}_S(A, M)$ by $\mathcal{F}$; this is a subset of $\mathcal{H}$. Furthermore, given $\varphi \in (A, M)$ let $\varphi(\mathcal{F}^0) = \{\varphi(F^0) : F \in \mathcal{F}\}$. Like for $\mathcal{H}(a)$ we define the intersection property (I.P.) for $\varphi(\mathcal{F}^0)$. This holds, for instance, if $\mathrm{Im}\varphi$ is finitely cogenerated and this, in turn, if M_R is finitely cogenerated or if $\mathrm{Soc}M_R$ is essential in M and A is linearly compact. Finally we denote the set of simple composition factors of A by $\mathcal{S}(A)$ and we call ${}_SM_R$ $\mathcal{S}(A)$-dualizing if for every $D \in \mathcal{S}(A)$ such that $(D, M) \neq 0$ the S-module (D, M) is simple and the canonical map $D \to ((D, M_R),{}_S M)$ is an isomorphism.

Theorem 4.2. *We assume that*
a) $\mathcal{H}$ satisfies AB5 and $\mathcal{H}(a)$ has I.P. for all $a \in A$.*
a') $\mathcal{F}^0$ satisfies AB5 and $\varphi(\mathcal{F}^0)$ has I.P. for all $\varphi \in (A, M)$.*
b) ${}_SM_R$ is $\mathcal{S}(A)$-dualizing.
c) For all $U \in \mathcal{F}$ and $\varphi \in (A, M)$ with $U^0 \subsetneqq \varphi^0$ we have Soc$(\varphi^0/U^0) \neq 0$. Then $U = U^{00}$ for all submodules U of ${}_S(A, M)$.

This, in particular, shows that all submodules of ${}_S(A, M)$ are H-subgroups. Hence, if M_R is p.i., then ${}_S(A, M)$ is linearly compact.

Theorem 4.3. *We assume that*
a) $\mathcal{H}$ satisfies AB5 and $\mathcal{H}(a)$ has I.P. for all $a \in A$.*
b) ${}_SM_R$ is $\mathcal{S}(A)$-dualizing.
c') For all $U \in \mathcal{H}$ and $\varphi \in (A, M)$ with $U^0 \subsetneqq \varphi^0$ we have Soc$(\varphi^0/U^0) \neq 0$.
Then

1) $U = U^{00}$ for all $U \in \mathcal{H}$.

2) $A/(A, M)^0$ is semiartinian and for all submodules C of A containing $(A, M)^0$ we have $C = C^{00}$. (Note that $(A, M)^0$ is the intersection of the kernels of all $h \in (A, M)$.)

The next corollary is shown by iterated application of Theorem 4.2. The last two corollaries are special cases; they extend the original results [2, 3] of Ánh.

Corollary 4.4. *Let* M_R *be a module,* $S = End_R(M)$ *and* $\overline{R} = End_S(M)$*. We assume that*
a) M_R *is p.i. and* ${}_SM$ *is finitely cogenerated.*
*a') * ${}_SM$ *is p.i. and* M_R *is finitely cogenerated.*
b) ${}_SM_R$ *is* $\mathcal{S}(M_R)$*-dualizing.*
Then the following hold:
1) For all left ideals L *of* S *we have* $L = Ann_S Ann_M(L)$*;* ${}_SS$ *is linearly compact;* ${}_SM$ *is an injective cogenerator;* $M_{\overline{R}}$ *is linearly compact, absolutely pure and it contains an isomorphic copy of each simple right* $\overline{R}$*-module;* $Soc(M_R) = Soc(M_{\overline{R}}) = Soc({}_SM)$*.*
2) If moreover $R \cong \overline{R}$*, then in addition to 1) we have* $N = Ann_R Ann_M(N)$ *for all right ideals* N *of* R*,* R_R *and* ${}_SM$ *are linearly compact and* M_R *is an injective cogenerator.*

Corollary 4.5. *We assume that* R *is finitely cogenerated on either side and that the* R*-duals of all simple left and right ideals are simple. If* R *is p.i. on either side then* R *is an injective cogenerator, hence also linearly compact on either side.*

Corollary 4.6. *Let* R *be a commutative p.i. ring such that* $SocR$ *is essential and a finite direct sum of non-isomorphic simple ideals. Then* R *is an injective cogenerator and linearly compact.*

To conclude we want to add that proofs of Theorems 4.2 and 4.3 as well as a detailed study of H-subgroups will appear in separate articles.

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