

WEAK-TYPE ESTIMATES FOR HAAR SHIFT OPERATORS: SHARP POWER ON THE A_p CHARACTERISTIC

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ABSTRACT. As a corollary to our main result we will deduce sharp A_p inequalities for T being either the Hilbert transform in dimension $d = 1$, the Beurling transform in dimension $d = 2$, or a Riesz transform in any dimension $d \geq 2$. Let T_* denote the maximal truncations of these operators. We prove weighted weak-type $L^p(w)$ inequalities.

$$\|T_* f\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p} \|f\|_{L^p(w)}, \quad 1 < p < 2,$$

$$\|T_* f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{1 \vee (p-1)^{-1}} \|f\|_{L^p(w)}, \quad 1 < p < \infty.$$

These estimates are sharp in the power of the A_p characteristic of the weight w , and match the best possible bounds without the truncations. They hold for certain kinds of paraproducts as well. Critical to this argument are these elements (1) extrapolation, (2) a recent argument on the A_2 bound [12] (3) a certain weak-type L^1 inequality for maximal truncations (4) and a recent characterization of two-weight inequalities for maximal truncations of singular integrals [13].

1. INTRODUCTION

We are interested in weighted estimates for singular integral operators, and cognate operators, with a focus on sharp estimates in terms of the A_p characteristic of a weight w . In particular, we will prove the result below for maximal truncations of singular integrals.

1.1. Theorem. *For T being either the Hilbert transform in dimension $d = 1$, the Beurling transform in dimension $d = 2$, or a Riesz transform in any dimension $d \geq 2$, we have the estimate*

$$(1.2) \quad \|T_* f\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p} \|f\|_{L^p(w)}, \quad 1 < p < 2,$$

$$(1.3) \quad \|T_* f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{1 \vee (p-1)^{-1}} \|f\|_{L^p(w)}, \quad 1 < p < \infty.$$

Here, we are using the A_p characteristic of the weight, defined by

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1.4. **Definition.** For w a positive function (a weight) on \mathbb{R}^d we define the A_p characteristic of w to be

$$\|w\|_{A_p} := \sup_Q |Q|^{-1} \int_Q w \, dx \cdot \left[|Q|^{-1} \int_Q w^{-1/(p-1)} \, dx \right]^{p-1}, \quad 1 < p < \infty,$$

where the supremum is over all cubes in \mathbb{R}^d . For $p = 1$, we interpret this condition as

$$\|w\|_{A_1} := \sup_x \frac{Mw(x)}{w(x)}$$

where M is the maximal function.

By T_* , we mean the following. Write $Tf(x) = \text{p.v.} \int K(y)f(x-y) \, dy$ for kernel K on \mathbb{R}^d . Then,

$$T_*f(x) := \sup_\epsilon \left| \int_{|y|>\epsilon} K(y)f(x-y) \, dy \right|.$$

The weak-type inequalities (1.2) are new, even for untruncated T . They are also sharp in the power of the A_p characteristic, even for the untruncated operator. Perhaps it is surprising that the maximal truncations are no worse in $L^p(w)$ norm: The impact of taking the maximal truncations is felt through the constants that are only a function of p , and not the weight. Andrei Lerner has a result on the strong-type norm for maximal truncations [14, Corollary 1.4], of a general class of Calderón-Zygmund operators.

The restriction to a special class of singular integrals comes from the method of proof. We study a special class of Calderón-Zygmund operators first identified by S. Petermichl [23], the so-call *Haar shifts*. The Hilbert, Riesz and Beurling operators are appropriate averages of Haar shifts. Indeed, the prior works [2, 9, 12, 24, 25] on the strong type inequalities have all used Haar shifts. They have also used deep two-weight inequalities, which we use as well.

This definition of a Haar shift operator is more general than some.

1.5. **Definition.** We say that T is a *Haar shift operator of index τ* iff

$$Tf = \sum_{Q \in \mathcal{Q}} \langle f, g_Q \rangle \gamma_Q,$$

where the functions γ_Q satisfy these conditions:

- (1) γ_Q is supported on Q ,
- (2) γ_Q is constant on dyadic subcubes $Q' \subset Q$ with $|Q'| \leq 2^{-\tau d}|Q|$,
- (3) $\|\gamma_Q\|_\infty \leq |Q|^{-1/2}$,
- (4) T extends to a bounded operator on L^2 : $\|Tf\|_2 \lesssim \|f\|_2$.

We assume that g_Q satisfy the first four conditions above. Further define

$$(1.6) \quad \begin{aligned} T_*f &:= \sup_{\epsilon > 0} |T_\epsilon f|, \\ T_\epsilon f &:= \sum_{\substack{Q \in \mathcal{Q} \\ |Q|^{1/d} \leq \epsilon}} \langle f, g_Q \rangle \gamma_Q. \end{aligned}$$

The point of the conditions in the definition, especially point (5) above, is that T be not only an $L^2(dx)$ bounded operator, but that it also be a Calderón-Zygmund operator. In particular, it will admit a weak- $L^1(dx)$ bound that depends only on the index τ and the norm of T on L^2 . See Proposition 3.11.

Note that the condition is asymmetric with respect to g_Q and γ_Q . The point here is that, in dimension $d = 1$, this definition is general enough to encompass *both* of these operators:

$$\begin{aligned} f &\mapsto \sum_{I \in \mathcal{Q}} \epsilon_I \langle f, h_I \rangle h_I, & |\epsilon_I| &\leq 1, \\ f &\mapsto \sum_{I \in \mathcal{Q}} \langle f, h_I \rangle h_{I_{\text{left}}}, \\ f &\mapsto \sum_{I \in \mathcal{Q}} \frac{\langle b, h_I \rangle}{\sqrt{|I|}} \langle f, h_I \rangle \mathbf{1}_I, & b &\in BMO. \end{aligned}$$

Here, h_I denotes the usual L^2 -normalized Haar function supported on I , and in the middle line I_{left} denotes the left-half of the dyadic interval I . The first line is a Haar multiplier, the second is a Haar shift, from which the Hilbert transform can be recovered [23], and the third is a paraproduct operator with symbol $b \in BMO$. Paraproducts are the operators considered by Beznosova [2]. In particular, our main result will extend the results of that paper to higher dimensions, weak- L^p estimates, and maximal truncations.

This is our main result:

1.7. Theorem. *The inequalities (1.2) and (1.3) hold for T be a Haar shift operator of index τ . The implied constant depends only dimension d , the index τ of the operator, and the norm $\|T\|_{2 \rightarrow 2}$.*

Essential to the proof of this result are four elements.

- (1) Certain extrapolation arguments allow one to restrict attention to special values of p in (1.2). For the strong type inequalities, only the case of $p = 2$ need be considered. For the weak-type inequality, it suffices to consider the case of p such that p' is an integer. Both points are used in the proof.
- (2) Characterizations of two-weight inequalities for T_* proved in [13], are essential. These results, and some antecedents, are recalled in § 2. Particularly important is the point that it suffices to check certain *testing conditions* which are far less complex than the full norm inequalities. (Indeed, all the inequalities sharp in the A_p characteristic for singular integrals have been proved using two-weight inequalities, see [2, 12, 24–26].)
- (3) Corona decompositions which have recently been used in a new proof of the strong-type inequalities for *untruncated* inequalities, see [12], and Definition 4.1 below.

- (4) Weak- L^1 estimates for the operators, and their duals, are essential. For the operator T_* , this is a well-known consequence of Definition 1.5. We also need for the ‘dual’ of T_* , which is explained in detail in § 3. The estimate is Theorem 3.5, and is not a typical estimate. We give a proof, which uses a Lemma from C. Fefferman’s proof of Carleson’s Theorem [10], also see [18, Section 10].

That Theorem 1.7 implies Theorem 1.1 follows from the fact that appropriate averages of Haar shifts give the Hilbert transform [23], the Beurling transform [9], and the Riesz transforms [22].

We note that there is a conjecture here concerning the $p = 1$ end point estimate. One formulation of this conjecture is as follows. See [15, 16] the best current information. Unfortunately, our methods will not shed much light on this conjecture.

1.8. Weak Muckenhoupt Wheeden Conjecture. *We have the following weak-type inequality for a Calderón-Zygmund operator T and non-negative weights w :*

$$\|Tf\|_{L^{1,\infty}(w)} \lesssim \|w\|_{A_1} \|f\|_{L^1(w)}.$$

The paper of Petermichl [24] has a detailed discussion of the history of such estimates, and the introduction of [12] points to several references since. Concerning weak-type estimates, we are only aware of the results of Lerner [14], which proves results for a more general singular integral type kernels, but does not achieve the sharp exponent in the A_p characteristic.

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2. SHARP RESULTS, TWO WEIGHT INEQUALITIES

Our proofs depend upon some results about two-weight inequalities. We recall them here. Let us state a well-known fact as this proposition, in which a one-weight inequality is turned into a two-weight inequality.

2.1. Proposition. *Let T be a sublinear operator acting on functions in \mathbb{R}^d . Let w be a weight, and $1 < p < \infty$. For any $N > 0$, the inequalities below are equivalent:*

$$\begin{aligned} \|Tf\|_{L^p(w)} &\leq N \|f\|_{L^p(w)}, \\ \|Tf\sigma\|_{L^p(w)} &\leq N \|f\|_{L^p(\sigma)}, \quad \sigma = w^{1-p'}. \end{aligned}$$

The same equivalence holds for the weak-type inequality.

Define the maximal function by

$$Mf(x) = \sup_{t>0} (2t)^{-1} \int_{-t}^t |f(x-y)| dy.$$

Buckley [3] has shown

2.2. Theorem. *We have the inequalities*

$$(2.3) \quad \|Mf\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p}^{1/p} \|f\|_{L^p(w)},$$

$$(2.4) \quad \|Mf\|_{L^p(w)} \lesssim \|w\|_{A_p}^{1/(p-1)} \|f\|_{L^p(w)}.$$

Let us state the characterization of the weak inequality for the maximal function given by Sawyer [27].

2.5. Weak Type Maximal Function Inequalities.. *Let $1 \leq p < \infty$. These two conditions are equivalent:*

$$(2.6) \quad \begin{aligned} & \|M(f\sigma)\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(\sigma)} \\ & \|w, \sigma\|_{A_p} := \sup_{Q \in \mathcal{Q}} \left[\frac{w(Q)}{|Q|} \right]^{1/p} \left[\frac{\sigma(Q)}{|Q|} \right]^{1/p'} < \infty. \end{aligned}$$

In the last definition, we have the dual index p' appearing.

2.7. Strong Type Maximal Function Inequalities.. *Let $1 < p < \infty$. These two conditions are equivalent:*

$$\|M(f\sigma)\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

$$\|M(\mathbf{1}_Q\sigma)\|_{L^p(w)} \lesssim \sigma(Q)^{1/p}.$$

More exactly, we have

$$(2.8) \quad \begin{aligned} & \|M(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(w)} \simeq \|\sigma, w\|_{M,p} \\ & \|\sigma, w\|_{M,p} := \sup_Q \sigma(Q)^{-1/p} \|\mathbf{1}_Q M(\mathbf{1}_Q\sigma)\|_{L^p(w)} \end{aligned}$$

We state a special case of the results from [13], providing a sufficient condition for a two-weight weak-type inequality for T_* .

2.9. Weak Type Inequalities for T_* .. *Let T and T_* be as in Definition 1.5. We have the inequality*

$$(2.10) \quad \begin{aligned} & \|T_*(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^{p,\infty}(w)} \lesssim \|w, \sigma\|_{A_p} + \|\sigma, w\|_{T_*,p}, \\ & \|\sigma, w\|_{T_*,p} := \sup_Q \sup_{\|f\|_{L^p(\sigma)} \leq 1} \int_Q T_*(f\sigma\mathbf{1}_Q) w(dx). \end{aligned}$$

Here, $\|w, \sigma\|_{A_p}$ is defined in (2.6).

There is also a (much harder) version of this result for the strong type inequality.

2.11. Strong Type Inequalities for T_* . *Let T and T_* be as in Definition 1.5. We have the inequality*

$$(2.12) \quad \|T_*(\cdot\sigma)\|_{L^p(\sigma)\rightarrow L^p(w)} \lesssim \|\sigma, w\|_{M,p} + \|w, \sigma\|_{M,p'} + \|\sigma, w\|_{T_*,p} + \|\sigma, w\|_{T,p}$$

$$(2.13) \quad \|\sigma, w\|_{T,p} := \sup_Q \sigma(Q)^{-1/p} \sup_{\|f\|_{L^\infty(\sigma)} \leq 1} \|\mathbf{1}_Q T_*(\mathbf{1}_Q f \sigma)\|_{L^p(w)}$$

On the right in (2.12), the first two terms characterize the boundedness of the maximal function and the dual inequality, see (2.8); the third term characterizes the weak-type inequality; and the fourth condition is dual to the third.

We have stated both of these theorems as sufficient conditions for the two-weight inequalities. Different forms of characterizations of these inequalities can be given, for which we refer the reader to [13].

It is interesting to note that the prior work [12] used a result parallel to Theorem 2.11, namely the two-weight $T1$ theorem of Nazarov-Treil-Volberg [21]. Our techniques, as were many of the papers we cite in this work, draw inspiration from the line of investigation opened up by the work of Nazarov-Treil-Volberg [19].

A simple remark about the A_p condition is this. With $w \in A_p$ and $\sigma = w^{1-p'}$, we have $\sigma \in A_{p'}$, and

$$(2.14) \quad \|\sigma\|_{A_{p'}} = \|w\|_{A_p}^{p'-1}.$$

3. LINEARIZATIONS AND A WEAK L^1 INEQUALITY

We use the method of linearizing maximal operators, a familiar method in the context of the maximal function. We would like, at different points, to treat T_* as a linear operator. While it is not a linear operator, T_* is a pointwise supremum of the linear truncation operators T_ϵ , and as such, the supremum can be linearized with measurable selection of the truncation parameter ϵ .

3.1. Definition. We say that L is a linearization of T_* if there are measurable functions $\epsilon(x) : \Lambda \rightarrow (0, \infty)$ such that

$$(3.2) \quad Lf(x) = \mathbf{1}_\Lambda(x) T_{\epsilon(x)} f(x), \quad x \in \mathbb{R}^d.$$

In this definition, we are using notation from (1.6), and we are specifically permitting $\epsilon(x)$ to be defined on a subset $\Lambda \subset \mathbb{R}^d$. For fixed f we can always choose a linearization L so that $\mathbf{1}_\Lambda(x) T_* f(x) \leq 2|L f(x)|$ for all x .

A key advantage of L is that it is a linear operator, as opposed to a sublinear one. As a linear operator, it has an adjoint, with the adjoint given by the formal expression

$$(3.3) \quad L^* \phi(y) = \sum_{Q \in \mathcal{Q}} \langle \phi \mathbf{1}_\Lambda \mathbf{1}_{\{|Q|^{1/d} \leq \epsilon(x)\}}, \gamma_Q \rangle g_Q(y).$$

The third testing condition in (2.10) has a more convincing formulation in the linearizations. It is equivalent to

$$(3.4) \quad \|\mathbf{1}_Q L^*(\mathbf{1}_Q g w)\|_{L^{p'}(\sigma)} \leq \|\sigma, w\|_{T_{*,p} w(Q)}^{1/p'}, \quad \|g\|_\infty \leq 1.$$

And, this holds uniformly over all choices of linearizations, and bounded functions g . This is form of the testing condition that we will verify.

A central tool here that we shall need is the following weak-type L^1 inequality for L^* . We are only aware of this result being mentioned as a parenthetical remark in [18, Section 10]. Accordingly, we will give a proof of it here.

3.5. Theorem. *We have the following uniform estimate over all linearizations*

$$\|L^* f\|_{1,\infty} \lesssim \|f\|_1.$$

3.1. Proof of Theorem 3.5.

3.1.1. Initial Considerations. The Tree Lemma. The obvious approach to prove this weak-type inequality is by the method of Calderón-Zygmund Decomposition. But, this method simply will not work in the current setting. (The form of the adjoint in (3.3) will not preserve the mean zero properties of the ‘bad’ function, preventing the use of this proof technique.) And so we will adopt a non-traditional method of proof.

Let us first remark, as is well-known, that we have $\|T_* f\|_p \lesssim \|f\|_p$ for $1 < p < \infty$. Taking the linearization into account and dualizing, this means that we have

$$(3.6) \quad \|L^* f\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

Indeed, the only thing that is needed here is the *a priori* assumption that T is a bounded operator on $L^2(dx)$, and the ‘size’ and ‘smoothness’ assumptions on the kernel. The size condition is given by condition (3) in Definition 1.5, and the smoothness condition is given by condition (2). To prove the L^1 end point version of this estimate, we appeal a particular refinement of (3.6) that goes back to C. Fefferman’s proof [10] of the Carleson theorem of Fourier series.

We need additional definitions. Let us regard the stopping time which gives the truncation $\varepsilon(x)$ as in (3.2) as fixed. For a subset $\mathcal{Q}' \subset \mathcal{Q}$ let us set

$$L_{\mathcal{Q}'} f(x) := \mathbf{1}_\Lambda(x) \sum_{Q \in \mathcal{Q}'} \langle \phi, \gamma_Q \rangle g_Q \mathbf{1}_{\{|Q|^{1/d} \leq \varepsilon(x)\}}.$$

We now define

$$\text{dense}(\mathcal{Q}') := \sup_{Q' \in \mathcal{Q}'} \sup_{Q: Q \supset Q'} \frac{|Q \cap \{x \in \Lambda : |Q|^{1/d} \leq \varepsilon(x)\}|}{|Q|}.$$

In this definition, note that we are taking $Q' \in \mathcal{Q}'$, but only requiring that $Q \supset Q'$. The ‘Tree Lemma’ of [10] gives us:

3.7. Lemma. *We have the estimates, universal in choice of measurable $\varepsilon(\cdot)$ and \mathcal{Q}' .*

$$\|L_{\mathcal{Q}'} f\|_p \lesssim \text{dense}(\mathcal{Q}')^{1/p} \|f\|_p$$

Proof. Let us set $\delta = \text{dense}(\mathcal{Q}')$. We use this variant of the maximal function. Let \mathcal{P} be a collection of disjoint dyadic cubes. To each $P \in \mathcal{P}$, associate a subset $E_P \subset P$ with $|E_P| \leq \delta|P|$. We have this maximal function estimate.

$$(3.8) \quad \|M_{\mathcal{P}} \phi\|_p \lesssim \delta^{1/p} \|\phi\|_p \quad 1 < p < \infty,$$

$$M_{\mathcal{P}} \phi := \sum_{P \in \mathcal{P}} \mathbf{1}_{E_P}(x) \sup_{Q: Q \supset P} \mathbb{E}_Q |\phi|.$$

The proof of (3.8) is straight forward.

$$\begin{aligned} \|M_{\mathcal{P}} \phi\|_p^p &= \sum_{P \in \mathcal{P}} |E_P| \sup_{Q: Q \supset P} [\mathbb{E}_Q |\phi|]^p \\ &\leq \delta \sum_{P \in \mathcal{P}} |P| \sup_{Q: Q \supset P} [\mathbb{E}_Q |\phi|]^p \\ &\leq \delta \|M \phi\|_p^p \lesssim \delta \|\phi\|_p^p. \end{aligned}$$

This proves (3.8).

We now pass to the selection of \mathcal{P} and $\{E_P: P \in \mathcal{P}\}$. Take \mathcal{P} to be the minimal $P \in \mathcal{Q}$ such that $P^{(1)}$ contains some element of \mathcal{Q}' . Here, $P^{(1)}$ denotes the parent of P : The minimal element of \mathcal{Q} that strictly contains P . Observe that if Q^* is a maximal element of \mathcal{Q}' , those elements of $P \in \mathcal{P}$ that are contained in Q^* also partition Q^* . Set

$$E_P := \{x \in P \cap \Lambda : |P^{(1)}|^{1/d} \leq \varepsilon(x)\}.$$

Let us argue that $|E_P| \leq 2\delta|P|$. The parent $P^{(1)}$ contains a $Q \in \mathcal{Q}'$. Hence, by definition,

$$\begin{aligned} \frac{|E_P|}{|P|} &\leq \frac{|\{x \in P^{(1)} \cap \Lambda : |P^{(1)}|^{1/d} \leq \varepsilon(x)\}|}{|P|} \\ &= 2 \frac{|\{x \in P^{(1)} \cap \Lambda : |P^{(1)}|^{1/d} \leq \varepsilon(x)\}|}{|P^{(1)}|} \\ &\leq 2 \text{dense}(Q) \leq 2\delta. \end{aligned}$$

Next, observe that the support of $L f(x)$ is contained in $\bigcup\{E_P: P \in \mathcal{P}\}$. Indeed, if $L f(x) \neq 0$, let $Q \in \mathcal{Q}'$ be a cube that contains x . It follows from the definition of \mathcal{P} that there must be a cube $P \in \mathcal{P}$ with $x \in P \subsetneq Q$. We necessarily have $|P^{(1)}|^{1/d} \leq |Q|^{1/d} \leq \varepsilon(x)$. Thus, $x \in E_P$, and the claim is proved.

Finally, let us connect the operator $L_{\mathcal{Q}'}$ and the maximal function $M_{\mathcal{P}}$. It will be convenient here, and below, to define

$$(3.9) \quad T_{\mathcal{Q}'} f = \sum_{Q \in \mathcal{Q}'} \langle f, \gamma_Q \rangle g_Q.$$

For the moment, let us assume that g_Q has mean zero (which is not required by Definition 1.5). Then, for any point $x \in Q_0 \subsetneq Q_1$

$$(3.10) \quad \sum_{\substack{Q \in \mathcal{Q}' \\ |Q_1|^{1/d} > |Q|^{1/d} > |Q_0|^{1/d}}} \langle f, \gamma_Q \rangle g_Q(x) = \mathbb{E}_{Q_0} \left[\sum_{Q \in \mathcal{Q}'} \langle f, \gamma_Q \rangle g_Q \right] - \mathbb{E}_{Q_1} \left[\sum_{Q \in \mathcal{Q}'} \langle f, \gamma_Q \rangle g_Q \right].$$

This permits us to write

$$|L_{\mathcal{Q}'} f(x)| = \mathbf{1}_{\Lambda}(x) \left| \sum_{Q \in \mathcal{Q}'} \langle f, \gamma_Q \rangle g_Q \mathbf{1}_{|Q|^{1/d} \leq \varepsilon(x)} \right| \leq 2 M_{\mathcal{P}}(T_{\mathcal{Q}'} f).$$

Now, in the case that g_Q need not have mean zero, we take these steps. We first ‘separate scales.’ That is, we fix an integer $0 \leq t < \tau$, and assume that $g_Q \neq 0$ implies that $\log|Q|^{1/d} \bmod \tau = t$. Condition (2) in Definition 1.5 then implies that an analog of (3.10) holds, and so this case reduces to the one where g_Q has mean zero.

To conclude, we estimate

$$\begin{aligned} \|L f\|_p &\leq \|M_{\mathcal{P}}(T_{\mathcal{Q}'} f)\|_p \\ &\lesssim \delta^{1/p} \|T_{\mathcal{Q}'} f\|_p \\ &\lesssim \delta^{1/p} \|f\|_p. \end{aligned}$$

The proof of the Lemma is complete. \square

This proposition is well-known:

3.11. Proposition. *A Haar shift operator T_* with index τ maps $L^1(dx)$ into $L^{1,\infty}(dx)$ with norm depending only on τ .*

We need a version of the John-Nirenberg inequality, which says that a ‘uniform L^0 condition implies exponential integrability.’

3.12. Lemma. *This holds for all integers τ . Let w be a weight on \mathbb{R}^d . Let $\{\phi_Q : Q \in \mathcal{Q}\}$ be functions so that for all dyadic cubes Q we have*

- (1) ϕ_Q is supported on Q and is constant on each sub-cube $Q' \subset Q$ with $|Q'| = 2^{-\tau d} |Q|$;
- (2) $\|\phi_Q\|_{\infty} \leq 1$;

(3) for all dyadic cubes Q , we have

$$\left| \left\{ \left| \sum_{Q': Q' \subset Q} \phi_{Q'} \right| > 1 \right\} \right| \leq 2^{-\tau d - 1} |Q|.$$

It then follows that we have the estimate

$$\left| \left\{ \sup_{\substack{Q': Q' \subset Q \\ |Q'| > \epsilon}} \left| \sum \phi_{Q'} \right| > 2\tau t \right\} \right| \leq \tau 2^{-t+1} |Q|, \quad t > 1.$$

3.1.2. *Main Steps in Proof of Theorem 3.5: Decomposition of f and \mathcal{Q} .* For integers $k \in \mathbb{Z}$, let us set

$$\begin{aligned} \mathcal{Q}_k &:= \{Q \in \mathcal{Q} : 2^{k-1} < \inf_{x \in Q} Mf(x) \leq 2^k\}, \\ f_k &:= \sum_{Q \in \mathcal{Q}_k} \Delta_Q f. \\ (3.13) \quad \Delta_Q f &:= \left\{ \sum_{\substack{Q' \subset Q \\ 2^d |Q'| = |Q|}} \mathbb{E}_{Q'} f \right\} - \mathbb{E}_Q f \end{aligned}$$

The top line is a decomposition of the set of dyadic cubes. The second line is essentially a Haar projection associated to the collections of cubes, and Δ_Q is the martingale difference associated to the cube Q . (We adopt this definition, as it allows us to not specifically define the collection of Haar functions on \mathbb{R}^d .) With this choice is $f = \sum_k f_k$. The basic properties of this decomposition are:

3.14. **Proposition.** *We have these estimates for the functions $\{f_k : k \in \mathbb{Z}\}$.*

$$(3.15) \quad |F_k| \lesssim 2^{-k}, \quad F_k := \text{supp}(f_k),$$

$$(3.16) \quad \|f_k\|_p \lesssim 2^{k/p'}, \quad 1 < p < \infty.$$

Proof. The first estimate follows from the weak-type estimate on the maximal function. To see the second, observe the following. The martingale differences in (3.13) are instance of operators that meet the definition of the operators we consider in Definition 1.5. In particular (as is well-known) they satisfy a weak- L^1 inequality, as is claimed in Proposition 3.11. Now observe this: For any $Q_0 \in \mathcal{Q}_k$, we have

$$\left\| \sum_{\substack{Q \in \mathcal{Q}_k \\ Q \subset Q_0}} \Delta_Q f \right\|_{1, \infty} \lesssim \|f \mathbf{1}_{Q_0}\|_1 \lesssim 2^k |Q_0|.$$

This follows from the definition of \mathcal{Q}_k : The maximal function takes satisfies $M|f|(x) \simeq 2^k$ for some $x \in Q_0$, so the average of f on Q_0 is at most 2^k .

But, then the Haar shift structure and the John-Nirenberg Lemma, Lemma 3.12, implies that we have an *exponential* distributional estimate, which clearly implies (3.16). \square

3.1.3. *Adding Up the Elements of the Decomposition.* We have assembled the main tools of our proof of Theorem 3.5. Recall that $\|f\|_1 = 1$. It suffices to show that

$$(3.17) \quad |\{L^* f > C\}| \lesssim 1,$$

where C will be an absolute constant that we will pick below.

We pass to the decomposition of f as in Proposition 3.14. For $k \leq 0$, the argument is quite simple. For $C_1 = \sum_{k \leq 0} 2^{-k/2}$ we estimate as follows, where we use (3.16) with $p = 4$.

$$\begin{aligned} \left| \left\{ \sum_{k=-\infty}^0 f_k > C_1 \right\} \right| &\leq \sum_{k=-\infty}^0 \{f_k > 2^{-k/2}\} \\ &\lesssim \sum_{k=-\infty}^0 2^{2k} \|f_k\|_4^4 \\ &\lesssim \sum_{k=-\infty}^0 2^{-k} \lesssim 1. \end{aligned}$$

This is half of the estimate in (3.17).

The estimate for $k > 0$ we need the Tree Lemma, Lemma 3.7. Consider $L^* f_k$. In this definition, as f_k is supported on a set of small measure, due to (3.15), it follows that the choice of function $\varepsilon(\cdot)$ that enters into the definition of L , can be restricted to the set F_k , since we are considering the adjoint operator here. We therefore define

$$F_k^* := \{M \mathbf{1}_{F_k} > 2^{-(1-\eta)k}\}, \quad 0 < \eta < 1.$$

It follows that

$$(3.18) \quad \sum_{k=1}^{\infty} |F_k^*| \lesssim 1.$$

We define $\mathcal{Q}'_k := \{Q \in \mathcal{Q} : Q \not\subset F_k^*\}$. The important points to observe are that

$$\begin{aligned} L^* f_k(x) &= L_{\mathcal{Q}'_k}^* f_k(x), \quad x \notin F_k^*, \\ \text{dense}(\mathcal{Q}'_k) &\lesssim 2^{-(1-\eta)k}. \end{aligned}$$

In the last line, the line that is essential to this argument, we are defining density relative to the function $\varepsilon(x)$ restricted to the set F_k .

3.19. *Remark.* It is this step that is indicative of the so-called restricted weak-type approach. See [17, 18].

For $1 < p < 2$ to be selected, we can then estimate

$$\begin{aligned} \left| \left\{ \sum_{k=1}^{\infty} L_{\mathcal{Q}'_k}^* f_k > \sum_{k=1}^{\infty} 2^{-\eta k} \right\} \right| &\leq \sum_{k=1}^{\infty} |\{L_{\mathcal{Q}'_k}^* f_k > 2^{-\eta k}\}| \\ &\leq \sum_{k=1}^{\infty} 2^{\eta p k} \|L_{\mathcal{Q}'_k}^* f_k\|_p^p \\ &\leq \sum_{k=1}^{\infty} 2^{\eta p k} \text{dense}(\mathcal{Q}'_k) \|f_k\|_p^p \\ &\leq \sum_{k=1}^{\infty} 2^{(\eta p - 1 + \eta + p - 1)k}. \end{aligned}$$

It is clear that we can choose $0 < \eta < 1$ and $1 < p < 2$ so that this last sum is finite. This estimate with (3.18) complete the proof of (3.17). The proof of Theorem 3.5 is complete.

4. INITIAL CONSIDERATIONS

We collect together facts useful to us. From this point on, by abuse of notation, we will write

$$w(E) = \int_E w \, dx,$$

and similarly for the weight $\sigma := w^{-p'+1}$, which is the weight dual to w .

4.1. Definition. Let $\mathcal{Q}' \subset \mathcal{Q}$ be any collection of dyadic cubes, and μ a positive measure. Call $(\mathcal{S} : \mathcal{Q}'(S))$ a μ -corona decomposition of \mathcal{Q}' if these conditions hold.

- (1) For each $Q \in \mathcal{Q}'$ there is a member of \mathcal{S} that contains Q , and letting $\lambda(Q) \in \mathcal{S}$ denote the minimal cube which contains Q we have

$$4 \frac{\mu(\lambda(Q))}{|\lambda(Q)|} \geq \frac{\mu(Q)}{|Q|}.$$

- (2) For all $S \subsetneq S' \in \mathcal{S}$

$$\frac{\mu(S)}{|S|} > 4 \frac{\mu(S')}{|S'|}.$$

We set $\mathcal{Q}'(S) := \{Q \in \mathcal{Q}' : \lambda(Q) = S\}$. The collections $\mathcal{Q}'(S)$ partition \mathcal{Q}' . We refer to \mathcal{S} as the *stopping cubes* for \mathcal{Q}' .

Decompositions of this type appear in a variety of questions. We are using terminology which goes back to (at least) David and Semmes [6, 7]. A subtle corona decomposition is central to [28], and the paper [1] includes several examples in the context of dyadic analysis. Indeed, the corona definition and Lemma 4.6

below are the primary tools in [12], which proves the strong-type inequalities for the un-truncated singular integrals.

Observe that we have the inequality

$$(4.2) \quad \sum_{S \in \mathcal{S}} \frac{w(S)}{|S|} \mathbf{1}_S \lesssim M(w \mathbf{1}_{Q_0}).$$

This follows immediately from the choice of stopping cubes in Definition 4.1: At each point x on the left, the non-zero summands are terms in a geometric series, and hence the sum is comparable to its maximal summand. A somewhat more subtle, but known fact is this.

4.3. Lemma. *Let \mathcal{S} be associated with corona decomposition for a collection of dyadic cubes contained in a given cube Q_0 and a weight w .*

$$(4.4) \quad \sum_{\substack{S \in \mathcal{S} \\ S \subset Q_0}} w(S) \lesssim \|w\|_{A_p} w(Q_0).$$

We turn to the essence of this approach. Let us fix $1 < p < \infty$, and $w \in A_p$. Let $\sigma = w^{1-p'}$ be the dual measure. Fix a cube Q_0 . We hold the A_p ratios constant. Define

$$(4.5) \quad \mathcal{Q}_{Q_0, a} := \left\{ Q \subset Q_0 : 2^{a-1} \leq \frac{w(Q)}{|Q|} \left[\frac{\sigma(Q)}{|Q|} \right]^{p-1} \leq 2^a \right\}$$

where $0 \leq a \leq \log_2 \|w\|_{A_p}$.

Let us denote by $T_{\mathcal{Q}', *}$ the maximal truncations of the sums in (3.9). We can define a linearization $L_{\mathcal{Q}'}$ of the maximal truncations $T_{\mathcal{Q}', *}$ of sums in (3.9). And we can consider the dual $L_{\mathcal{Q}'}^*$. For these operators, we have these distributional estimates.

4.6. Lemma. *Apply the corona decomposition to $\mathcal{Q}_{Q_0, a}$ and the measure ω . Uniformly over $S \in \mathcal{S}$, and functions f supported on Q_0 with $\|f\|_\infty = 1$, we have these inequalities:*

$$(4.7) \quad |\{x \in S : L_{\mathcal{Q}(S)}^*(f\omega) > Kt \frac{\omega(S)}{|S|}\}| \lesssim e^{-t} |S|,$$

$$(4.8) \quad w(\{x \in S : L_{\mathcal{Q}(S)}^*(f\omega) > Kt \frac{\omega(S)}{|S|}\}) \lesssim e^{-t} w(S),$$

Proof. For integers $b \geq 0$, and $S \in \mathcal{S}$ let

$$(4.9) \quad \mathcal{Q}_b(S) := \left\{ Q \in \mathcal{Q}_{Q_0, a}(S) : 2^{-b+1} \frac{\omega(S)}{|S|} < \frac{\omega(Q)}{|Q|} \leq 2^{-b+2} \frac{\omega(S)}{|S|} \right\}.$$

We are suppressing the dependence of this definition on the parameter a , which pigeonholes the A_p characteristic, and the dyadic cube Q_0 , see (4.5). For any cube $Q \in \mathcal{Q}(S)$, let $\mathcal{Q}_b(S, Q) := \{Q' \in \mathcal{Q}_b(S) : Q' \subset Q\}$. We have for any cube

$Q \in \mathcal{Q}_b(S)$

$$\|L_{\mathcal{Q}_b(S,Q)}^*(f\omega)\|_{1,\infty} \lesssim \|f\omega \mathbf{1}_Q\|_1 \lesssim \omega(Q) \lesssim 2^{-b} \frac{\omega(S)}{|S|} |Q|.$$

For a fixed cube \overline{Q} , we use the estimate above on maximal cubes in $\mathcal{Q}_b(S, \overline{Q})$, to see that

$$\|L_{\mathcal{Q}_b(S,\overline{Q})}^*(f\omega)\|_{1,\infty} \lesssim 2^{-b} \frac{\omega(S)}{|S|} |\overline{Q}|.$$

This estimate is universal in \overline{Q} , whence Lemma 3.12 will give us an improvement on this distributional estimate

$$(4.10) \quad |\{x \in S : L_{\mathcal{Q}_b(S)}^*(f\omega) > Kt 2^{-b} \frac{\omega(S)}{|S|}\}| \lesssim e^{-t} |S|.$$

Clearly, we can sum in b to derive (4.7).

To prove (4.8) for the operator $L_{\mathcal{Q}_b(S)}^*$, consider the event

$$\mathcal{E}(t) := \{x \in S : L_{\mathcal{Q}_b(S)}^*(f\omega) > Kt 2^{-b} \frac{\omega(S)}{|S|}\}$$

The dyadic structure is decisive, as well as the fact that we are considering the adjoint of the maximal truncation operator. This event is the union of children of dyadic cubes in $\mathcal{Q}_b(S)$. If we denote by $\mathcal{E}^*(t)$ the event that we obtain by passing from these children to the parents, it also satisfies the estimate (4.10): $|\mathcal{E}^*(t)| \lesssim e^{-t} |S|$. But, for the parents P that make up the event $\mathcal{E}^*(t)$, we have held the A_p ratio constant, as in (4.5). And in (4.9), we have held the ratio $\omega(P)/|P|$ constant. Hence, $\sigma(P)$ is proportional to $|P|$, so that (4.10) holds for w -measure as well. (The constant of proportionality depends upon L , but occurs on both sides of the distributional inequality, so is not important for us to calculate.) Hence, we see that (4.8) holds for $L_{\mathcal{Q}(S)}^*$. \square

We need the corresponding inequalities for T_* .

4.11. Lemma. *Apply the corona decomposition to $\mathcal{Q}_{Q_0,a}$ and the measure σ . Uniformly over $S \in \mathcal{S}$, and functions f supported on Q_0 with $\|f\|_\infty = 1$, we have these inequalities:*

$$(4.12) \quad |\{x \in S : T_{\mathcal{Q}(S),*}(f\sigma) > Kt \frac{\sigma(S)}{|S|}\}| \lesssim e^{-t} |S|,$$

$$(4.13) \quad w(\{x \in S : T_{\mathcal{Q}(S),*}(f\sigma) Kt \frac{w(S)}{|S|}\}) \lesssim e^{-t} w(S),$$

Here, by $T_{\mathcal{Q},*}$ we mean the maximal truncations of the sums $T_{\mathcal{Q}'}$ as defined in (3.9).

Proof. Following the proof of Lemma 4.6, one can prove

$$\begin{aligned} |\{x \in S : T_{\mathcal{Q}(S)}(f\sigma) > Kt \frac{\sigma(S)}{|S|}\}| &\lesssim e^{-t} |S|, \\ w(\{x \in S : T_{\mathcal{Q}(S)}(f\sigma) Kt \frac{w(S)}{|S|}\}) &\lesssim e^{-t} w(S). \end{aligned}$$

One only needs the weak-type estimate for $T_{\mathcal{Q}(S)}$, but these are well-known. (These inequalities are also proved in [12].) But, there is also the following refinement: The inequalities above hold, with the same constants, with $T_{\mathcal{Q}(S)}$ replaced by $T_{\mathcal{P}}$, where \mathcal{P} is *any* subset of $\mathcal{Q}(S)$.

To address the maximal truncations, let us observe this: For all $\lambda > 0$, we can choose $\mathcal{P} \subset \mathcal{Q}(S)$ so that

$$(4.14) \quad \{x \in S : T_{\mathcal{Q}(S)}(f\sigma) > \lambda\} = \{x \in S : T_{\mathcal{P}}(f\sigma) > \lambda\}.$$

That is, the uniformity of the distributional estimates over \mathcal{P} give us distributional estimates over the maximal truncations. This equality, once proved, gives the Lemma. Indeed, for each $x \in S$, define

$$\varepsilon(x) := \inf \left\{ \epsilon : \left| \sum_{\substack{Q \in \mathcal{Q}(S) \\ |Q|^{1/d} > \epsilon}} \langle f, \gamma_Q \rangle g_Q(x) \right| > \epsilon \right\}.$$

Clearly, by definition, we have

$$\{x \in S : T_{\mathcal{Q}(S)}(f\sigma) > \lambda\} = \left\{ x : \left| \sum_{\substack{Q \in \mathcal{Q}(S) \\ |Q|^{1/d} \geq \varepsilon(x)}} \langle f, \gamma_Q \rangle g_Q(x) \right| > \lambda \right\}.$$

Moreover, it follows from the definition of $\varepsilon(x)$ that for all $Q \in \mathcal{Q}(S)$, if $|Q|^{1/d} \geq \varepsilon(x)$ for any point $x \in Q$, then the inequality holds *for all* $x \in Q$. Thus, to prove (4.14), we take $\mathcal{P} = \{Q \in \mathcal{Q}(S) : |Q|^{1/d} \geq \inf_{x \in Q} \varepsilon(x)\}$. \square

5. PROOF OF THE STRONG TYPE INEQUALITIES

To be specific, we are to prove the Theorem

5.1. Theorem. *Let T be a Haar-shift operator as in Definition 1.5. Let $1 < p < \infty$ and let $w \in A_p$. We have*

$$\|T_* f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\max\{1, 1/(p-1)\}} \|f\|_{L^p(w)}.$$

Extrapolation makes our task simpler. By the main result of [8], we need only prove this Theorem in the case of $p = 2$. Namely, with $\sigma = w^{-1}$ being the dual measure, we should check

$$\|T_* f\sigma\|_{L^2(w)} \lesssim \|w\|_{A_2} \|f\|_{L^2(\sigma)}.$$

Appealing to Theorem 2.11, we have four testing conditions to check. The first two concern the maximal function. But, we have by (2.4)

$$\|M(f\sigma)\|_{L^2(w)} \lesssim \|w\|_{A_2} \|f\|_{L^2(\sigma)},$$

and clearly the same inequality holds with the roles of σ and w reversed.

The remaining two conditions concern T_* . The first of these concerns the condition of (2.10). But, we prefer the equivalent formulation of (3.4), which requires us to prove the inequality

$$\|L^*(\mathbf{1}_Q g w)\|_{L^2(\sigma)} \lesssim \|w\|_{A_2} w(Q)^{1/2}, \quad \|g\|_\infty = 1.$$

One can follow the argument in [12, Section 4, beginning at (4.10)], or the argument in the next section.

The second estimate is testing condition (2.13), that is we should check that

$$\|\mathbf{1}_Q T_*(\mathbf{1}_Q f \sigma)\|_{L^2(w)} \lesssim \|w\|_{A_2} \sigma(Q)^{1/2}, \quad \|f\|_\infty = 1.$$

But, with the two distributional estimates (4.12) and (4.13), we can repeat the argument of [12, Section 4, beginning at (4.10)] with very little change.

6. THE PROOF OF THE WEAK-TYPE INEQUALITIES

To be specific, we are to prove the Theorem

6.1. Theorem. *Let T be a Haar-shift operator as in Definition 1.5. Let $1 < p < \infty$ and let $w \in A_p$. We have*

$$\|T_* f\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p} \|f\|_{L^p(w)}.$$

We will only consider the case of p such that p' is an integer. Extrapolation will then prove the result as stated. We remark that our argument will be rather inefficient in p , we do not think that this argument can deliver the sharp growth in p , and so we will not attempt to track the dependence of p' .

6.2. Remark. Concerning the extrapolation, the extrapolation of weak-type estimates is well known, and follows from the formulation of an extrapolation result in an ‘operator free’ fashion, see [4, Theorem 2.1]. This formulation can be used in the strong-type extrapolation result we cited above [8] to yield the extrapolation result we need for Theorem 6.1 above. The details of this approach are contained in [5, Theorem 3.2.1], which is in close parallel to the main result of [8]. We are grateful to Cristina Pererya for help with this point.

We will ‘separate scales’. Namely we consider an operator T as in Definition 1.5, satisfying the additional condition that if $Q \subsetneq Q'$ and $g_Q, g_{Q'} \neq 0$ then we must have $|Q| < 2^{-\tau d} |Q'|$. This assumption can be imposed without loss of generality.

By Theorem 2.9, we have two testing conditions to check. The first arises from the weak-type estimate for the maximal function, but by (2.3), this norm is $\|w\|_{A_p}^{1/p} \leq \|w\|_{A_p}$, so we turn to the second testing condition. This is the condition (2.10), which we prefer to check in the form of (3.4). That is, we should check that

$$(6.3) \quad \|\mathbf{1}_Q L^*(\phi w \mathbf{1}_Q)\|_{L^{p'}(\sigma)} \lesssim \|w\|_{A_p} w(Q)^{1/p'}, \quad \|\phi\|_\infty = 1.$$

We will follow the broad outlines of the argument in [12, Section 4, beginning at (4.10)]. In particular, this argument calls for expanding the left-hand side above, which is why we insist on p' being an integer.

We fix a cube Q_0 on which we are to check (6.3). Let us first check that the ‘large scales’ satisfy the desired estimate. It follows from Definition 1.5, that we have

$$\begin{aligned} \int \left[\sum_{Q: Q \supset Q_0} |\langle w \mathbf{1}_{Q_0}, \gamma_Q \rangle g_Q| \right]^{p'} \sigma(dx) &\lesssim \left[\frac{w(Q_0)}{|Q_0|} \right]^{p'} \sigma(Q_0) \\ &\lesssim \|w\|_{A_p}^{p'} w(Q_0), \end{aligned}$$

as follows from inspection of the A_p definition. Here, we have imposed absolute values inside the sum, that there is no need to consider maximal truncations.

We can assume that in the definition of \mathbb{T} , we have $g_Q = \gamma_Q \equiv 0$ if $Q \not\subset Q_0$. We use the definition of $\mathcal{Q}_{Q_0, a}$ in (4.5). Apply the corona decomposition to this collection of cubes and weight w . This generates the stopping cubes \mathcal{S} and the corresponding collections $\mathcal{Q}_{Q_0, a}(S)$.

Our goal is to show that

$$(6.4) \quad \int_{Q_0} \left[\sum_{S \in \mathcal{S}} |L_S^*| \right]^{p'} \sigma(dx) \lesssim 2^{a(p'-1)} \|w\|_{A_p} w(Q_0)$$

where by abuse of notation we write $L_S := L_{\mathcal{Q}_{Q_0, a}(S)}$ for $S \in \mathcal{S}$.

We set notation to expand the p' -th power of the integrand in (6.4). We set notation for the expansion of the integrand in (6.4), which will be indexed by partitions of the integer p' .

$$(6.5) \quad \begin{aligned} \mathfrak{S} &:= \left[\sum_{S \in \mathcal{S}} |L_S^*| \right]^{p'} \lesssim \sum_{\vec{p} \in \mathbf{P}} \mathfrak{S}_{\vec{p}}, \\ \mathfrak{S}_{\vec{p}} &:= \sum_{S_1 \in \mathcal{S}} \sum_{\substack{S_2 \in \mathcal{S} \\ S_2 \subsetneq S_1}} \cdots \sum_{\substack{S_t \in \mathcal{S} \\ S_t \subsetneq S_{t-1}}} \prod_{u=1}^t |L_{S_u}|^{p_u}, \quad \vec{p} = (p_1, \dots, p_t), \\ \mathbf{P} &:= \bigcup_{t=1}^{p'} \left\{ \vec{p} = (p_1, \dots, p_t) \in \{1, 2, \dots, p'\}^t : \sum_{u=1}^t p_u = p' \right\}. \end{aligned}$$

In the line (6.5), the implied constant depends only on p' , and can be taken to be $(p')!$. Let us set $\overline{\mathfrak{S}} = \int \mathfrak{S} \sigma(dx)$, and likewise for $\overline{\mathfrak{S}}_{\vec{p}}$.

Here is what we will show. We have

$$(6.6) \quad \overline{\mathfrak{S}}_{(p')} \lesssim 2^{a(p'-1)} \|w\|_{A_p} w(Q_0),$$

$$(6.7) \quad \overline{\mathfrak{S}}_{\vec{p}} \lesssim \overline{\mathfrak{S}}^{1-p_t/p'} \left[2^{a(p'-1)} \|w\|_{A_p} w(Q_0) \right]^{p_t/p'} \quad \vec{p} = (p_1, \dots, p_t) \in \mathbf{P} - \{(p')\}.$$

The first line says that the term associated with the trivial partition (p') satisfies the desired estimate. And the second says that the terms associated with non-trivial partitions satisfy a slightly different estimate. A moment's thought shows that this proves (6.4). Indeed, for $\epsilon = \epsilon(p)$, we must have one element $\vec{p} \in \mathbf{P}$ with $\epsilon \bar{\mathfrak{G}} \leq \bar{\mathfrak{G}}_{\vec{p}}$, and then the two inequalities above prove our Theorem, after a trivial summation in $0 \leq a \leq \log_2 \|w\|_{A_p}$.

Proof of (6.6). We use (4.8);, which in this case is:

$$(6.8) \quad \sigma(\{x \in S : |L_S^*| > Kt \frac{w(S)}{|S|}\}) \lesssim e^{-t} \sigma(S).$$

It gives us

$$\begin{aligned} \sum_{S \in \mathcal{S}} \int_S |L_S^*|^{p'} \sigma(dx) &\lesssim \sum_{S \in \mathcal{S}} \left[\frac{w(S)}{|S|} \right]^{p'} \sigma(S) && \text{(by (4.2))} \\ &\lesssim 2^{a(p'-1)} \sum_{S \in \mathcal{S}} w(S) && \text{(by (2.14) \& (4.5))} \\ &\lesssim 2^{a(p'-1)} \|w\|_{A_p} w(Q_0). && \text{(by (4.4))} \quad \square \end{aligned}$$

Proof of (6.7). Let $\vec{p} = (p_1, \dots, p_t) \in \mathbf{P}$ where $t > 1$. The point to exploit if we have $S \subsetneq S'$ with $S, S' \in \mathcal{S}$ we have $L_{S'}^*$ constant on the interval S , due to the separation of scales. We will denote this value by $L_{S'}^*(S)$. This permits us to use the distributional estimate (6.8) to estimate the L^1 norm, which gives us the estimate below, in which we set $\vec{q} = (p_1, \dots, p_{t-1})$, that is the partition \vec{p} , with the last place taken off.

$$\begin{aligned} \bar{\mathfrak{G}}_{\vec{p}} &= \sum_{S_1 \in \mathcal{S}} \sum_{\substack{S_2 \in \mathcal{S} \\ S_2 \subsetneq S_1}} \cdots \sum_{\substack{S_t \in \mathcal{S} \\ S_t \subsetneq S_{t-1}}} \prod_{u=1}^{t-1} |L_{S_u}^*(S_t)| \int |L_{S_t}^*| \sigma(dx) \\ &= \sum_{S_1 \in \mathcal{S}} \sum_{\substack{S_2 \in \mathcal{S} \\ S_2 \subsetneq S_1}} \cdots \sum_{\substack{S_t \in \mathcal{S} \\ S_t \subsetneq S_{t-1}}} \prod_{u=1}^{t-1} |L_{S_u}^*(S_t)| \cdot \left[\frac{w(S_t)}{|S_t|} \right]^{p_t} \sigma(S_t) \\ &\lesssim \int \mathfrak{G}_{\vec{q}} \cdot M_S(w \mathbf{1}_{Q_0})^{p_t} \sigma(dx) && \text{(by (4.2))} \\ (6.9) \quad &\lesssim \|\mathfrak{G}_{\vec{q}}\|_{L^{(p'/p_t)'(\sigma)}} \|M_S(w \mathbf{1}_{Q_0})^{p_t}\|_{L^{p'/p_t}(\sigma)} && \text{(by Hölder's)} \\ &M_S g(x) := \sup_{S \in \mathcal{S}} \mathbf{1}_S \mathbb{E}_S g. \end{aligned}$$

We estimate the two terms in (6.9) separately. We can use a rather crude estimate on the term $\mathfrak{G}_{\vec{q}}$. Since the length of \vec{q} is $\sum_{u=1}^{t-1} p_u = p' - p_t$, we can estimate

$$(6.10) \quad \|\mathfrak{G}_{\vec{q}}\|_{L^{(p'/p_t)'(\sigma)}} \lesssim \left\| \left[\sum_{S \in \mathcal{S}} L_S^* \right]^{p'-p_t} \right\|_{L^{(p'/p_t)'(\sigma)}} \lesssim \bar{\mathfrak{G}}^{1-p_t/p'}.$$

The last line follows by inspection, using $(p'/p_t)' = p'/(p' - p_t)$.

For the second term in (6.9), we estimate using (4.2) again

$$\begin{aligned}
 \|M_S(w\mathbf{1}_{Q_0})^{p_t}\|_{L^{p'/p_t}(\sigma)}^{p'/p_t} &\lesssim \sum_{S \in \mathcal{S}} \left[\frac{w(S)}{|S|} \right]^{p'} \sigma(S) \\
 &\lesssim 2^{a(p'-1)} \sum_{S \in \mathcal{S}} w(S) && \text{(by (2.14))} \\
 (6.11) \qquad &\lesssim 2^{a(p'-1)} \|w\|_{A_p} w(Q_0) && \text{(by (4.4))}
 \end{aligned}$$

Combining (6.9), (6.10) and (6.11), we prove (6.7). Our proof is complete. \square

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