THE NON-LINEAR DIRICHLET PROBLEM IN HADAMARD MANIFOLDS

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Abstract. We prove existence theorems for the Dirichlet problem for hypersurfaces of constant special Lagrangian curvature in Hadamard manifolds. The first results are obtained using the continuity method and approximation and then refined using two iterations of the Perron method. The a-priori estimates used in the continuity method are valid in any ambient manifold.

1. Introduction

This paper treats the problem of finding immersed hypersurfaces satisfying prescribed boundary and curvature conditions inside manifolds of strictly negative sectional curvature.
This is an old geometric problem. The simplest version is Plateau's problem (see, for example, [1]), which requires minimal hypersurfaces with specified boundary. In this case, the curvature condition (minimality) is linear in terms of the shape operator of the immersion. The more general linear problem is that of finding hypersurfaces of constant mean curvature with specified boundary, for which a substantial litterature exists.

The next interesting problem concerns hypersurfaces of constant Gaussian curvature. This is much harder, since Gaussian curvature is a non-linear function of the shape operator. However, as it arises from a character on the algebra of matrices (i.e. the determinant), the partial differential equations involved are nonetheless much simpler than in the most general case. Various results exist, using various different techniques (the following list is not exhaustive): constant Gaussian curvature surfaces which are graphs over hyperplanes in $\mathbb{R}^n$ are obtained using the continuity method by Caffarelli, Nirenberg and Spruck in [7] and by Guan in [10]; hypersurfaces whose boundary is the boundary of a given convex set in $\mathbb{R}^n$ are then obtained by Spruck and Guan using the Perron method in [11]; graphs over horospheres and open subsets of the ideal boundary of $\mathbb{H}^n$ are obtained using the continuity method again by Rosenberg and Spruck in [22] while Guan and Spruck obtain more general results, again in $\mathbb{H}^n$ using a mixture of the continuity

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method and the Perron method in [12] and [13]; a slightly different species of local existence results is obtained using the Implicit Function Theorem by Mazzeo and Pacard in [21]; and finally more general graphs are obtained in 3-dimensional Hadamard manifolds by Labourie in [17] using the theory of pseudo-holomorphic curves and these results are further developed in the case of $\mathbb{H}^3$ by the author in [27] and [28].

Gaussian curvature constitutes the simplest non-linear curvature, but there are many other interesting examples (see, for example, [12] and [13]). Existence results for constant curvature hypersurfaces for these different notions of curvature are interesting for various reasons and have varied applications. In general they yield dimensional reductions of geometric problems, since the space of constant curvature hypersurfaces is typically a finite dimensional manifold in contrast to, for example, the space of convex immersions, which is much more complicated. The following are a few applications of these types of results: in [18], Labourie uses constant Gaussian curvature surfaces to study the structure of 3-dimensional hyperbolic manifolds; in [23], Schlenker and Krasnov use constant mean curvature surfaces to study the relationship between the Teichmueller space of a surface and its moduli space of hyperbolic metrics; in [24], the author uses constant special Lagrangian curvature hypersurfaces to obtain geometric results concerning the structure of hyperbolic ends; in [2], Andersson, Barbot, Beguin and Zeghib use constant mean curvature hypersurfaces to study flat, de-Sitter and anti de-Sitter spacetimes; and, in a similar vein, in [19] and [20], Loftin uses the existence results [8] of Cheng and Yau for solutions of the Monge-Ampère equation to construct affine structures on convex projective manifolds.

This paper studies hypersurfaces of constant special Lagrangian (SL) curvature. SL curvature was introduced by the author in [29] and [25], and is defined in this paper in section 2. As its name might suggest, it is closely related to the special Legendrian structure of the unitary bundle of the ambient manifold, and is derived from the theory of Calibrated Geometries developed by Harvey and Lawson in [15]. The interest of SL curvature is twofold. Firstly, like Gaussian curvature, it is intimately related to convexity, and thus provides a natural tool in the study of convex problems (which is employed to advantage in the case of hyperbolic ends and flat conformal structures in [24]), and secondly it is regular, in the sense that a sequence of hypersurfaces of constant SL curvature only degenerates in one simple way, which can often then be excluded by geometric considerations. We underline that this extra regularity greatly simplifies the proof of strong existence results for SL curvature which remain very difficult for all other non-linear curvatures that the author is aware of. Crucially, Gaussian curvature - the number one competitor to SL curvature, as it where - only exhibits this property only when the dimension of the ambient manifold is equal to 3, where it coincides with the SL curvature anyway.

SL curvature is only defined for convex immersed hypersurfaces and depends on an angle parameter, $\theta \in [0, n\pi/2[$. We thus denote it by R_θ. We only concern ourselves with the case where $\theta \in [(n-1)\pi/2, n\pi/2[$, since it is here that R_θ interacts well with convexity (more precisely, it vanishes along the boundary of the cone of positive definite symmetric two forms and is positive in its interior). The case where $\theta > (n-1)\pi/2$ is more regular and, in general, results will be obtained for this case and then extended to the case where $\theta = (n-1)\pi/2$ by compactness. However, when $\theta = (n-1)\pi/2$, R_θ has the simplest form and the most interesting geometric properties. For example, when $n = 2$:

$$R_\theta = K^{1/2},$$

where K is the Gaussian curvature, and when $n = 3$:

$$R_\theta = (K/H)^{1/2},$$

where H is the mean curvature. In other words, in these cases, SL curvature reduces to more familier notions of curvature. In higher dimensions, $R_{(n-1)\pi/2}$ has a more complicated expression, but still exhibits the same properties: specifically, sequences of constant SL curvature hypersurfaces degenerate in exactly the same way as constant Gaussian curvature surfaces do in 3-dimensional ambient manifolds (see [17]). Alongside its close associations to symplectic geometry, it is for this reason that SL curvature should be considered as an alternative higher dimensional generalisation of Gaussian curvature, in analogy to the way in which the richer theory of symplectic geometry compared to that of volume preserving maps allows symplectic structures to be considered as alternative higher dimensional generalisations of 2-dimensional volume.

Throughout the rest of the introduction, we shall use rescaled SL curvature, $\hat{R}_\theta$, given by:

$$\hat{R}_\theta = \tan(\theta/n)R_\theta.$$

This is chosen so that the rescaled SL curvature of a horosphere in hyperbolic space equals 1, thus satisfying geometric intuition. Nevertheless, to save on multiplicative factors, throughout the rest of the paper, R_θ will be used.

We now present the main result of this paper. We recall that a smoothly immersed hypersurface is said to be (locally) convex if and only if its shape operator is everywhere positive definite. In this case, in a neighbourhood of every point, it coincides with an open subset of the boundary of a smooth convex set. A more general notion of convex immersion is provided in section 2.

Theorem 1.1. *Let M be an $(n+1)$-dimensional manifold of negative sectional curvature bounded above by -1. Let $N \subseteq M$ be a compact, convex immersed hypersurface. Suppose that the diameter of immersions homotopic to N is bounded below by some $\epsilon > 0$. Choose $\theta \in [(n-1)\pi/2[$ and $r \in]0, 1[$. Then:*

(i) *If* $\theta > (n-1)\pi/2$*, then there exists a smooth, convex, immersed submanifold* $N_{r,\theta} \in M$*, isotopic by convex immersions to* N *such that:*

$$R_\theta(N_{r,\theta}) = r.$$

(ii) *If* $\theta = (n-1)\pi/2$*, then the same result holds provided that, in addition,* N *is not homeomorphic to the sphere bundle* $S^{n-1} \times S^1$*.*

Remark. The hypotheses if (i) are satisfied if, for example, N is homotopically non-trivial and M is compact or convex co-compact.

Remark. By comparing this theorem to an earlier result, [26], of the author, we see that, when M is compact and 3-dimensional, given a two dimensional homotopy class, under very general conditions there exist infinitely many isotopy inequivalent constant curvature convex immersions within that class. There is no a-priori reason not to expect such degeneration also in higher dimensions.

This theorem follows by an application of the Perron method to the second result of this paper. We begin by introducing some notation. Let M be an $(n+1)$ dimensional manifold. Let $\Sigma = (S, i)$ be a smooth convex immersed hypersurface in M. Let N_Σ be the outward pointing normal over Σ. Let Ω be an open subset of Σ. Let $\mathsf{N}_{\partial\Omega}$ be the outward pointing normal of $\partial\Omega$ in Σ. Define $N\Omega$ and $N\partial\Omega$ by:

$$N\Omega = \{\mathsf{N}_\Sigma(p) \text{ s.t. } p \in \Omega\},$$
$$N\partial\Omega = \{\mathsf{V}_p \text{ s.t. } p \in \partial\Omega \ \& \ \langle V_p, \mathsf{N}_{\partial\Omega}(p)\rangle, \langle V_p, \mathsf{N}_\Sigma(p)\rangle \geq 0\}.$$

We define $\hat{N}\Omega$ by:

$$\hat{N}\Omega = N\Omega \cup N\partial\Omega.$$

We call $\hat{N}\Omega$ the extended normal of Ω. Ω embeds naturally as an open subset of $\hat{N}\Omega$. Let n denote the canonical immersion of $\hat{N}\Omega$ into UM.

Let $\Sigma' = (S', i')$ be another (not necessarily smooth) convex immersed hypersurface in M. We say that Σ' is a graph over $\hat{N}\Omega$ if there exists:

(i) a relatively compact open subset $U \subseteq \hat{N}\Omega$ such that $\Omega \subseteq U$;

(ii) a homeomorphism $\alpha : S' \to U$; and

(iii) a continuous function $f : U \to [0, \infty[$, such that f vanishes along ∂U, and for all $p \in S'$:

$$i'(p) = \mathrm{Exp}(f(p)(n \circ \alpha)(p)).$$

Now let M be an $(n+1)$-dimensional Hadamard manifold of sectional curvature bounded above by -1. Let Σ be a smooth, convex immersed hypersurface in M. Let $\Omega \subseteq \Sigma$ be an open subset and let $\hat{\Sigma}$ be a convex immersed hypersurface in M which is a graph over the extended normal of Σ:

Theorem 1.2. *Choose* $\theta \in [(n-1)\pi/2, n\pi/2[$. *Choose* $0 \leq R_0 < R_1 \leq 1$. *Suppose that* $\hat{R}_\theta(\Sigma) \leq R_0$ *and* $\hat{R}_\theta(\hat{\Sigma}) \geq R_1$ *in the weak (Alexandrov) sense. If* $\theta > (n-1)\pi/2$, *then, for all* $r \in [R_0, R_1]$, *there exists an immersed hypersurface* Σ_r *in* M *such that:*

(i) Σ_r *is a graph over* $\hat{N}\Omega$;

(ii) Σ_r *lies below* $\hat{\Sigma}$ *as a graph over* $\hat{N}\Omega$;

(iii) Σ_r *is smooth away from the boundary; and*

(iv) $\hat{R}_\theta(\Sigma_r) = r$.

If $\theta = (n-1)\pi/2$, *then the same result holds provided that, in addition,* $\hat{\Sigma}$ *is uniformly strictly convex.*

Remark. This result constitutes a higher dimensional generalisation of [17].

Remark. A simple modification of Theorem 1.2 yields the case where $\hat{\Sigma}$ is the finite boundary of a hyperbolic end (see [24]), and other more general objects of non-constant curvature. This yields in particular the existence results of [24] in much greater generality than is required in that paper.

Remark. This result is proven in two stages. In the first stage, the continuity method is used to obtain existence in a restricted case, and closely follows the approach towards the non-linear Dirichlet problem for functions over open subsets of $\mathbb{R}^n$ used in [4], [6] and [7]. The second stage then uses the Perron method to extend this to the general case and closely follows the work [11] of Guan and Spruck on hypersurfaces of constant Gaussian curvature in $\mathbb{R}^n$, although numerous features of SL curvature allow for a simpler exposition in our case.

Remark. The application of the continuity method itself consists of two stages: local deformation and compactness. Only the local deformation stage requires the ambient manifold to have strictly negative sectional curvature. Importantly, the compactness stage follows from a-priori estimates that are valid in any manifold.

The paper is arranged as follows:

(i) various background concepts are introduced and studied in section 2. Special Lagrangian curvature is introduced and is shown to be defined in terms of a homogeneous, concave function. The properties of convex subsets of Riemannian manifolds are studied in detail, and it is shown how mollifiers may be used to produce convex sets with certain desired properties;

(ii) in section 3, which is the most innovative part of the paper, we follow the approach of [4], [6] and [7] to determine a-priori C^2 bounds on graphs of constant special Lagrangian curvature in the case when $\theta > (n-1)\pi/2$. The key departure from [4], [6] and [7] is the use of entirely geometric means in the construction of the required barrier functions, which as a consequence interact well with the underlying geometry of the ambient manifold. Once these estimates are established, the continuity method is applied to obtain Theorem 3.22, which forms a precursor to Theorem 1.2. Special cases of particular interest - Corollaries 3.23 and 3.24 - are also proven. Finally, we remark that, although SL curvature is especially amenable to the above analysis, given the generality of the results of [4], [6] and [7], one would hope that these geometric constructions (especially Lemma 3.11) may also be of use in the study of other curvatures.

(iii) in section 4, inspired by [17], we introduce extended normals and graphs over extended normals. We then use the Perron method along with Theorem 3.22 to prove Theorem 1.2; and

(iv) in section 5, we introduce the concept of pseudo-immersions as a compactification of the space of convex immersions. These are used as an important tool in the proof of Theorem 1.1, which is carried out using the Perron method and Theorem 1.2.

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2. Preliminaries

2.1. **Immersed Submanifolds and Special Lagrangian Curvature.** Let M be a smooth Riemannian manifold. An **immersed submanifold** is a pair $\Sigma = (S, i)$ where S is a smooth manifold and $i : S \to M$ is a smooth immersion. An **immersed hypersurface** is an immersed submanifold of codimension 1. We give S the unique Riemannian metric i^*g which makes i into an isometry. We say that Σ is **complete** if and only if the Riemannian manifold (S, i^*g) is.

Let UM be the unitary bundle of M (i.e the bundle of unit vectors in TM. In the cooriented case (for example, when I is convex), there exists a unique exterior normal vector field N over i. We denote $\hat{\imath} = \mathsf{N}$ and call it the **Gauss lift** of i. Likewise, we call the manifold $\hat{\Sigma} = (S, \hat{\imath})$ the **Gauss lift** of Σ.

The special Lagrangian curvature, which is only defined for strictly convex immersed hypersurfaces, is defined as follows. Denote by $\text{Symm}(\mathbb{R}^n)$ the space of symmetric matrices over $\mathbb{R}^n$. We define $\Phi : \text{Symm}(\mathbb{R}^n) \to \mathbb{C}^*$ by:

$$\Phi(A) = \text{Det}\ (I + iA).$$

Since Φ never vanishes and $\text{Symm}(\mathbb{R}^n)$ is simply connected, there exists a unique analytic function $\tilde{\Phi} : \text{Symm}(\mathbb{R}^n) \to \mathbb{C}$ such that:

$$\tilde{\Phi}(I) = 0, \qquad e^{\tilde{\Phi}(A)} = \Phi(A) \qquad \forall A \in \text{Symm}(\mathbb{R}^n).$$

We define the function arctan : $\mathrm{Symm}(\mathbb{R}^n) \to (-n\pi/2, n\pi/2)$ by:

$$\arctan(A) = \mathrm{Im}(\tilde{\Phi}(A)).$$

This function is trivially invariant under the action of $O(\mathbb{R}^n)$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then:

$$\arctan(A) = \sum_{i=1}^{n} \arctan(\lambda_i).$$

For $r > 0$, we define:

$$\mathrm{SL}_r(A) = \arctan(r^{-1}A).$$

If A is positive definite, then SL_r is a strictly decreasing function of r. Moreover, $SL_\infty = 0$ and $SL_0 = n\pi/2$. Thus, for all $\theta \in]0, n\pi/2[$, there exists a unique $r > 0$ such that:

$$SL_r(A) = \theta.$$

We define $R_\theta(A) = r$. R_θ is also invariant under the action of $O(n)$ on the space of positive definite, symmetric matrices.

Let M be an oriented Riemannian manifold of dimension $n + 1$. Let $\Sigma = (S, i)$ be a strictly convex, immersed hypersurface in M. For $\theta \in]0, n\pi/2[$, we define $R_\theta(\Sigma)$ (the **θ-special Lagrangian curvature** of Σ) by:

$$R_\theta(\Sigma) = R_\theta(A_\Sigma),$$

where A_Σ is the shape operator of Σ.

2.2. **Properties of Special Lagrangian Curvature.** R_θ is an analytic homogenous function of order 1. Importantly:

Lemma 2.1. *For all θ, R_θ is a concave function over the set of positive definite symmetric matrices.*

Remark. This property is necessary for the application of the Perron method. In the following proof, we explicitely determine the second derivative. However, a simpler, more geometric argument may also be employed. Indeed, the function SL_r is concave. Moreover $\mathrm{SL}_r(A) \geq \theta$ if and only if $R_\theta(A) \geq r$. It follows that $R_\theta^{-1}([r, +\infty[)$ is convex for all $r > 0$, and the result follows since the Hessian of a homogeneous function is, up to a factor, the second second fundamental form of its level sets.

Proof. Define the function σ over the space of symmetric matrices by:

$$\sigma(A) = \mathrm{Arg}(\mathrm{Det}\ (\mathrm{Id} + iA)).$$

Trivially:

$$D\sigma_A(M) = \mathrm{Tr}(\mu^{-1}M),$$
$$D^2\sigma_A(M, M) = -2\mathrm{T}r(\mu^{-1}AM\mu^{-1}M),$$

where $\mu = \mathrm{Id} + A^2$. Choose $\theta \in]0, n\pi/2[$. Define the function r over the space of symmetric matrices such that:

$$\sigma(r(A)^{-1}A) = \theta.$$

Define μ_r and ϕ_r by:

$$\mu_r = \mathrm{Id} + r^{-2}A^2, \qquad \phi_r = \mathrm{Tr}(\mu_r A).$$

Using the chain rule and the formula for $D\sigma$ and $D^2\sigma$ yields:

$$D^2 r_A(M, M) = \frac{-2}{r\phi_r}\mathrm{Tr}(\mu_r^{-1}A\tilde{M}\mu_r^{-1}\tilde{M}),$$

where:

$$\tilde{M} = M - \frac{1}{\phi_r}\mathrm{Tr}(\mu_r^{-1}M)A.$$

Thus, when A is positive definite, for all M:

$$D^2 r_A(M, M) \leq 0,$$

The result follows. □

For $\theta > (n-1)\pi/2$, $R_\theta(A)$ approximates the smallest eigenvalue of A:

Lemma 2.2. *Let $\lambda_1(A)$ denote the smallest eigenvalue of the matrix A. For all $\theta \in]0, n\pi/2[$, there exists K_1 such that:*

$$R_\theta(A) \geq K_1\lambda_1(A).$$

For all $\theta > (n-1)\pi/2$, there exists K_2 such that:

$$R_\theta(A) \leq K_2\lambda_1(A).$$

Remark. Observe that the second relation is no longer valid in the case where $\theta \leq (n-1)\pi/2$.

Remark. In particular, when $\theta > (n-1)\pi/2$, constant special Lagrangian curvature yields uniform lower bounds on the principal curvatures.

Proof. Let $K_1 = R_\theta(\mathrm{Id})$. Then:

$$R_\theta(A) = \lambda_1(A)R_\theta(A/\lambda_1(A)) \geq \lambda_1(A)R_\theta(\mathrm{Id}) = K_1\lambda_1(A).$$

The first result follows. Let $\lambda_1 = \lambda_1(A)$ and $r = R_\theta(A)$. Then:

$$r^{-1}\lambda_1 \geq \arctan(r^{-1}\lambda_1) \geq \theta - (n-1)\pi/2.$$

Thus, if $K_2 := (\theta - (n-1)\pi/2)^{-1} < \infty$, then:

$$R_\theta(A) = r \leq K_2\lambda_1 = K_2\lambda_1(A).$$

The second result follows. □

The case where $\theta = (n-1)\pi/2$ is of particular interest. Here R_θ has the simplest form and the most interesting geometric properties. For example, when $n = 2$:

$$R_{\pi/2} = K^{1/2},$$

where K is the Gaussian curvature, and when $n = 3$:

$$R_\pi = (K/H)^{1/2},$$

where H is the mean curvature. However, the case where $\theta > (n-1)\pi/2$ is more regular. In the sequel, results for $\theta = (n-1)\pi/2$ are obtained by first treating this case, and then taking limits.

It is important to note that, although R_θ is more appealing geometrically, SL_r is analytically simpler. In the sequel, results for R_θ constant will often be proven for SL_r constant, which is trivially equivalent.

2.3. **Convex Conditions and Convex Immersions.** Let M be an $(n+1)$-dimensional Riemannian manifold. Let TM and $UM \subseteq TM$ be the tangent and unitary bundles respectively over M. Let $\pi : TM \to M$ be the canonical projection.

Let $\mathrm{Symm}^+(\mathbb{R}^n)$ denote the space of symmetric, positive definite matrices over $\mathbb{R}^n$. Let X be an open subset of $\mathrm{Symm}^+(\mathbb{R}^n)$. We say that X defines a homogeneous convex property if and only if:

(i) X is convex;

(ii) X is invariant under the action of $\mathrm{S}O(n)$; and

(iii) X is homogeneous in the sense that, for all $\lambda \in [1, \infty[$, $\lambda X \subseteq X$.

Let $K \subseteq M$ be a convex set. We say that K posseses the property X if and only if, for all $p \in \partial K$, and for every supporting normal N_p to K at p, there exists a smooth hypersurfaces Σ such that:

(i) Σ is an exterior tangent to K at p;

(ii) N_p is the outward pointing normal to Σ at p; and

(iii) if A is the second fundamental form of Σ at p, then $A \in X$.

Lemma 2.3. *Let X define a homogeneous convex property. Let $K, K' \subseteq M$ be convex sets. If K and K' possess the property X, then so does $K \cap K'$.*

Proof. It suffices to check the condition at $p \in \partial K \cap \partial K'$. Let N_p and N'_p be supporting normals to K and K' respectively at p. If $\mathsf{N}_p = \mathsf{N}'_p$, then $K \cap K'$ possesses the property X in the direction of $\mathsf{N}_p = \mathsf{N}'_p$. Likewise, if $\mathsf{N}_p = \mathsf{N}'_p$, then $K \cap K'$ is a single point. We thus assume that they are distinct and not colinear.

Let $\pi : \mathrm{Symm}^+(\mathbb{R}^n) \to \mathrm{Symm}^+(\mathbb{R}^{n-1})$ be the projection defined by restriction to the subspace. Define $X' = \pi(X)$. Trivially, X' is open, convex and homogeneous.

Let Σ_p and Σ'_p be smooth convex hypersurfaces at p as in the definition of possession of property X. Σ_p and Σ'_p are transverse at p. Let A and A' be the second fundamental forms at p of Σ_p and Σ'_p respectively. Define $\Gamma = \Sigma_p \cap \Sigma'_p$. Near p, Γ is a smooth submanifold. For $s, t > 0$ such that $s + t = 1$, define $\mathsf{N}_{s,t}$ by:

$$\mathsf{N}_{s,t} = s\mathsf{N}_p + t\mathsf{N}'_p.$$

Let A_Γ be the second fundamental form of Γ at p. A_Γ depends on a choice of normal vector to Γ at p:

$$A_\Gamma(\mathsf{N}_p) = \pi(A) \in X'$$

$$A_\Gamma(\mathsf{N}'_p) = \pi(A') \in X'$$

Thus, for all $s, t > 0$ such that $s + t = 1$:

$$A_\Gamma\left(\frac{1}{\|\mathsf{N}_{s,t}\|}\mathsf{N}_{s,t}\right) = \frac{1}{\|\mathsf{N}_{s,t}\|}\left(sA_\Gamma(\mathsf{N}_p) + tA_\Gamma(\mathsf{N}'_p)\right) \in X'.$$

Thus, for all $s, t > 0$ such that $s + t = 1$, there exists an immersed hypersurface $\Sigma_{s,t}$ such that:

(i) $\Sigma_{s,t}$ is an exterior tangent to $K \cap K'$ at p;

(ii) $\mathsf{N}_{s,t}/\|\mathsf{N}_{s,t}\|$ is the outward pointing normal to $\Sigma_{s,t}$ at p; and

(ii) if $A_{s,t}$ is the second fundamental form of $\Sigma_{s,t}$ at p, then $A_{s,t} \in X$.

Since the set of supporting normals to $K \cap K'$ at p is a convex set whose boundary is contained in the union of the sets of supporting normals to K and K' at p, the result follows. □

Let $K \subseteq M$ be a convex set. For $\epsilon > 0$, we say that K is ϵ-convex if and only if, for every $p \in \partial K$, for every supporting normal N_p to K at p, and for every $0 < \epsilon' < \epsilon$, there exists a smooth convex hypersurface Σ such that:

(i) Σ is an exterior tangent to K at p;

(ii) N_p is the outward pointing normal to Σ; and

(iii) the second fundamental form of Σ at p is bounded below by $\epsilon'\mathrm{Id}$.

This is a homogeneous convex property, and is thus preserved under the intersection of convex sets.

Choose $\theta \in [0, n\pi/2[$ and $r > 0$. We say that $R_\theta(\partial K) \geq r$ in the weak sense if and only if, for every $p \in \partial K$, for every supporting normal N_p to K at p, and for every $0 < r' < r$, there exists a smooth convex hypersurface Σ such that:

(i) Σ is an exterior tangent to K at p;

(ii) N_p is the outward pointing normal to Σ; and

(iii) $R_\theta(\Sigma)(p) \geq r'$.

Since is is also a homogeneous convex property, Lemma 2.3 yields:

Lemma 2.4. *Choose $\theta \in [0, n\pi/2[$ and $r > 0$. Let $K, K' \subseteq M$ be convex sets. If $R_\theta(\partial K), R_\theta(\partial K') \geq r$ in the weak sense, then $R_\theta(\partial K \cap K') \geq r$ in the weak sense.*

If $K \subseteq M$ is a convex set, and $U \subseteq \partial K$ is an open subset of the boundary, let $\mathcal{N}(U)$ denote the set of supporting normals to K over U. Let $(N, \partial N)$ be a compact n-dimensional manifold with boundary. A convex immersion of N into M is a pair $(\varphi, \hat{\varphi})$ where:

(i) $\varphi : N \to M$ and $\hat{\varphi} : N \to UM$ are $C^{0,1}$ mappings such that $\pi \circ \hat{\varphi} = \varphi$; and

(ii) for every $p \in N$, there exists a convex set $K \subseteq M$ such that $\varphi(p) \in \partial K$ and neighbourhoods $U \subseteq N$ and $V \subseteq \partial K$ of p and $\varphi(p)$ respectively such that $\hat{\varphi}$ restricts to a homeomorphism from U to $\mathcal{N}(U)$.

In the sequel, we will denote the convex immersion simply by φ. ϵ-convex immersions are defined in an analogous manner.

Lemma 2.5. *Suppose that $n \geq 2$ and $\partial N \neq \emptyset$. Let $K \subseteq M$ be a convex subset. Let $\varphi : N \to M$ be a convex immersion. Suppose there exists an open subset $U \subseteq N$ and a point $p \in \partial K$ such that:*

(i) *$\varphi(U) = \partial K \setminus \{p\}$; and*

(ii) *$\varphi(\partial U) = \{p\}$.*

Then $\hat{\varphi}$ defines a homeomorphism between N and $\mathcal{N}(\partial K)$.

Proof. Choose $p \in \partial U$. Let $U' \subseteq N$ and $K' \subseteq M$ be a neighbourhood of p in N and a convex subset of M respectively as in the definition of convex immersions. The complement of $\varphi^{-1}(\{p\})$ in U' has only one connected component. However:

$$\partial(U \cap U') \subseteq \varphi^{-1}(\{p\}) \cap U'.$$

Since U is not contained in $\varphi^{-1}(\{p\})$ it follows that:

$$U' \setminus \varphi^{-1}(\{p\}) \subseteq U.$$

In particular $N = U \cup U'$. $\hat{\varphi}$ therefore defines a covering map from N to $\mathcal{N}(\partial K)$. Since $n \geq 2$ and the latter is homeomorphic to an n-dimensional sphere, $\hat{\varphi}$ is a homeomorphism, and the result follows. □

Finally we recall the following Geometric Maximum Principle.

Lemma 2.6. *Let M be a Riemannian manifold and let $\Sigma = (S, i)$ and $\Sigma' = (S', i')$ be C^0 convex, immersed hypersurfaces in M. For $\theta \in]0, n\pi/2[$, let R_θ and R'_θ be the θ-special Lagrangian curvatures of Σ and Σ' respectively. If $p \in S$ and $p' \in S'$ are such that $q = i(p) = i'(p')$, and Σ' is an interior tangent to Σ at q, then:*

$$R_\theta(p) \leq R'_\theta(p').$$

Proof. See [24]. □

2.4. **Distance Functions.** Let M be a Riemannian manifold. Let $p \in M$ be a point. Let $\Sigma \subseteq M$ be a strictly concave, smooth, immersed hypersurface passing through p. Let d_Σ be the signed distance in M to Σ.

Lemma 2.7. *There exists a neighbourhood, U of p in M such that d_Σ is concave in U.*

Proof. For $t \in \mathbb{R}$, let $\Sigma_t = d_\Sigma^{-1}(\{t\})$. For U a sufficiently small neighbourhood of p and for t small, the intersection of Σ_t with U is smooth and concave. For all t, let N_t and II_t be the unit normal vector and the second fundamental form respectively of Σ_t. Then, for all X tangent to Σ_t:

$$\begin{aligned} \mathrm{Hess}(d_\Sigma)(X,X) &= A_t(X,X), \\ \mathrm{Hess}(d_\Sigma)(X,\mathsf{N}) &= 0, \\ \mathrm{Hess}(d_\Sigma)(\mathsf{N},\mathsf{N}) &= 0. \end{aligned}$$

The result follows. □

Let $U \subseteq M$ be an open subset. Let Σ be a hypersurface in M. Let N and II^Σ be the unit normal and the second fundamental form respectively of Σ. Let Hess^Σ be the Hessian for smooth functions defined on Σ. Trivially, we obtain:

Lemma 2.8. *Let $\phi : U \to \mathbb{R}$ be smooth. Then:*

$$\mathrm{Hess}^\Sigma(f) = \mathrm{Hess}(f) - df(\mathsf{N})II^\Sigma.$$

Corollary 2.9. *Suppose that Σ is convex and $\langle \nabla\phi, \mathsf{N}\rangle \geq 0$. If ϕ is concave as a function over U, then it is also concave as a function over $\Sigma \cap U$.*

Remark. Let Σ be a convex surface with smooth boundary. Let $p \in \partial\Sigma$ and Σ' be a concave surface tangent to $\partial\Sigma$ at p such that Σ locally lies in its exterior near p. Let $d_{\Sigma'}$ now be the distance function to Σ'. We see that $d_{\Sigma'}$ acts as a barrier for Laplacians derived from the Hessian of Σ. This will play a central rôle later.

Let d_p be the distance to p in M. For all r, let Σ_r be the sphere of radius r about p. We recall:

Lemma 2.10. *If X is tangent to Σ_r and N is the unit exterior normal to Σ_r, then, near p:*

$$\begin{aligned} \mathrm{Hess}(d_p)(X,X) &= r^{-1}(1+O(r^2))\langle X,X\rangle, \\ \mathrm{Hess}(d_p)(X,\mathsf{N}) &= 0, \\ \mathrm{Hess}(d_p)(\mathsf{N},\mathsf{N}) &= 0. \end{aligned}$$

Corollary 2.11.

$$\mathrm{Hess}(d_p^2/2) = \langle\cdot,\cdot\rangle + O(r_p^2).$$

2.5. **Regularising Convex Sets.** We recall the definition of mollifiers:

Definition 2.12. *Let M be a Riemannian manifold. A mollifier of M is a smooth, positive function $\varphi : TM \to [0,+\infty[$ such that:*

(i) *for all $p \in M$:*

$$\int_{T_pM} \varphi \mathrm{Vol}_p = 1,$$

where Vol_p is the volume form of T_pM;

(ii) *$\varphi(v_p) = 0$ for $\|v_p\| > 1$; and*

(iii) *φ is preserved by parallel transport of M.*

We construct mollifiers as follows. Let $\psi : [0,\infty[\to [0,\infty[$ be a smooth, positive function such that:

$$t \leq 1/2 \Rightarrow \psi(t) = 1, \qquad t \geq 1 \Rightarrow \psi(t) = 0.$$

Let $\lambda > 0$ be a positive constant and define $\varphi : TM \to [0,\infty[$ by:

$$\varphi(v_p) = \lambda\psi(\|v_p\|).$$

φ is trivially preserved by parallel transport. If λ is chosen such that the integral of φ over any (and thus every) tangent space is equal to 1, then φ is a mollifier.

If φ is a mollifier, we define $(\varphi_\epsilon)_{\epsilon>0} : TM \to [0,+\infty[$ by:

$$\varphi_\epsilon(v_p) = \epsilon^{-n}\varphi(\epsilon^{-1}v_p).$$

Using mollifiers, we obtain:

Lemma 2.13. *Let M be a Riemannian manifold. Choose $\theta \in [(n-1)\pi/2, n\pi/2[$ and $r > 0$. Let $\Sigma \subseteq M$ be a compact, convex immersed hypersurface such that $R_\theta(\Sigma) \geq r$ in the weak sense. If $\theta > (n-1)\pi/2$, then, for all $\delta > 0$ there exists a smooth, convex hypersurface Σ' (which may be chosen arbitrarily close to Σ in the C^0 sense) such that:*

$$R_\theta(\Sigma') \geq r - \delta.$$

If $\theta = (n-1)\pi/2$, the the same result holds provided that the second fundamental form of Σ is bounded below in the weak sense.

Remark. Mollification preserves homogeneous convex conditions up to a small error. This is the content of the proof.

Proof. Let φ be a mollifier of M. Let $\mathrm{Exp} : TM \to M$ be the exponential map of M. We work locally and therefore assume that there exists a unique geodesic between any two points, $x, y \in M$. Let $\tau_{y,x}$ be parallel transport from x to y along this geodesic.

Define $f : M \to \mathbb{R}$ by:

$$f(p) = d(p, \Sigma).$$

This function is convex in a small neighbourhood of Σ. We restrict to this neighbourhood for the rest of the proof. f is a locally $C^{1,1}$ function away from Σ. In particular, $\mathrm{Hess}(f)$ is measurable and bounded in every compact subset of the complement of Σ. For $\epsilon > 0$, define $f_\epsilon : M \to \mathbb{R}$ by:

$$f_\epsilon(p) = \int_{T_pM} (f \circ \mathrm{Exp})(V_p)\varphi_\epsilon(V_p)\mathrm{Vol}_p.$$

Trivially, $(f_\epsilon)_{\epsilon>0} \to f$ in the C^1 sense as $\epsilon \to 0$. It remains to show that the second derivative of f_ϵ has the desired properties for ϵ sufficiently small.

We construct an approximation for the Hessian. For $\epsilon > 0$, define $A_\epsilon \in \Gamma(\mathrm{Symm}(TM))$ by:

$$A_\epsilon(p) = \int_{T_pM} (\mathrm{Exp}_p^*\mathrm{Hess}f)\varphi_\epsilon\mathrm{Vol}_p.$$

For $t > 0$, let Σ_t be the level hypersurface of f with value t. Let $\delta_1 > 0$ be small. There exists $T_0 > 0$ such that, for $t < T_0$, $R_\theta(\Sigma_t) > r - \delta_1$ (this is the stage that requires the supplementary condition when $\theta = (n-1)\pi/2$). Let $p \in M$ be such that $0 < f(p) < T_0$. Let $X_p \in T_pM$ be a unit vector orthogonal to ∇f at p. Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic such that $\partial_t\gamma(0) = X_p$. For all t, define the vector field X_t, such that, for all $V \in T_{\gamma(t)}M$:

$$X_t(\mathrm{Exp}_{\gamma(t)}(V)) = D\mathrm{Exp}_{\gamma(t)} \cdot \partial_t\gamma(t).$$

For V in T_pM, we define $c_V : \mathbb{R} \to M$ by:

$$c_V(t) = (\mathrm{Exp} \circ \tau_{\gamma(t),\gamma(0)})(V).$$

By Taylor's Theorem:

$$\begin{aligned}(f \circ c_V)(t) &= (f \circ c_V)(0) + df(\partial_t c_V)(0)t + \int_0^t (t-s)\partial_t^2(f \circ c_V)(s)ds\\ &= (f \circ c_V)(0) + df(\partial_t c_V)(0)t\\ &\qquad + \int_0^t (t-s)\mathrm{Hess}(f)(\partial_t c_V, \partial_t c_V)ds + \int_0^t (t-s)df(\nabla_{\partial_t c_V}\partial_t c_V)ds.\end{aligned}$$

Let $\eta > 0$ be small. Trivially, $c_V \to \gamma$ in the C^∞ sense as $\|V\| \to 0$. Thus, since $\mathrm{Hess}(f)$ is bounded, there exists $\epsilon_0 > 0$ such that, for $\|V\| < \epsilon_0$ and for all $t \in]-r, r[$:

$$(f \circ c_V)(t) \geq (f \circ c_V)(0) + df(\partial_t c_V)(0)t + \int_0^t (t-s)(\mathrm{Hess}(f) - \eta)_{c_V(s)}(X_s, X_s)ds.$$

For $\epsilon < \epsilon_0$, since $\varphi\mathrm{Vol}_p$ is invariant under parallel transport:

$$\begin{aligned}(f_\epsilon \circ \gamma)(t) &= \int_{T_pM} \varphi_\epsilon(V)(f \circ \mathrm{Exp} \circ \tau_{\gamma(t),\gamma(0)})(V)\mathrm{Vol}_p \\ &= \int_{T_pM} \varphi_\epsilon(V)(f \circ c_V)(t)\mathrm{Vol}_p \\ &\geq \int_{T_pM} \varphi_\epsilon(V)((f \circ c_V)(0) + tdf(\partial_t c_V)(0))\mathrm{Vol}_p \\ &\qquad + \int_0^t (t-s)(\mathrm{Hess}(f) - \eta)_{c_V(s)}(X_s, X_s)ds\mathrm{Vol}_p.\end{aligned}$$

However:

$$\begin{aligned}(f_\epsilon \circ \gamma)(0) &= \int_{T_pM} \varphi_\epsilon(V)(f \circ \mathrm{Exp} \circ \tau_{\gamma(t),\gamma(0)})(V)\mathrm{Vol}_p|_{t=0} \\ &= \int_{T_pM} \varphi_\epsilon(V)(f \circ c_V)(0)\mathrm{Vol}_p\end{aligned}$$

Moreover:

$$\begin{aligned}df_\epsilon(\partial_t\gamma(0)) &= \partial_t \int_{T_pM} \varphi_\epsilon(V)(f \circ \mathrm{Exp} \circ \tau_{\gamma(t),\gamma(0)})(V)\mathrm{Vol}_p|_{t=0} \\ &= \int_{T_pM} \varphi_\epsilon(V)(\partial_t(f \circ c_V)(t)|_{t=0})\mathrm{Vol}_p \\ &= \int_{T_pM} \varphi_\epsilon(V)df(\partial_t c_V(0))\mathrm{Vol}_p.\end{aligned}$$

Finally, by definition:

$$A_\epsilon(\gamma(s))(\partial_t\gamma, \partial_t\gamma) = \int_{T_pM} \varphi_\epsilon(V)\mathrm{Hess}(f-\eta)_{c_V(s)}(X_s, X_s)ds)\mathrm{Vol}_p.$$

Thus, for $\epsilon < \epsilon_0$:

$$(f_\epsilon \circ \gamma)(t) \geq (f_\epsilon \circ \gamma)(0) + df_\epsilon(\partial_t\gamma(0))t + \int_0^t (t-s)(A_\epsilon - \eta)(\gamma(s))(\partial_t\gamma, \partial_t\gamma)ds.$$

Consequently:

$$\mathrm{Hess}(f_\epsilon)(p) \geq A_\epsilon - \eta.$$

Let E_p be the orthogonal complement of ∇f_ϵ at p. Let E be the distribution obtained by parallel transport of E_p along geodesics leaving p. Let $A|_E$ be the restriction of $\mathrm{Hess}(f)$ to E. Since f_ϵ tends to f in the C^1 sense, and since $\mathrm{Hess}(f)$ is bounded, for q sufficiently close to p.

$$R_\theta(A(q)|_E) \geq r - \delta/2.$$

However, R_θ is a concave function. Thus, for ϵ sufficiently small:

$$R_\theta(A_\epsilon(p)|_E) \geq r - \delta/2.$$

And so:

$$R_\theta(\|\nabla f_\epsilon\|^{-1}\mathrm{Hess}(f_\epsilon)|_E) \geq r - \delta.$$

Since these estimates may be calculated locally uniformly, the result follows by taking an appropriate level subset of f_ϵ, for ϵ sufficiently small. □

3. The Continuity Method

3.1. **First Order Control.** Let M be a Riemannian manifold. Let Exp denote the exponential mapping of M. Let H be a smooth convex hypersurface. Let N_H be the outward pointing unit normal over H. Let $\Omega \subseteq H$ be an open set. We will say that a $C^{0,1}$ hypersurface Σ is a graph over Ω if and only if there exists a $C^{0,1}$ function $f : \overline{\Omega} \to [0, +\infty[$ and a homeomorphism $\varphi : \overline{\Omega} \to \Sigma$ such that:

(i) f vanishes along $\partial\Omega$ (i.e. $\partial\Sigma = \partial\Omega$); and
(ii) for all $p \in \Omega$:

$$\varphi(p) = \mathrm{Exp}_p(f(p)\mathsf{N}_H(p)).$$

We refer to f as the graph function of Σ. Consider the family of graphs over Ω. We define the partial order ">" on this family such that if Σ and Σ' are two graphs over Ω and f and f' are their respective graph functions, then:

$$\Sigma > \Sigma' \Leftrightarrow f(p) > f'(p) \text{ for all } p \in \Omega.$$

Since $\partial\Omega$ is smooth, for all $p \in \partial\Omega$, the set of supporting hyperplanes to $\partial\Omega$ at p is parametrised by $\mathbb{R}$. Supporting hyperplanes may be locally considered as graphs over Ω, and we obtain an analogous partial order on this set, which we also denote by $>$.

Choose $\theta \in](n-1)\pi/2, n\pi/2[$. Suppose now that $R_\theta(H) = R_0$ where $R_0 \geq 0$ is constant. Let $\hat{\Sigma}$ be a $C^{0,1}$ convex hypersurface which is a graph over Ω. Suppose moreover that $R_\theta(\hat{\Sigma}) \geq R_1$ in the weak sense. Let $R_0 < r_1 < r_2 < \cdots < r_\infty < R_1$ be a sequence of positive real numbers and let $(\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of graphs over Ω such that:

(i) for all $n \in \mathbb{N}$, Σ_n is a smooth convex hypersurface such that $R_\theta(\Sigma_n) = r_n$;
(ii) for all $n \in \mathbb{N}$, $\Sigma_n < \hat{\Sigma}$; and
(iii) for all $n > m$, $\Sigma_n > \Sigma_m$.

Lemma 3.1. *There exists a $C^{0,1}$ convex hypersurface with boundary Σ_0 which is C^∞ in its interior such that:*

(i) *Σ_0 is a graph over Ω;*
(ii) *$\hat{\Sigma} > \Sigma_0$; and*
(iii) *The sequence of graph functions $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the $C^{0,\alpha}$ sense over $\overline{\Omega}$ and in the C^∞_{loc} sense over Ω.*

Moreover, if Σ_0 *is smooth up to the boundary:*

(iv) *for every* $p \in \partial\Sigma$, $T_p\hat{\Sigma} > T_p\Sigma_0$.

Proof. For all n, let f_n be the graph function of Σ_n. $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded above by the graph function of $\hat{\Sigma}$. Since $(f_n)_{n\in\mathbb{N}}$ is strictly increasing, there exists f_∞ to which this sequence converges pointwise. For all $n \in \mathbb{N}\cup\{\infty\}$, define U_n by:

$$U_n = \left\{\mathrm{Exp}_p(t\mathsf{N}_H(p)) \text{ s.t. } p \in \overline{\Omega} \text{ and } 0 \leq t \leq f_n(p)\right\}.$$

Trivially, for all $i < j$, $U_i \subseteq U_j$, and:

$$U_\infty = \bigcup_{i=1}^{\infty} U_n.$$

Since U_n is convex (away from H) for all n, so is U_∞. Moreover, the supporting hyperplanes of U_∞ are transverse to the normal geodesics leaving H. Indeed, let $p \in \partial U_\infty$ be a point where the supporting hyperplane is not transverse to the normal geodesic leaving H. Since $\hat{\Sigma}_0 > \Sigma_n$ for all n, $p \notin \partial\Omega$. Let $q \in \Omega$ be such that $\mathrm{Exp}(f_\infty(q)\mathsf{N}(q)) = p$. Let γ be the geodesic segment joining q to p. γ lies inside U_∞. Moreover, it is tangent to ∂U_∞ at p. Consequently, it lies in the boundary of U_∞ and thus defines a continuous path in ∂U_∞ from p to q which does not intersect $\partial\Omega$. This is absurd.

Since ∂U_∞ is the graph of f_∞, it follows that f_∞ is $C^{0,1}$ and that $(f_n)_{n\in\mathbb{N}}$ converges to f_∞ in the $C^{0,\alpha}$ sense. This proves (i) and the first half of (iii).

Let $\epsilon > 0$. Let $p \in \Omega$ be such that $d(p, \partial\Omega) > \epsilon$. For all $n \in \mathbb{N}\cup\{\infty\}$, define $p_n \in \Sigma_n$ by:

$$p_n = \mathrm{Exp}(f_n(p)\mathsf{N}_H(p)).$$

Trivially $(p_n)_{n\in\mathbb{N}}$ converges to p_∞. Choose $\epsilon > 0$. For all n, let B_n be the ball of radius ϵ about p_n in Σ_n. By Theorem 1.4 of [25], for ϵ sufficiently small, there exists an immersed hypersurface Σ'_∞ containing p_∞ such that (B_n, p_n) subconverges to $(\Sigma'_\infty, p_\infty)$ in the C^∞ pointed Cheeger/Gromov sense. Trivially, $\Sigma'_\infty \subseteq \Sigma_\infty$. Thus every subsequence of $(f_n)_{n\in\mathbb{N}}$ subconverges in the C^∞_{loc} sense to f_∞. This proves the second half of (iii).

In order to prove (ii), suppose the contrary. Then Σ_∞ intersects $\hat{\Sigma}$ non trivially at an interior point, p, say. Since $\hat{\Sigma} \geq \Sigma_\infty$, Σ_∞ is an interior tangent to $\hat{\Sigma}$ at this point, which is absurd by the Geometric Maximum Principal (Lemma 2.6).

(iv) Follows in a similar manner from the Geometric Maximum Principal, and this concludes the proof. □

Let N_n be the unit normal vector field over Σ_n. Since $\partial\Omega$ is smooth, for all $n \in \mathbb{N}\cup\{\infty\}$, Σ_n has only one supporting hyperplane at any point of $\partial\Omega$. N_n therefore extends continuously to $\partial\Omega$.

By compactness, there exists $\epsilon > 0$ such that, for all $p \in \partial\Omega$ and for all q in $B_\epsilon(p)$, there exists a unique geodesic joining p to q. For such p and q, let $\tau_{q,p}$

denote parallel transport from p to q along this geodesic. The following uniform modulus of continuity is of subtle importance in the sequel:

Lemma 3.2. *There exists a continuous function $\delta : [0,\infty[\to [0,\infty[$ such that for all $n \in \mathbb{N}$, for all $p \in \partial\Omega$ and for all $q \in B_\epsilon(p)$, if N' is a supporting normal to Σ_n at p, then:*

$$\|\tau_{q,p}\mathsf{N}_n(p) - \mathsf{N}'\| < \delta(d(p,q)).$$

Proof. The normal to a convex set is continuous whereever it is uniquely defined. Likewise, for a sequence of convex sets converging towards a limit, the normal converges at any point in the limit where it is uniquely defined. Uniformity of convergence follows by compactness, and the result follows. □

3.2. **Constructing Barriers I.** Let M be an $(n+1)$-dimensional manifold. Let $H \subseteq M$ be a smooth convex hypersurface such that:

$$R_\theta(H) = R_0,$$

where R_0 is constant. Let $\Omega \subseteq H$ be a bounded open subset of H with smooth boundary. Let $\hat{\Sigma} \subseteq M$ be a convex hypersurface such that $\partial\hat{\Sigma} = \partial\Omega =: \Gamma$ and such that $R_\theta(\hat{\Sigma}) \geq R_1$ in the weak sense. Suppose that Γ is strictly convex as a subset of M with respect to the outward pointing normal to Γ in $\hat{\Sigma}$.

For N a normal vector to Γ, let $A_\Gamma(\mathsf{N})$ be the second fundamental form of Γ in the direction of N. Thus, if X and Y are vector fields tangent to Γ:

$$A_\Gamma(\mathsf{N})(X,Y) = -\langle \nabla_X Y, \mathsf{N}\rangle.$$

Proposition 3.3. *Let $p \in \Gamma$. Let N_0 be the outward pointing normal to H at p. Let N_1 be the outward pointing normal to Γ in $\hat{\Sigma}$ at p. For $s,t \in [0,1]$ such that $s+t=1$, $A_\Gamma(s\mathsf{N}_0 + t\mathsf{N}_1)$ is strictly positive definite.*

Remark. Unit vectors colinear to vectors of the form $s\mathsf{N}_1 + t\mathsf{N}_1$ for $s,t \geq 0$ will be said to lie between the outward normals of H and $\partial\hat{\Sigma}$.

Remark. In particular, $\partial\Omega$ is strictly convex as a subset of M with respect to any normal lying between the normals of H and $\partial\hat{\Sigma}$

Proof. By definition, $A_\Gamma(\mathsf{N}_0)$ and $A_\Gamma(\mathsf{N}_1)$ are both strictly positive definite. The result follows by convexity of the set of positive definite quadratic forms. □

Corollary 3.4. *The normal to $\hat{\Sigma}$ at p points above H.*

Proof. Otherwise, if $\hat{\mathsf{N}}$ is the outward pointing normal to $\hat{\Sigma}$ at p, then $-\hat{\mathsf{N}}$ lies between the normals of H and $\partial\hat{\Sigma}$. Γ is therefore strictly concave with respect to $\hat{\mathsf{N}}$. This is absurd, since $\hat{\Sigma}$ is strictly convex. □

Let $p \in \partial\Omega$. Let N_p be a normal vector to Γ at p lying between the outward normals of H and $\partial\hat{\Sigma}$. Let $\lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $A_\Gamma(\mathsf{N}_p)$. We define

$\mu(p, \mathsf{N}_p, r, \theta) \in]0, \infty]$ by:

$$\mu(p, \mathsf{N}_p, r, \theta) = \sup\{m > 0 \text{ s.t. } SL_r(\lambda_1, \ldots, \lambda_{n-1}, m) < \theta\}.$$

μ is continuous in p, N_p, r and θ. Suppose that $\mu(p, \mathsf{N}_p, r, \theta) = +\infty$. We aim to construct a barrier for hypersurfaces of constant special Lagrangian curvature equal to r whose boundary is Γ and whose normal at p is N_p.

Near p, let N'_H be the parallel transport of the upwards pointing normal of H at p. Define the set $\hat{\Omega} \subseteq M$ near p by:

$$\hat{\Omega} = \{\text{Exp}(t\mathsf{N}'_H(q)) \text{ s.t. } t \in -]\epsilon, \epsilon[, q \in \Omega\}.$$

$\hat{\Omega}$ may be considered as the solid vertical cylinder over Ω. Define the real valued function d_H and d_Ω over a neighbourhood of p by:

$$d_H(q) = d(q, H), \qquad d_\Omega(q) = d(q, \partial\hat{\Omega}),$$

where d_Ω is chosen to be positive inside Ω. Observe that $(\nabla d_H, \nabla d_\Omega)$ forms an orthonormal basis of the space of normal vectors to Γ at p. Let $K > 0$ be such that $\nabla d_H - K\nabla d_\Omega$ is parallel to N_p. Define the real valued function Φ_0 in a neighbourhood of p by:

$$\Phi_0 = d_H - Kd_\Omega.$$

Since N_p lies between the normals to H and $\partial\hat{\Sigma}$ at p, by Proposition 3.3, there exists a strictly convex hypersurface H' which is a strict exterior tangent to Γ at p such that:

$$T_pH' = T_p\partial\Omega \oplus \langle \mathsf{N}_p \rangle.$$

Let $A_{H'}$ be the second fundamental form of H'. We may choose H' such that $\|A_{H'}\|$ is arbitrarily small at p. We define $d_{H'}$ by:

$$d_{H'}(q) = d(q, H').$$

For any two functions f and g with non-colinear derivatives at p, define the the $(n-2)$-dimensional distribution $E(f, g)$ near p by:

$$E(f, g) = \langle \nabla f, \nabla g \rangle^{\perp}.$$

Let $e_1, \ldots, e_{n-1}$ be an orthonormal basis for $T_p\Gamma$ with respect to which $A_\Gamma(\mathsf{N}_p)$ is diagonal. We extend this to a local frame in TM such that, at p, for all X and all i:

$$\nabla_X e_i = -\text{Hess}(d_{H'})(e_i, X)\nabla d_{H'} - (1 + K^2)^{-1}\text{Hess}(\Phi_0)(e_i, X)\nabla\Phi_0.$$

Define the distribution E near p to be the span of $e_1, \ldots, e_{n-1}$.

Proposition 3.5. *If D represents the Grassmannian distance between two $(n-2)$- dimensional subspaces then:*

$$D(E, E(\Phi_0, d_{H'})) = O(d_p^2).$$

Proof. At p:

$$\langle \nabla d_{H'}, \nabla\Phi\rangle = 0.$$

Thus, for every vector X at p, and for all i, by definition of e_i:

$$\begin{aligned}\langle \nabla_X e_i, \nabla d_{H'}\rangle &= -\mathrm{Hess}(d_{H'})(e_i, X)\\ &= -\langle e_i, \nabla_X\nabla d_{H'}\rangle\\ \Rightarrow X\langle e_i, \nabla d_{H'}\rangle &= 0.\end{aligned}$$

Likewise:

$$X\langle e_i, \nabla\Phi\rangle = 0.$$

The result now follows. □

For any smooth function f and any non-negative function l, we define $\mathrm{SL}'_r(f, l, E)$ by:

$$\mathrm{SL}'_r(f, l, E) = \sum_{i=1}^{n-1} \arctan\left(\frac{1}{r\sqrt{1+l^2}}\lambda_i(f, E)\right),$$

where $(\lambda_i(f, E))_{i\leq i\leq(n-1)}$ are the eigenvalues of the restriction of $\mathrm{Hess}(f)$ to E.

Proposition 3.6. *Let f be such that $f(p), \nabla f(p) = 0$ and the restriction of $\mathrm{Hess}(f)$ to H' is positive definite. There exists a function x such that $x(p)$, $\mathrm{Hess}(x)(p) = 0$ and:*

$$\mathrm{SL}'_r(\Phi_0 + x(d_{H'} - f), \|\nabla\Phi_0\|, E) \leq \theta - \pi/2 + O(d_p^2).$$

Proof. By definition of Φ_0 and N_p, at p:

$$\mathrm{SL}'_r(\Phi_0, \|\nabla\Phi_0\|, E) \leq \theta - \pi/2.$$

The Hessian of xf vanishes at p. Likewise, the Hessian of the second order term $xd_{H'}$ vanishes on $(\nabla d_{H'})^\perp$ at p and thus so does its restriction to E. It follows that $x(d_{H'} - f)$ does not affect SL'_r at p. Thus, for all x, at p:

$$\mathrm{SL}'_r(\Phi_0 + x(d_{H'} - f), \|\nabla\Phi_0\|, E) \leq \theta - \pi/2.$$

Denote $l = \sqrt{1 + \|\nabla\Phi_0\|^2}$. For $1 \leq i \leq n-1$, define μ_i by:

$$\mu_i = \frac{(rl)^{-1}}{1 + (rl)^{-2}\lambda_i^2}.$$

Define A and B by:

$$A = \sum_{i=1}^{n-1}\mu_i f_{;ii}, B = \sum_{i=1}^{n-1}\mu_i d_{H';ii}.$$

Since $f_{;ij}$ is positive definite, $A > 0$. Likewise, since H' is concave, $B < 0$. Define the vectors X and Y at p by:

$$\begin{aligned}X &= \nabla\mathrm{SL}'_r(\Phi, \|\nabla\Phi_0\|, E),\\ Y &= \nabla\mathrm{SL}'_r(\Phi + x(d_{H'} - f), \|\nabla\Phi_0\|, E).\end{aligned}$$

Denote $P = x(d_{H'} - f)$. At p:

$$\mathrm{Hess}(P) = \nabla x \otimes \nabla d_{H'} + \nabla d_{H'} \otimes \nabla x.$$

At p, for all i, by definition, $\langle e_i, \nabla d_{H'}\rangle = 0$. Likewise $\langle \nabla\Phi, \nabla d_{H'}\rangle = 0$. Thus, recalling the formula for ∇e_i:

$$\begin{aligned}X\mathrm{Hess}(P)(e_i,e_j) &= (\nabla_X \mathrm{Hess}(P))(e_i,e_j) + \mathrm{Hess}(P)(\nabla_X e_i, e_j) + \mathrm{Hess}(P)(e_i, \nabla_X e_j)\\ &= (\nabla_X \mathrm{Hess}(P))(e_i,e_j) - \mathrm{Hess}(d_{H'})(X,e_i)x_{;j} - \mathrm{Hess}(d_{H'})(X,e_j)x_{;i}.\end{aligned}$$

Extend $(e_i)_{1\leq i\leq n-1}$ to a basis for T_pM by defining:

$$e_0 = \mathsf{N}_p, \qquad e_n = \nabla d_{H'}.$$

Then, with respect to this basis:

$$(Y-X)_k = -(A-B)x_{;k} - 2\sum_{i=i}^{n-1}\mu_i x_{;i} f_{;ik} + N_{ik}x_{;k},$$

where $N = O(\delta)$. Consider the linear map, M, given by:

$$(MV)_k = (A-B)V_k + 2\sum_{i=1}^{n-1}\mu_i f_{;ik} V_i.$$

Suppose that $MV = 0$, then:

$$\begin{aligned}&\sum_{k=1}^{n-1}(MV)_k\mu_k V_k = 0\\ \Rightarrow &\sum_{k=1}^{n-1}(A-B)\mu_k V_k^2 + 2\sum_{i,j=1}^{n-1}(\mu_i V_i)(\mu_j V_j) f_{;ik} = 0.\end{aligned}$$

Since $(A-B) > 0$ and $f_{;ij}$ is positive definite, it follows that:

$$V_k = 0 \text{ for all } 1 \leq k \leq n-1.$$

This in turn yields:

$$(A-B)V_0 = (A-B)V_n = 0.$$

And so $V = 0$. It follows that M is invertible. There therefore exists x such that, at p:

$$\nabla \mathrm{SL}'_r(\Phi_0 + x(d_{H'} - f), \|\nabla\Phi_0\|) = 0.$$

The result follows. □

For $M > 0$, we define Φ by:

$$\Phi = \Phi_0 + x(d_{H'} - f) + Md_{H'}^2.$$

Proposition 3.7. *If D represents the Grassmannian distance between two $(n-2)$-dimensional subspaces then:*

$$D(E(\Phi_0, d_{H'}), E(\Phi, d_{H'})) = O(d_p^2) + O(d_{H'}).$$

Proof. Since xf is of order 3 at p:

$$\nabla\Phi = \nabla\Phi_0 + (x + 2Md_{H'})\nabla d_{H'} + O(d_p^2) + O(d_{H'}).$$

Thus:

$$\langle \nabla\Phi, \nabla d_{H'}\rangle = \langle \nabla\Phi_0 + O(d_p^2) + O(d_{H'}), \nabla d_{H'}\rangle,$$

where $\langle\cdot,\cdot\rangle$ here represents the subspace generated by two vectors. The result follows. $\square$

Corollary 3.8. *If D represents the Grassmannian distance between two $(n-2)$-dimensional subspaces then:*

$$D(E, E(\Phi, d_{H'})) = O(d_p^2) + O(d_{H'}).$$

Proof. This follows by the triangle inequality and Proposition 3.5. $\square$

Proposition 3.9. *Suppose that $M\epsilon^2 < 1$, $0 \leq d_{H'} < \epsilon^2$ and $0 \leq d_p < \epsilon$. Then:*

$$\|\nabla\Phi\|^2 \geq \|\nabla\Phi_0\|^2 + O(\epsilon^2).$$

Proof. We examine each of the terms seperately. Trivially:

$$\|\nabla x(d_{H'} - f)\|^2 \geq 0.$$

And:

$$\|\nabla M d_{H'}^2\|^2 = 4M^2 d_{H'}^2.$$

We now consider the interaction terms. Recalling that $0 \leq d_{H'} < \epsilon^2$ and $0 \leq d_p < \epsilon$:

$$\langle \nabla\Phi_0, \nabla x(d_{H'} - f)\rangle = x\langle\nabla\Phi_0, \nabla d_{H'}\rangle + O(\epsilon^2).$$

Recalling that $\langle\nabla\Phi_0, \nabla d_{H'}\rangle = O(\epsilon)$, we obtain:

$$\langle \nabla\Phi_0, \nabla x(d_{H'} - f)\rangle = O(\epsilon^2).$$

Likewise:

$$\langle \nabla\Phi_0, \nabla M d_{H'}^2\rangle = Md_H O(\epsilon).$$

Finally, since $M\epsilon^2 < 1$:

$$\langle \nabla x(d_{H'} - f), \nabla M d_{H'}^2\rangle = 2Md_{H'}x + O(\epsilon^2).$$

Combining these terms yields:

$$\|\nabla\Phi\|^2 \geq \|\Phi_0\|^2 + Md_{H'}(Md_{H'} - O(\epsilon)) + O(\epsilon^2).$$

However:

$$Md_{H'}(Md_{H'} - O(\epsilon)) \geq -O(\epsilon^2).$$

the result now follows. $\square$

Proposition 3.10. *Let $\epsilon > 0$. If $M\epsilon^2 < 1$, $d_{H'} < \epsilon^2$ and $d_p < \epsilon$, then:*

$$\mathrm{SL}'_r(\Phi, \|\nabla\Phi\|, E(\nabla\Phi, \nabla d_{H'})) \leq \theta - \pi/2 + O(\epsilon^2).$$

Proof. Define Φ_1 by:

$$\Phi_1 = \Phi_0 + x(d_{H'} - f).$$

By Proposition 3.6:

$$\mathrm{SL}'_r(\Phi_1, \|\nabla\Phi_0\|, E) \leq \theta - \pi/2 + O(\epsilon^2).$$

Since $\mathrm{Hess}(\Phi(1)) = O(1)$, by Proposition 3.9 and Corollary 3.8:

$$\mathrm{SL}'_r(\Phi_1, \|\nabla\Phi\|, E(\nabla\Phi, \nabla d_{H'})) \leq \theta - \pi/2 + O(\epsilon^2).$$

Differentiating $Md_{H'}^2$ yields:

$$\mathrm{Hess}(Md_{H'}^2) = 2M\nabla d_{H'} \otimes \nabla d_{H'} + 2Md_{H'}\mathrm{Hess}(d_{H'}).$$

The first term vanishes along $(\nabla d_{H'})^{\perp}$, and the second term is negative, and thus does not affect the inequality either. The result now follows. □

3.3. **Boundary Lower Bounds for the Normal.** Let M, H, $\hat{\Sigma}$ and Ω be as in the preceeding section. Let Σ be a C^0 convex hypersurface such that:

(i) Σ lies between Ω and $\hat{\Sigma}$;
(ii) the interior of Σ is smooth; and
(iii) $\partial\Sigma = \partial\Omega = \Gamma$.

Let N_Σ be the exterior normal to Σ (which is continuous).

Lemma 3.11. *There exists $K > 0$ such that if $R_\theta(\Sigma) = r$, then $\mu(p, \mathsf{N}_\Sigma(p), r, \theta) < K$ for all $p \in \partial\Sigma = \Gamma$.*

Proof. We assume the contrary and obtain a contradiction. By continuity, there exists $p \in \Gamma$ such that:

$$\mu(p, \mathsf{N}_\Sigma(p), r, \theta) = +\infty.$$

Define d_H, d_Ω, Φ_0 and K as in the previous section. Since Σ is convex and $\partial\Sigma = \Gamma$ is smooth, N_Σ is continuous at p. Since, by definition $\mathsf{N}_\Sigma(p) = \nabla(d_H - Kd_\Omega)(p)$, (c.f. Lemma 3.2) there exists a continuous function $\delta : [0,\infty[\rightarrow [0,\infty[$ such that $\delta(0) = 0$ and, along Σ:

$$\|\pi(\nabla\Phi_0)(q)\| \leq \delta(d_p(q)),$$

where π is the orthogonal projection onto $T\Sigma$. Define H', $d_{H'}$ as in the previous section. For $\epsilon > 0$ small, define U_ϵ by:

$$U_\epsilon = \left\{q \in M \text{ s.t. } d_p(q) < \epsilon, d_{H'}(q) < \epsilon^2\right\}.$$

Along $\partial\Sigma = \Gamma$, $\Phi_0 = 0$. Recall that any convex set is $C^{0,1}$. Thus, along $\partial U_\epsilon \cap \Sigma$, $d_\Sigma(q, \partial\Sigma) = O(d_{H'}) = O(\epsilon^2)$, where d_Σ is the Riemannian distance inside Σ, and so:

$$\Phi(q) = \delta(\epsilon)O(\epsilon^2),$$

along $\partial U_\epsilon \cap \Sigma$. Since Γ is strictly convex and lies strictly inside H', there exists a function f such that:

(i) $f(p), \nabla f(p) = 0$ and the restriction of $\mathrm{Hess}(f)(p)$ to H' is positive definite; and

(ii) $d_{H'} - f = O(d_p^3)$ along Γ.

We define Φ as in the previous section. Along $\partial\Sigma \cap U_\epsilon = \Gamma \cap U_\epsilon$:

$$\Phi(q) \geq M d_{H'}^2 - O(\epsilon^4).$$

This is positive for sufficiently large M. Likewise, along $\partial U_\epsilon \cap \Sigma$:

$$\Phi(q) \geq M d_{H'}^2 - \delta(\epsilon) O(\epsilon^2).$$

There thus exists $K_1 > 0$ independant of ϵ such that, if $M = K_1 \delta(\epsilon)\epsilon^2$, then $\Phi \geq 0$ along $\partial U_\epsilon \cap \Sigma$.

Let A be the restriction of $\|\nabla\Phi\|^{-1}\mathrm{Hess}(\Phi)$ to $\nabla\Phi^\perp$. Let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. Let $\lambda_1' \leq \cdots \leq \lambda_n'$ be the eigenvalues of the restriction of A to $\nabla\Phi^\perp \cap \nabla d_{H'}^\perp$. By the Minimax Principal, for $1 \leq i \leq (n-1)$:

$$\lambda_i < \lambda_i'.$$

Thus, by Proposition 3.10, there exists $K_2 > 0$, also independant of ϵ such that:

$$\sum_{i=1}^{n-1} \arctan(r\lambda_i) \leq \theta - \pi/2 + K_2\epsilon^2.$$

However:

$$\lambda_n = O(M).$$

There thus exists $K_3 > 0$, independant of ϵ, such that:

$$\begin{aligned} \mathrm{SL}_r(A) &\leq \theta + (K_2\epsilon^2 - K_3 M^{-1}) \\ &= \theta + \epsilon^2(K_2 - K_1K_3\delta(\epsilon)^{-1}). \end{aligned}$$

Since $\delta(\epsilon)$ tends to 0 as ϵ tends to 0, there exists $\eta > 0$ such that, for ϵ sufficiently small, throughout U_ϵ:

$$SL_r(A) \leq \theta - \eta < \theta.$$

It follows that if $\Sigma_t = \Phi^{-1}(\{t\})$ for all t, then:

$$\mathrm{SL}_r(\Sigma_t \cap U_\epsilon) \leq \theta - \eta < \theta.$$

At p, $\nabla^\Sigma\Phi = 0$. Thus, reducing ϵ further if necessary, we may deform Φ slightly to Φ' (by subtracting a very small multiple of d_Ω, for example) such that Φ' is non-negative along $\partial(\Sigma \cap U_\epsilon)$, Φ is strictly negative over a non trivial subset of $\Sigma \cap U_\epsilon$, and, if $\Sigma_t' = (\Phi')^{-1}(\{t\})$ for all t, then:

$$SL_r(\Sigma_t' \cap U_\epsilon) \leq \theta - \eta/2 < \theta.$$

Let $p \in \Sigma$ be the point where Φ is minimised. Let $t_0 = \Phi'(p)$. Since p lies in the interior of Σ, Σ'_{t_0} is smooth at this point. Σ'_{t_0} is an interior tangent to Σ at p. In particular, Σ'_{t_0} is convex near p and:

$$R_\theta(\Sigma'_t \cap U_\epsilon) < r,$$

which is absurd, by the Geometric Maximum Principle (Lemma 2.6), and the result follows. □

Using limits yields:

Corollary 3.12. *There exists $K > 0$ such that if Σ is a smooth hypersurface lying between Ω and $\hat{\Sigma}$ such that $\partial\Sigma = \partial\Omega = \Gamma$ and $R_\theta(\Sigma) = r \in]R_0, R_1[$ is constant, then:*

$$\mu(p, \mathsf{N}_\Sigma(p), r, \theta) < K \text{ for all } p \in \Gamma.$$

3.4. **Constructing Barriers II.** Let M be an $(n+1)$-dimensional manifold. Let $\delta_0 \geq 0$ be small. Let $H \subseteq M$ be a smooth convex hypersurface such that:

$$R_\theta(H) = R_0,$$

where R_0 is constant. Let $\Omega \subseteq H$ be a bounded open subset of H. Let $\hat{\Sigma} \subseteq M$ be a convex hypersurface such that $\partial\hat{\Sigma} = \partial\Omega =: \Gamma$ and such that $R_\theta(\hat{\Sigma}) > R_1$ in the weak sense. Suppose that Γ is strictly convex as a subset of M with respect to the outward pointing normal to Γ in $\hat{\Sigma}$.

Let Σ be a smooth immersed convex hypersurface lying between Ω and $\hat{\Sigma}$ such that $\partial\Sigma = \Gamma$ and:

$$R_\theta(\Sigma) = r \in [R_0, R_1].$$

Let N, II and A be the unit normal, the second fundamental form and the shape operator respectively of Σ. Define B over Σ such that:

$$B^{ij}(\delta_{jk} + r^{-2}A^2_{jk}) = \delta^i{}_j.$$

Define the operator $\Delta^B : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by:

$$\Delta^B f = B^{ij} f_{;ij},$$

where $f_{;ij}$ is the Hessian of f with respect to Levi-Civita covariant derivative of Σ. We aim to construct barriers for Δ^B at any point of $\partial\Sigma$.

Let $p \in \partial\Sigma$ be a point. There exists a strictly convex hypersurface Σ' tangent to $\partial\Sigma = \Gamma$ at p such that Σ lies in its interior and the normal to Σ at p also points into its interior.

Lemma 3.13. *Let $d_{\Sigma'}$ denote the distance to Σ'. Let U be a neighbourhood of p such that, throughout $\Sigma \cap U$:*

$$\langle \nabla d_{\Sigma'}, \mathsf{N} \rangle \geq 0.$$

Then, throughout $\Sigma \cap U$:

(i) $d_{\Sigma'} \geq 0$, *and*

(ii) $\Delta^B d_{\Sigma'} \leq 0$.

Remark. U depends only on the modulus of continuity for N near p.

Proof. See Corollary 2.9. □

Lemma 3.14. *Let $\theta \in](n-1)\pi/2, n\pi/2[$ be an angle. Let $r \in]0,\infty[$. Suppose that $R_\theta(\Sigma) = r$. There exists $\epsilon, \delta > 0$ which only depend on θ and r such that, throughout $B_\delta(p) \cap \Sigma$:*

$$\Delta^B d_p^2 \geq \epsilon.$$

Proof. By Lemma 2.8 and Corollary 2.11:

$$(d_p^2)_{;ij} = 2\delta_{ij} - 2A_{ij} d_p \langle \nabla d_p, \mathsf{N} \rangle + O(d_p^2).$$

By Lemma 2.2, there exists $K_1 > 0$, which only depends on θ and r such that:

$$Tr(B) \geq \frac{1}{K_1}.$$

Thus, throughout Σ:

$$\Delta^B (d_p^2) \geq \frac{2}{K_1} - O(d_p).$$

There exists δ, which only depends on K such that, for $d_p < \delta$, the error term is less than $\frac{1}{K}$ in magnitude. The result now follows. □

3.5. **Second Order Boundary Estimates.** Let f be the signed distance to Σ. f is a real valued function which is smooth in a nieghbourhood of Σ. By definition:

$$\|\nabla f\| = 1.$$

For X, Y tangent to Σ:

$$\text{Hess}(f)(X,Y) = II(X,Y),$$

where II is the second fundamental form of Σ. Let $p \in \partial\Sigma$. Let X be a vector field over $\mathbb{H}^{n+1}$ which is tangent to $\partial\Sigma$ (but not necessarily tangent to Σ). Define $\varphi = Xf$. For any Y tangent to Σ:

$$Y\varphi = \text{Hess}(f)(X,Y) + \langle \nabla f, \nabla_Y X \rangle = II(X,Y) + \langle \nabla f, \nabla_Y X \rangle.$$

Thus, a-priori bounds on X and φ yield a-proiri bounds on II.

Lemma 3.15. *For $X, Y \in T\Sigma$:*

$$(\nabla_{\mathsf{N}} \text{Hess}(f))(X,Y) = \langle R_{\mathsf{N}X}\mathsf{N}, Y \rangle - \langle A^2 X, Y \rangle.$$

Proof. Define $\Phi : \Sigma\times]-\epsilon,\epsilon[\rightarrow M$ by:

$$\Phi(p,t) = \mathrm{Exp}_p(t\mathsf{N}(p)).$$

Pulling back through Φ, we identify M with $\Sigma\times]-\epsilon,\epsilon[$ and N with ∂_t. In particular, if X is tangent to Σ, then $[X,\mathsf{N}]=0$. Trivially:

$$\nabla f = \mathsf{N}.$$

Thus:

$$\mathrm{Hess}(f)(X,Y) = \langle\nabla_Y\mathsf{N},X\rangle = A(X,Y).$$

Bearing in mind that $\nabla_{\mathsf{N}}\mathsf{N}=0$:

$$\begin{aligned}(\nabla_N\mathrm{Hess}(f))(X,Y) &= \mathsf{N}\langle\nabla_X\mathsf{N},Y\rangle - A(\nabla_{\mathsf{N}}X,Y) - A(X,\nabla_{\mathsf{N}}Y)\\ &= \langle\nabla_{\mathsf{N}}\nabla_X\mathsf{N},Y\rangle + \langle\nabla_X\mathsf{N},\nabla_{\mathsf{N}}Y\rangle - A(\nabla_{\mathsf{N}}X,Y) - A(X,\nabla_{\mathsf{N}}Y)\\ &= \langle R_{\mathsf{N}X}\mathsf{N},Y\rangle + \langle\nabla_X\mathsf{N},\nabla_Y\mathsf{N}\rangle - A(\nabla_X\mathsf{N},Y) - A(X,\nabla_Y\mathsf{N})\\ &= \langle R_{\mathsf{N}X}\mathsf{N},Y\rangle - \langle A^2X,Y\rangle.\end{aligned}$$

The result follows. □

Lemma 3.16. *There exists $K>0$, which only depends on X, r, θ and the structure of M such that, throughout Σ:*

$$\left|\Delta^B\varphi\right| \leq K.$$

Proof. Let Latin indices represent directions in TM and let Greek indices represent directions in $T\Sigma$. Let ν represent the exterior normal direction to Σ.

Let ; denote covariant differentiation with respect to the Levi-Civita covariant derivative of M. Let $O(1)$ represent terms bounded in terms of X, r, θ or the structure of M. Recall that $A_{\alpha\beta}=f_{;\alpha\beta}$. This is symmetric in α and β. By definition of curvature:

$$f_{;\alpha\beta k} = f_{;\alpha k\beta} + R_{\beta k\alpha}{}^l f_{;l} = f_{;k\alpha\beta} + O(1).$$

Since $f_{;\nu k}=0$ for all k, for all X,Y,Z tangent to Σ:

$$(\nabla\mathrm{Hess}(f))(Y,Z;X) = (\nabla^\Sigma\mathrm{Hess}(f))(Y,Z;X).$$

Thus, differentiating $R_\theta(A)=r$ along Σ yields:

$$\begin{aligned}&B^{\alpha\beta}f_{;\alpha\beta\gamma} = 0\\ \Rightarrow\quad &B^{\alpha\beta}f_{;\gamma\alpha\beta} = O(1).\end{aligned}$$

We remark in passing that it is at this stage that the differential condition on f (and therefore Σ) is used. We now consider the normal derivative. By Lemma 3.15:

$$\begin{aligned}&f_{;\alpha\beta\nu} = R_{\nu\alpha\nu\beta} - (A^2)_{\alpha\beta}\\ \Rightarrow\quad &f_{;\nu\alpha\beta} = -(A^2)_{\alpha\beta}.\end{aligned}$$

Thus:

$$\left|B^{\alpha\beta}f_{;\nu\alpha\beta}\right| \leq nr^2.$$

Thus, for all k:

$$\left|B^{\alpha\beta}f_{;k\alpha\beta}\right| = O(1).$$

We now consider $\varphi = X^k f_{;k}$:

$$B^{\alpha\beta}\varphi_{;\alpha\beta} = B^{\alpha\beta}X^k{}_{;\alpha\beta}f_{;k} + 2B^{\alpha\beta}X^k{}_{;\alpha}f_{;k\beta} + B^{\alpha\beta}X^k f_{;k\alpha\beta}.$$

Since $\|B\| \leq 1$, the first term is controlled by a-priori bounds on $\nabla^2 X$. The third term is controlled by a-priori bounds on X and the preceeding discussion. We now control the second term. Recalling that $f_{;\nu k} = 0$ for all k:

$$B^{\alpha\beta}X^k{}_{;\alpha}f_{;k\beta} = B^{\alpha\beta}A_{\gamma\beta}X^\gamma{}_{;\alpha}.$$

Since $\|BA\| \leq 1$, this term is controlled by a-priori bounds on ∇X. Finally, since ∇f is the unit normal to Σ, by Lemma 2.8:

$$\mathrm{Hess}^\Sigma(\varphi)_{\alpha\beta} = \varphi_{;\alpha\beta} - A_{\alpha\beta}.$$

Thus:

$$\Delta^B\varphi = B^{\alpha\beta}(\varphi)_{;\alpha\beta} - B^{\alpha\beta}A_{\alpha\beta}.$$

Since $\|AB\| \leq 1$, the result follows. □

Lemma 3.17. *There exists K, which only depends on M, H, $\hat{\Sigma}$, r, θ and the modulus of continuity of Σ near $\partial\Sigma$ such that, along $\partial\Sigma$:*

$$\|A\| \leq K.$$

Proof. Let $p \in \partial\Omega$. The normal to Σ at p lies between the normals to H and $\partial\hat{\Sigma}$ at p. There thus exists a convex hypersurface, H', which is an exterior tangent to $\partial\Omega$ at p and such that the normal to Σ at p points into H'. Define $d_{H'}$ by:

$$d_{H'}(q) = d(q, H').$$

By Lemma 3.13, there exists a neighbourhood U_1 of p, which only depends on $\hat{\Sigma}$ and the modulus of continuity of Σ at the boundary, such that, throughout $\Sigma \cap U_1$:

$$\Delta^B d_{H'} \leq 0.$$

Define d_p by:

$$d_p(q) = d(q, p).$$

By Lemma 3.14, there exists $\epsilon > 0$ and a neighbourhood U_2 of p such that, throughout $\Sigma \cap U_2$:

$$\Delta^B d_p^2 \geq \epsilon.$$

Let f be the perpindicular distance to Σ. Let X be a vector field tangent to $\partial\Omega$. Consider the function $\varphi = Xf$. φ vanishes along $\partial\Omega$. Since $\|\nabla f\| = 1$, there exists $K_1 > 0$, which only depends on X such that, throughout Σ:

$$|\varphi| = \|Xf\| \leq K_1.$$

By Lemma 3.16, there exists $K_2 > 0$ such that, throughout Σ:

$$\left|\Delta^B\varphi\right| \leq K_2.$$

Choosing $\delta > 0$ such that $B_\delta(p) \subseteq U_1 \cap U_2$, there exists $A_- > 0$ such that, throughout $B_\delta(p) \cap \Sigma$:

$$\Delta^B(\varphi - Ad_p^2) \leq 0.$$

There exists $B_- > 0$ such that:

(i) $\Delta^B(\varphi + B_- d_{H'} - A_- d_p^2) \leq 0$ throughout $B_\delta(p) \cap \Sigma$; and
(ii) $(\varphi + B_- d_{H'} - A_- d_p^2) \geq 0$ along $\partial(B_\delta(p) \cap \Sigma)$.

Thus, by the maximum principal, throughout $B_\delta(p) \cap \Sigma$:

$$\varphi \geq B_- d_{H'} - A_- d_p^2.$$

Likewise, reducing δ if necessary, there exists B_+ and A_+ such that, throughout $B_\delta(p) \cap \Sigma$:

$$\varphi \leq B_+ d_{\Sigma'} - A_+ d_p^2.$$

We thus obtain a-priori bounds on $\nabla\varphi$ at p. Since X is arbitrary, this yields a-priori bounds on $\mathrm{Hess}(f)(X, Y)$ for all pairs of vectors $X, Y \in T_p\Sigma$ where at least one of X or Y is tangent to $\partial\Sigma$. Since the second fundamental form of Σ is the restriction to $T\Sigma$ of the hessian of f (and since $\|\nabla f\| = 1$), we obtain a-priori bounds on $A(X, Y)$ for all such pairs of vectors.

Let $(e_1, \ldots, e_{n-1})$ be an orthonormal basis of $T_p\partial\Sigma$ which diagonalises the restriction of A. Let e_n be the inward pointing normal of $\partial\Sigma$ at p. With respect to this basis, there exists $0 < \lambda_1 < \cdots < \lambda_{n-1}$ and $M > 0$ such that:

$$A = \begin{pmatrix} D & O(1) \\ O(1) & M \end{pmatrix},$$

where $D = \mathrm{Diag}(\lambda_1, \ldots, \lambda_{n-1})$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_{n-1}$. Let $\lambda'_1, \ldots, \lambda'_n$ by the eigenvalues of A. By Lemma 1.2 of [6]:

$$\lambda'_i = \begin{cases} \lambda_i + o(1) \text{ if } 1 \leq i \leq n-1 \\ M(1 + o(M^{-1})) \text{ if } i = n. \end{cases}$$

However, by Corollary 3.12, there exists K such that $\mu(p, \mathsf{N}_p, r, \theta) < K$. M therefore cannot become arbitrarily large. We thus obtain a-priori bounds on M, and the result now follows. □

3.6. **Second Order Interior Estimates.** Let M be a Riemannian manifold. Let $K \subseteq M$ be a compact subset. Let ∇ be the Levi-Civita covariant derivative over M and let R be the Riemann curvature tensor of M. Choose $\theta \in](n-1)\pi/2, n\pi/2[$ and $r > 0$. Let $K \subseteq M$ be a compact subset. Let $\Sigma \subseteq M$ be a smooth, convex, immersed hypersurface contained in K such that:

$$R_\theta(\Sigma) = r.$$

Let A and N be the second fundamental form and the exterior normal of Σ respectively.

Lemma 3.18. *Let H be the mean curvature of Σ. There exists $C > 0$ which only depends on r and the norms of R and ∇R over K such that:*

$$\Delta^B H \geq -C(1+H) + \sum_{i,j=1}^{n} \frac{(\lambda_i - \lambda_j)r^2\lambda_i\lambda_j}{(1+\lambda_j^2)}.$$

Proof. Choose $p \in \Sigma$ and let $e_1, \ldots, e_n$ be an orthonormal basis of eigenvectors for A in $T_p\Sigma$. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues of $r^{-1}A$. Let ; denote covariant differentiation with respect to the Levi-Civita covariant derivative of Σ. Let the index ν denote the direction normal to Σ. Differentiating the curvature condition twice yields:

$$\sum_{i=1}^{n} \frac{1}{(1+\lambda_i^2)} A_{iik} = 0,$$

$$\sum_{i=1}^{n} \frac{1}{(1+\lambda_i^2)} A_{iipq} = \sum_{i,j=1}^{n} \frac{r^{-2}(\lambda_i + \lambda_j)}{(1+\lambda_i^2)(1+\lambda_j^2)} A_{ij;p}A_{ij;q}.$$

Let R^Σ be the Riemann curvature tensor of Σ. By definition of curvature:

$$A_{ij;k} - A_{ik;j} = R_{kj\nu i},$$

$$A_{ij;kl} - A_{ij;lk} = R^\Sigma_{klij}\lambda_j + R^\Sigma_{klji}\lambda_i.$$

Thus, for all i and j:

$$\begin{aligned} A_{ii;jj} &= A_{ij;ij} + R_{ji\nu i;j} \\ &= A_{ij;ji} + R_{ji\nu i;j} + R^\Sigma_{ijij}\lambda_j + R^\Sigma_{ijji}\lambda_i \\ &= A_{jj;ii} + R_{ji\nu j;i} + R_{ji\nu i;j} + R^\Sigma_{ijij}\lambda_j + R^\Sigma_{ijji}\lambda_i. \end{aligned}$$

Applying the second derivative of the curvature condition:

$$\sum_{i,j=1}^{n} \frac{1}{(1+\lambda_j^2)} A_{jj;ii} \geq 0.$$

For any 1-form, ξ:

$$\xi_{i;j} = (\nabla\xi)_{ij} - A_{ij}\xi_\nu,$$

$$\xi_{\nu;j} = (\nabla_{\partial_j}\xi)(\mathsf{N}) + A_j{}^k\xi_k.$$

Thus:

$$R_{ji\nu j;i} = (\nabla R)_{ji\nu ji} - r\lambda_i(1-\delta_{ij})R_{j\nu\nu j} + r\lambda_i R_{jiij},$$

$$R_{ji\nu i;j} = (\nabla R)_{ji\nu ij} - r\lambda_j(1-\delta_{ij})R_{\nu i\nu i} + r\lambda_j R_{jiji}.$$

This yields:

$$\sum_{i,j=1}^{n}\frac{1}{(1+\lambda_j^2)}(R_{ji\nu j;i}+R_{ji\nu i;j})\geq\sum_{i,j=1}^{n}\frac{-r\lambda_i(1-\delta_{ij})}{(1+\lambda_j^2)}R_{j\nu\nu j}+\sum_{i,j=1}^{n}\frac{-r\lambda_j(1-\delta_{ij})}{(1+\lambda_j^2)}R_{\nu i\nu i}$$
$$+\sum_{i,j=1}^{n}\frac{r(\lambda_i-\lambda_j)}{(1+\lambda_j^2)}R_{ijji}-C_1,$$

where C_1 only depends on the norm of ∇R over K. The first and third terms on the right hand side is bounded by a multiple of H times the norm of R over K. Likewise, the second term is bounded in terms of the norm of R over K. Thus:

$$\sum_{i,j=1}^{n}\frac{1}{(1+\lambda_j^2)}(R_{ji\nu j;i}+R_{ji\nu i;j})\geq -C(1+H),$$

where C only depends on r and the norms of R and ∇R over K. Finally, since A is the shape operator of Σ:

$$R^{\Sigma}_{ijij}\lambda_j+R^{\Sigma}_{ijji}\lambda_i=(\lambda_i-\lambda_j)R_{ijji}+r^2(\lambda_i-\lambda_j)\lambda_i\lambda_j.$$

The result follows. □

Lemma 3.19. *There exists $D>0$, which only depends on r,θ and the norms of R and ∇R over K such that:*

$$H\geq D\Rightarrow\Delta^B H\geq 0.$$

Proof. Symmetrising the inequality obtained in Lemma 3.18 yields:

$$\Delta^B H\geq -C(1+H)+\sum_{i,j=1}^{n}F(\lambda_i,\lambda_j;r,C),$$

where F is given by:

$$F(x,y;r,C)=\frac{r^2xy}{2(1+x^2)(1+y^2)}(x^3+y^3-x^2y-y^2x).$$

Since $R_\theta(A)=r$, there exists $\epsilon>0$ which only depends on r and θ such that $\lambda_i\geq\epsilon$ for all i. We observe that, for all $x,t\geq 0$:

$$F(x,y;r,C)\geq 0.$$

Without loss of generality:

$$\lambda_1\geq H/n,\qquad \epsilon\leq\lambda_n\leq r\tan(\theta/n).$$

Consequently, $F(\lambda_1,\lambda_n;r,C)$ grows like H^2 as $H\rightarrow+\infty$. In particular, there exists $D>0$ such that, for $H\geq D$:

$$F(\lambda_1,\lambda_n;r,C)\geq C(1+H).$$

This is the desired value for D and the result follows. □

Proposition 3.20. *There exists K, which only depends on M, H, $\hat{\Sigma}$, r, θ and the modulus of continuity of Σ near $\partial\Sigma$ such that, throughout Σ:*

$$\|A\| \leq K.$$

Proof. By convexity, $\|A\| \leq H \leq n\|A\|$. If H acheives its maximum on the boundary, then, by Lemma 3.17:

$$\|A\| \leq H \leq nK.$$

If H acheives its maximum in the interior, then, by Lemma 3.19 and the Maximum Principal:

$$\|A\| \leq H \leq D.$$

The result follows. □

3.7. **The Dirichlet Problem I.** Let M be an $(n+1)$-dimensional manifold. Choose $\theta \in [(n-1)\pi/2, n\pi/2[$. Let $H \subseteq M$ be a smooth convex hypersurface such that:

$$R_\theta(H) = R_0.$$

Where R_0 is constant. Let $\Omega \subseteq H$ be a bounded open subset of H. Let $\hat{\Sigma} \subseteq M$ be a convex hypersurface such that $\partial\hat{\Sigma} = \partial\Omega =: \Gamma$ and such that $R_\theta(\hat{\Sigma}) \geq R_1$ in the weak sense, where $R_1 \leq 1/\tan^{-1}(\theta/n)$.

Proposition 3.21. *Let $f : \Omega \to \mathbb{R}$ be a smooth function. Define $\hat{f} : \Omega \to M$ by:*

$$\hat{f}(p) = \mathrm{Exp}(f(p)\mathsf{N}_H(p)),$$

where N_H is the unit exterior normal over H. If II denotes the second fundamental form of the graph of f, then:

$$\hat{f}^*II = -B^{-1}\mathrm{Hess}(f)B^{-1} + R,$$

where B is a symmetric positive definite matrix, R is a symmetric 2-form, and B and R are functions only of p, f and ∇f.

Proof. Define $\Phi : \Omega \times [0,\infty[\to M$ by:

$$\Phi(p,t) = \mathrm{Exp}(t\mathsf{N}_H(p)).$$

Let g_0 be the Riemannian metric over Ω. Since M is a Hadamard manifold, Φ is a local diffeomorphism. Let g be the Riemannian metric over M. $\hat{g} = \Phi^*g$ defines a Riemannian metric over $\Omega \times [0,\infty[$. With respect to $\hat{g}$, ∂_t has unit length and is orthogonal to $T\Omega$. Let $M(p,t)$ be a symmetric matrix such that, for all vectors tangent to $T\Omega$:

$$\hat{g}(X,Y) = g_0(M(p,t)X,Y).$$

Let ∇_0 be the Levi-Civita covariant derivative over Ω. Let ∇ be the Levi-Civita covariant derivative of $\hat{g}$. For X tangent to Ω, define $\hat{X}_f$ by:

$$\hat{X}_f = (X, \langle \nabla^0 f, X\rangle).$$

Define the symmetric matrix $B := B(p, f, \nabla f)$ such that, for all $X, Y \in T\Omega$:

$$g_0(BX, BY) = \hat{g}(\hat{X}_f, \hat{Y}_f).$$

Define $\hat{\mathsf{N}} := \hat{\mathsf{N}}(p, f, \nabla f)$ by:

$$\hat{\mathsf{N}}_f = (-M^{-1}(p, f)\nabla^0 f, 1).$$

$\hat{\mathsf{N}}_f$ is an outward normal to the graph of f. Define $\mu_f := \mu_f(p, f, \nabla f)$ by:

$$\mu_f = \|\hat{\mathsf{N}}_f\|.$$

For all $X \in T\Omega$:

$$\nabla_{\hat{X}_f}\hat{\mathsf{N}}_f = (-M^{-1}(p, f)\nabla^0_X\nabla^0(f), 0) + R_1(p, f, \nabla f)(X),$$

where R_1 is a term which only depends on p, f and ∇f. Thus:

$$\hat{g}(\nabla_{\hat{X}_f}\hat{\mathsf{N}}_f, \hat{Y}_f) = (-\nabla^0_X\nabla^0 f, 0) + R_2(p, f, \nabla f)(X, Y).$$

It follows that:

$$II = -\frac{1}{\mu_f}B^{-1}\text{Hess}(f)B^{-1} + R_3.$$

Where μ_f, B and R_3 only depend on p, f and ∇f. The result follows. □

We now obtain the first stage in the proof of Theorem 1.2:

Theorem 3.22. *Suppose that $\hat{\Sigma}$ is a graph over Ω and that Γ is strictly convex as a subset of M with respect to the outward pointing normal to Γ in $\hat{\Sigma}$. If $\theta > (n-1)\pi/2$, then, for all $r \in [R_0, R_1]$, there exists an immersed hypersurface $\Sigma_r \subseteq M$ such that:*

(i) *Σ_r is C^0 and C^∞ in its interior;*

(ii) *$\partial\Sigma_r = \Gamma$;*

(iii) *Σ_r is a graph over Ω lying below $\hat{\Sigma}$; and*

(iv) *$\hat{R}_\theta(\Sigma_r) = r$.*

Moreover, the same result holds for $\theta = (n-1)\pi/2$ provided that, in addition $\hat{\Sigma}$ is ϵ-convex, for some $\epsilon > 0$.

Remark. The hypotheses of this theorem are satisfied when the norm of the second fundamental form of H is small with respect to that of $\hat{\Sigma}$ and the normal of $\hat{\Sigma}$ is sufficiently bounded away from TH along Γ. Explicitely, if $\hat{\Sigma}$ is ϵ-convex, if the norm of the second fundamental form of H is bounded above by δ and if the angle between the normal to $\hat{\Sigma}$ and TH is bounded below by θ along Γ, then the hypotheses are satisfied provided that:

$$\epsilon\text{sin}(\theta) - \delta > 0.$$

Proof. Suppose that $\theta > (n-1)\pi/2$ and that $\hat{\Sigma}$ and Γ are smooth. By Lemma 2.13, the general case follows by approximation. Let $I \subseteq [R_0, R_1]$ be such that, for all $t \in I$, a solution exists which is smooth over $\overline{\Omega}$ and which lies strictly below $\hat{\Sigma}$ and whose supporting tangents along Γ also lie strictly below those of $\hat{\Sigma}$. By definition $R_0 \in I$. By Theorem 1.3 of [25], noting that $r > 1/\tan^{-1}(\theta/n)$, I is open.

Let $(r_n)_{n\in\mathbb{N}} \in I$ be an increasing sequence converging to $r_0 \in [R_0, R_1]$. For all n, let Σ_n be a solution with $R_\theta(\Sigma_n) = r_n$. For all n, let $f_n : \overline{\Omega} \to \mathbb{R}$ be the function of which Σ_n is the graph. By Lemma 3.1, there exists f_0 to which $(f_n)_{n\in\mathbb{N}}$ subconverges in the $C^{0,\alpha}$ sense for all α and whose graph lies below $\hat{\Sigma}$.

Proposition 3.20 yields uniform C^2 bounds on $(f_n)_{n\in\mathbb{N}}$. For all n, f_n satisfies an equation of the form:

$$F(p, \phi, D\phi, D^2\phi; r, \theta) = 0.$$

Since f_n is uniformly bounded in the C^2 sense, F is uniformly elliptic. By concavity of R_θ and Proposition 3.21, F is concave with respect to $D^2\phi$. Theorem 1 of [5] therefore yields uniform $C^{2,\alpha}$ bounds on $(f_n)_{n\in\mathbb{N}}$ for all α. Repeated application of Schauder's estimates then yield uniform C^k bounds on $(f_n)_{n\in\mathbb{N}}$ for all k. It follows that $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the C^∞ sense over $\overline{\Omega}$.

Thus, if Σ_0 is the graph of f_0, Σ_0 is smooth up to the boundary and $R_\theta(\Sigma_0) = r_0$. By part *(iv)* of Lemma 3.1, Σ_0 lies strictly below $\hat{\Sigma}$ and every supporting tangent to Σ_0 along Γ also lies strictly below those of $\hat{\Sigma}$. I is therefore closed, and existence for $r \in [R_0, R_1[$ follows by connectedness of the interval $[R_0, R_1]$. The case where $r = R_1$ is proven by taking limits, and the result follows. □

The special case where M is $(n + 1)$-dimensional hyperbolic space, $\mathbb{H}^{n+1}$, and H is a totally geodesic hypersurface is interesting in itself. In particular, the hypersurfaces thus obtained are unique:

Corollary 3.23. *Let $H \subseteq \mathbb{H}^{n+1}$ be a totally geodesic hypersurface. Let $\Omega \subset H$ be a bounded open subset. Choose $\theta \in [(n-1)\pi/2, n\pi/2[$ and let $\hat{\Sigma} \subseteq \mathbb{H}^{n+1}$ be a convex hypersurface which is a graph over Ω such that $\partial\hat{\Sigma} = \partial\Omega$ and:*

$$\hat{R}_\theta(\hat{\Sigma}) \geq R_1,$$

in the weak sense, where $R_1 \leq 1$. If $\theta > (n-1)\pi/2$, then, for all $r \in [0, R_1]$, there exists a unique immersed hypersurface $\Sigma_r \subseteq M$ such that:

(i) *Σ_r is C^0 and C^∞ in its interior;*
(ii) *$\partial\Sigma_r = \Gamma$;*
(iii) *Σ_r is a graph over Ω lying below $\hat{\Sigma}$; and*
(iv) *$\hat{R}_\theta(\Sigma_r) = r$.*

Moreover, the same result holds for $\theta = (n-1)\pi/2$ provided that, in addition, $\hat{\Sigma}$ is ϵ-convex, for some $\epsilon > 0$.

Proof. By Theorem 3.22, it remains to prove uniqueness. Choose $\theta \in [(n-1)\pi/2, n\pi/2[$ and $r \in [0, R_1]$. Let Σ_1 and Σ_2 be two distinct solutions such that:

$$R_\theta(\Sigma_1), R_\theta(\Sigma_2) = r.$$

Suppose that there exists $p \in \Sigma_2$ which lies strictly above Σ_1. By deforming Σ_1 slightly and moving it upwards by isometries of hyperbolic space, we obtain an immersed hypersurface, Σ_1' and a point $p' \in \Sigma_1'$ such that $R_\theta(\Sigma_1) < r$ and Σ_2 is an interior tangent to Σ_1' at p'. This contradicts the Geometric Maximum Principal. Σ_2 therefore lies below Σ_1. Likewise, Σ_1 lies below Σ_2 and they therefore coincide. The result follows. □

In particular, Ω satisfies the hypothesis of Corollary 3.23 when its shape operator is everywhere bounded below by Id:

Corollary 3.24. *Let $H \subseteq \mathbb{H}^{n+1}$ be a totally geodesic hypersurface. Let $\Omega \subset H$ be a bounded open subset. If $\partial\Omega$ is 1-convex, then, for all $\theta \in [(n-1)\pi/2, n\pi/2[$ and for all $r \in [0, 1]$, there exists a unique immersed hypersurface $\Sigma_r \subseteq M$ such that:*

(i) *Σ_r is C^0 and C^∞ in its interior;*
(ii) *$\partial\Sigma_r = \Gamma$;*
(iii) *Σ_r is a graph over Ω lying below $\hat{\Sigma}$; and*
(iv) *$\hat{R}_\theta(\Sigma_r) = r$.*

Remark. In fact, this illustrates a general feature of hyperbolic space: that the curvature of horospheres, which is equal to 1, provides a threshold for geometric results. This becomes particularly evident in the study of curvature flows (c.f. [16] and [3]), and constitutes an important distinction between hyperbolic space and Euclidean space. In both spaces, the curvature of totally geodesic hypersurfaces, which is equal to 0, forms one threshold, but, in Euclidean space, horospheres coincide with totally geodesic hypersurfaces, and so the horospheric threshold is absorbed into the totally geodesic one.

Proof. Let U' be the intersection of all horoballs in $\mathbb{H}^{n+1}$ containing Ω. Let U be the intersection of U' with one of the connected components of $\mathbb{H}^{n+1} \setminus H$. Define $\hat{\Sigma} = \partial U'$. $\hat{\Sigma}$ satisfies the hypotheses of Theorem 3.23 and the result now follows. □

4. The Perron Method

4.1. **Extended Normals.** Let M be a Hadamard manifold of sectional curvature bounded above by -1. Let UM be the unitary bundle of M. Let $\Sigma = (S, i)$ be a smooth convex immersed hypersurface in M. Let N_Σ be the outward pointing normal over Σ. Let Ω be an open subset of Σ. Let $\mathsf{N}_{\partial\Omega}$ be the outward pointing normal of $\partial\Omega$ in Σ. Define $N\Omega$ and $N\partial\Omega$ by:

$$N\Omega = \{\mathsf{N}(p) \text{ s.t. } p \in \Omega\},$$
$$N\partial\Omega = \{\mathsf{V}_p \text{ s.t. } p \in \partial\Omega \;\&\; \langle V_p, \mathsf{N}_{\partial\Omega}(p)\rangle, \langle V_p, \mathsf{N}_{\Sigma}(p)\rangle \geq 0\}.$$

We define $\hat{N}\Omega$ by:

$$\hat{N}\Omega = N\Omega \cup N\partial\Omega.$$

We call $\hat{N}\Omega$ the extended normal of Ω. Ω embeds naturally as an open subset of $\hat{N}\Omega$. Moreover, i extends naturally to an immersion $\hat{\imath} : \hat{N}\Omega \to UM$. We define $\Phi : \hat{N}\Omega \times [0, \infty[\to M$ by:

$$\Phi(p,t) = \mathrm{Exp}(t\hat{\imath}(p)).$$

Since M is a Hadamard manifold and Σ is convex, for every $p \in \hat{N}\Omega$ there exists a neighbourhood U of p in $\hat{N}\Omega$ such that the restriction of Φ to $U\times]0,\infty[$ is a homeomorphism onto its image. We refer to Φ as the end of Ω. The differential structure of M pulls back through Φ to a differential structure over $\hat{N}\Omega\times]0,\infty[$, which we also refer to as the end of Ω when there is no ambiguity. We denote it by $\mathcal{E}(\Omega)$. $\mathcal{E}(\Omega)$ is foliated by the geodesics normal to $\hat{\Omega}$. We refer to this foliation and the resulting vector field as the vertical foliation and vector field respectively.

The boundary of $\partial\mathcal{E}(\Omega)$ divides into two parts, which we denote by $\partial_w\mathcal{E}(\Omega)$ and $\partial_f\mathcal{E}(\Omega)$ and define as follows:

$$\partial_w\mathcal{E}(\Omega) = \left\{(p,t) \text{ s.t. } p \in \partial\hat{N}\Omega \;\&\; t \in [0,\infty[\right\},$$
$$\partial_f\mathcal{E}(\Omega) = \left\{(p,0) \text{ s.t. } p \in \hat{N}\Omega\right\}.$$

We refer to $\partial_w\mathcal{E}(\Omega)$ and $\partial_f\mathcal{E}(\Omega)$ as the wall and the floor respectively of $\mathcal{E}(\Omega)$. Trivially, $\partial_f\mathcal{E}(\Omega)$ is identified with $\hat{N}\Omega$. The experienced reader will be aware that $\mathcal{E}(\Omega)$ also has an ideal boundary at infinity. This will not concern us.

Let $U \subseteq \mathcal{E}(\Omega)$ be an open set. We say that U is convex if and only if the shortest geodesic in $\mathcal{E}(\Omega)$ joining any two points in U also lies in U. We say that U is boundary convex if and only if, for every boundary point $p \in \partial U$, there exists $r > 0$ such that $B_r(p) \cap U$ is convex.

If $U \subseteq \mathcal{E}(\Omega)$ is boundary convex, let δ_U be the distance function to U in $\mathcal{E}(\Omega)$.

Proposition 4.1. *Let $p \in \partial U$ and $r > 0$ be such that:*

$$d(p, \partial\mathcal{E}(\Omega)) > 2r.$$

Then, δ_U is convex in $B_r(p)$. In particular, $U \cap B_r(p)$ is convex.

Proof. For any $q \in B_r(p)$, the shortest geodesic joining q to U does not intersect $\partial\mathcal{E}(\Omega)$. The first assertion now follows from the fact that the distance to a convex set in a Hadamard manifold is a convex function. Since $B_r(p)$ is convex, the second assertion follows. □

Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of compact boundary convex subsets of $\mathcal{E}(\Omega)$. We say that $(U_n)_{n\in\mathbb{N}}$ is nested if and only if for all $i > j$, $U_i \subseteq U_j$. By classical point set topology, there exists a compact subset $U_0 \subseteq \mathcal{E}(\Omega)$ such that $(U_n)_{n\in\mathbb{N}}$ converges to U_0 in the Haussdorf sense.

Proposition 4.2. *$U_0 \setminus \partial\mathcal{E}(\Omega)$ is boundary convex away from $\partial\mathcal{E}(\Omega)$.*

Proof. Choose $p \in \partial U_0 \setminus \partial\mathcal{E}(\Omega)$. There exists $r > 0$ such that $d(p, \partial\mathcal{E}(\Omega)) > 3r$. There exists $(p_n)_{n\in\mathbb{N}} \in \mathcal{E}(\Omega)$ which converges to p such that, for all n:

$$p_n \in \partial U_n.$$

and:

$$d(p_n, \partial\mathcal{E}(\Omega)) > 2r.$$

For all $n \in \mathbb{N} \cup \{0\}$, define $\delta_n = \delta_{U_n}$. By the preceeding proposition, the restriction of δ_n to $B_r(p_n)$ is convex. Taking limits, it follows that δ_0 is convex, and so $U_0 \cap B_r(p)$ is convex. □

Proposition 4.3. *Choose $p_0 \in U_0 \setminus \partial\mathcal{E}(\Omega)$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence converging to p_0 such that $p_n \in \partial U_n$ for all n. For all n, let N_n be a supporting normal to U_n at p_n. If N_n converges to N_0, then N_0 is a supporting normal to U_0 at p_0.*

Proof. This follows from the analogous result in a Hadamard manifold. □

We now say that U is boundary ϵ-convex if and only if, for every boundary point $p \in \partial U$, there exists $r > 0$ such that $B_r(p) \cap U$ is ϵ-convex.

Lemma 4.4. *Let M be a Hadamard manifold of sectional curvature bounded above by -1. Choose $\epsilon > 0$. Let $\Sigma \subseteq M$ be a convex hypersurface whose second fundamental form is bounded below by ϵId in the weak sense. Let d_Σ be the distance to Σ in M. Let $\gamma : \mathbb{R} \to M$ be a curve lying on the outside of Σ of geodesic curvature is less than ϵ. For $d_\Sigma < \epsilon^{-1}$, d_Σ^2 is a convex function of γ.*

Proof. Choose $t \in \mathbb{R}$. Let $p \in \Sigma$ be the closest point to $\gamma(t)$. Let $\epsilon' < \epsilon$ be greater than the geodesic curvature of γ near t. Let d be the distance to Σ at $\gamma(t)$. Let $\eta : [0, d] \to M$ be the shortest geodesic segment from p to $\gamma(t)$. By definition of Σ, there exists a strictly convex hypersurface, Σ', which is an exterior tangent to Σ at p whose second fundamental form equals $\epsilon'\mathrm{Id}$ at p.

For $s > 0$, let Σ'_s be the hypersurface at constant distance s from Σ'. Let A'_s be the second fundamental form of Σ'_s at $\eta(s)$. By Lemma 3.15:

$$\nabla_{\partial_s} A_s = W_s - A_s^2,$$

where W_s is such that, for all X tangent to Σ_s:

$$W_s(X, X) = \langle R_{\partial_s X}\partial_s, X\rangle.$$

For all s, by definition of M:

$$W_s \geq \mathrm{Id}.$$

Thus:

$$A_s \geq \tanh(s + \hat{\epsilon})\mathrm{Id},$$

where $\hat{\epsilon}'$ is given by:

$$\tanh(\hat{\epsilon}') = \epsilon'.$$

Let $d_{\Sigma'}$ be the distance to Σ'. Then, along η:

$$\mathrm{Hess}(d'_\Sigma) = \tanh(s + \hat{\epsilon}')\mathrm{Id}^\perp,$$

where:

$$\mathrm{Id}^\perp = (\mathrm{Id} - \nabla d_{\Sigma'} \otimes \nabla d_{\Sigma'}).$$

Thus:

$$\begin{aligned} & \mathrm{Hess}({d'_\Sigma}^2) = 2d'_\Sigma \tanh(s + \hat{\epsilon}')\mathrm{Id}^\perp + 2\nabla d_{\Sigma'} \otimes \nabla d_{\Sigma'} \\ \Rightarrow \quad & \mathrm{Hess}({d'_\Sigma}^2)(X, X) \geq 2\mathrm{Min}(d'_\Sigma \tan h(s + \hat{\epsilon}'), 1)\|X\|^2. \end{aligned}$$

Thus, along γ at t:

$$\begin{aligned} (\partial_t^2 (d_{\Sigma'} \circ \gamma)^2) &\geq 2\mathrm{Min}(d \tanh(d + \hat{\epsilon}'), 1) - 2d\langle \nabla d_{\Sigma'}, \nabla_{\partial_t \gamma} \partial_t \gamma \rangle \\ &\geq 2\mathrm{Min}(d \tanh(d + \hat{\epsilon}'), 1) - 2d\epsilon' \\ &\geq 2\mathrm{Min}(d \tanh(d + \hat{\epsilon}') - d\epsilon', 0) \\ &= 0. \end{aligned}$$

It follows that $(d_{\Sigma'} \circ \gamma)^2$ is convex at t. Since:

$$(d_{\Sigma'} \circ \gamma)^2 \geq (d_\Sigma \circ \gamma)^2,$$

and since both functions are equal at t, the result now follows. □

Proposition 4.5. *If $(U_n)_{n \in \mathbb{N}}$ is ϵ-boundary convex for all n, then U_0 is also ϵ-boundary convex away from $\partial \otimes$.*

Proof. This follows by a similar reasoning as before, this time using Lemma 4.4 instead of the convexity of the distance from a geodesic to a convex set. □

4.2. **Graphs Over Extended Normals.** We extend the notion of graphs to extended normals. Let $\Sigma = (S, i)$ be a convex immersed submanifold. We say that Σ is a graph over $\hat{N}\Omega$ if there exists:

(i) a relatively compact open subset $\Omega_\Sigma \subseteq \hat{N}\Omega$ such that $\Omega \subseteq \Omega_\Sigma$;
(ii) a homeomorphism $\alpha : \Sigma \to \Omega_\Sigma$; and
(iii) a continuous function $f : \Omega_\Sigma \to [0, \infty[$,

such that f vanishes along $\partial \Omega_\Sigma$, and for all $p \in \Sigma$:

$$i(p) = \mathrm{Exp}(f(p)(\mathsf{N} \circ \alpha)(p)).$$

We call f and Ω_Σ the graph function and the graph domain respectively of Σ. We define U_f by:

$$U_f = \{(p,t) \text{ s.t. } p \in \Omega_\Sigma \ \&\ t \leq f(p)\}\,.$$

By definition of Σ, U_f is a boundary convex subset of $\mathcal{E}(\Omega_\Sigma)$.

Let Σ and Σ' be two graphs over $\hat{N}\Omega$. Let f, f' and Ω, Ω' be their respective graph functions and graph domains. We define the partial order "$\geq$" over the space of graphs over $\hat{N}\Omega$ such that $\Sigma \geq \Sigma'$ if and only if:

$$U_{f'} \subseteq U_f.$$

In other words, if and only if $\Omega' \subseteq \Omega$ and:

$$f|_{\Omega'} \geq f'.$$

If $\Sigma' \leq \Sigma$, then we say that Σ' lies below Σ.

If Σ is a graph over $\hat{N}\Omega$, we define $\mathrm{Vol}(\Sigma)$ to be the volume of U_Σ. By compactness, this is finite. Trivially:

$$\Sigma' \leq \Sigma \Rightarrow \mathrm{Vol}(\Sigma') \leq \mathrm{Vol}(\Sigma).$$

Moreover equality holds in the above relation if and only if $\Sigma = \Sigma'$.

Lemma 4.6. *Let $\Sigma_1 > \Sigma_2 > \ldots$ be a decreasing sequence of ϵ-convex immersed hypersurfaces which are graphs over $\hat{N}\Omega$. For all i, let f_i be the graph function of Σ_i. There exists an ϵ-convex immersed hypersurface Σ_0 such that:*

(i) *for all i, $\Sigma_i > \Sigma_0$;*
(ii) *Σ_0 is a graph over $\hat{N}\Omega$; and*
(iii) *if f_0 is the graph function of Σ_0 over $\hat{N}\Omega$, then f_0 is $C^{0,1}_{loc}$, and $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the $C^{0,\alpha}_{loc}$ sense for all α.*

Remark. Even without ϵ-convexity, the graph function of the limit would still be $C^{0,1}$ over Ω and the graph functions would also converge accordingly over this set. The ϵ-convexity is required to ensure that the limit function is also $C^{0,1}$ over $\Omega_0 \setminus \Omega$.

Proof. Trivially $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence. There thus exists f_0 to which this sequence converges pointwise. For all $n \in \mathbb{N}\cup\{0\}$, denote $U_n := U_{f_n}$. Trivially, $(U_n)_{n\in\mathbb{N}}$ is a nested sequence. $(U_n)_{n\in\mathbb{N}}$ therefore converges to U_0 in the Haussdorf sense. Since U_n is a graph over $\hat{N}\Omega$ for every n, Proposition 4.5 may be modified to show that U_0 is boundary ϵ-convex at every $p \in \partial U_0$ which does not lie in $\partial_w\mathcal{E}(\Omega)$.

For $p \in \hat{N}\Omega$ and for $n \in \mathbb{N}\cup\{0\}$, define $\hat{p}_n \in \mathcal{E}(\Omega)$ by:

$$\hat{p}_n = (p, f_n(p)).$$

For $\epsilon > 0$, define Ω_ϵ by:

$$\Omega_\epsilon = \left\{ p \in \hat{N}\Omega \text{ s.t. } f_0(p) \geq \epsilon \right\} \cup \Omega.$$

There exists $r_\epsilon > 0$ such that, for every $p \in \Omega_\epsilon$ and for all n:

$$d(\hat{p}_n, \partial_w \mathcal{E}(\Omega)) > 2r_\epsilon.$$

For all n, the supporting tangents to Σ_n over Ω_ϵ are uniformly bounded away from the vertical vector field. Indeed, suppose the contrary, then there exists $p \in \overline{\Omega}_\epsilon$ such that the vertical vector at $\hat{p}_0$ is tangent to Σ_0 at $\hat{p}_0$. However, by continuity, the geodesic segment joining p to $\hat{p}$ lies in U_0. This is a absurd, since Σ_0 is ϵ-convex at p.

Let $\gamma : I \to \Omega_\epsilon$ be a rectifiable curve. For all $n \in \mathbb{N} \cup \{0\}$, let $\gamma_n : I \to \Sigma_n$ be the lift of γ. Since $(f_n)_{n\in\mathbb{N}}$ is unformly bounded over Ω_ϵ, and since its slope is uniformly bounded, there exists B_ϵ, independant of γ, such that, for all $n \in \mathbb{N}$:

$$\text{Length}(\gamma_n) \leq B_\epsilon \text{Length}(\gamma).$$

It follows that the sequence $(\hat{f}_n)_{n\in\mathbb{N}} := (p, f_n)_{n\in\mathbb{N}}$ is uniformly Lipschitz over Ω_ϵ. Thus so is $(f_n)_{n\in\mathbb{N}}$. Consequently f_0 is $C^{0,1}$ over Ω_ϵ and $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the $C^{0,\alpha}$ sense over Ω_ϵ for all α.

Define Ω_0' by:

$$\Omega_0' = f^{-1}(]0, \infty[) \cup \Omega.$$

Let Ω_0 be the connected component of Ω_0' containing Ω. Then f_0 is $C^{0,1}_{\text{loc}}$ over Ω_0 and $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the $C^{0,\alpha}_{loc}$ sense over Ω_0 for all α. Moreover, f_0 extends to a continuous function over the closure of Ω_0 which vanishes on $\partial\Omega_0$. The result follows. □

The following lemma describes an important property of convex graphs which will be referred to as "fatness" in the sequel:

Lemma 4.7. *Let Σ be an ϵ-convex immersed hypersurface which is a graph over $\hat{N}\Omega$. Let $p \in \Sigma$ be an interior point. There exists $\eta > 0$ and a supporting normal N_p to Σ at p such that, for any other supporting normal N_p' to Σ at p:*

$$\langle \mathsf{N}_p, \mathsf{N}_p' \rangle \geq \eta.$$

Proof. Let $\mathcal{N}_p$ denote the set of supporting normals to Σ at p. $\mathcal{N}_p$ is a convex subset of the sphere of unit vectors over p. Let V_p be the vertical vector at p. Since Σ is a strictly convex graph, there exists $\eta_1 > 0$ such that, for every supporting normal $\mathsf{N}_p \in \mathcal{N}_p$:

$$\langle \mathsf{N}_p, V_p \rangle \geq \eta_1.$$

$\mathcal{N}_p$ is thus strictly contained in the hemisphere about V_p. We denote this hemisphere by H. If $V_p \in \mathcal{N}_p$, then the result follows with $\mathsf{N}_p = V_p$. Suppose therefore that $V_p \notin \mathcal{N}_p$. By convexity, there exists a totally geodesic subsphere S, orthogonal to ∂H such that $V_p \in S$ and $\mathcal{N}_p$ lies strictly to one side of S in H. Let S'

be obtained by rotating S about $S \cap H$ until it meets $\mathcal{N}_p$. Choose $\mathsf{N}_p \in S' \cap \mathcal{N}_p$. N_p has the desired properties, and the result follows. $\square$

4.3. **The Dirichlet Problem II.** Let M be an $(n+1)$-dimensional Hadamard manifold of sectional curvature bounded above by -1. Let Σ be a smooth convex immersed hypersurface in M. Let $\Omega \subseteq \Sigma$ be an open subset and let $\hat{\Sigma}$ be a convex immersed hypersurface in M which is a graph over the extended normal of Ω.

Proposition 4.8. *For all $p \in M$, for every normal vector N_p over p, for all $\theta \in [(n-1)\pi/2, n\pi/2[$, and for all sufficiently small $\epsilon > 0$, there exists $\delta > 0$ and an immersed hypersurface Σ of radius δ about p which is normal to N_p at p such that:*

$$R_\theta(\Sigma) = \epsilon \tan^{-1}(\theta/n) \text{ and } \|A_\Sigma\| \leq 2\epsilon,$$

where A_Σ is the second fundamental form of Σ.

Remark. Such disks will be refered to as δ-adapted disks. They are important for the use of the Perron method in the proof of Theorem 1.2.

Proof. We use the Implicit Function Theorem for elliptic operators. Let Σ_0 be an immersed hypersurface in M which is normal to N_p at p such that:

$$A_0 = \epsilon \mathrm{Id},$$

where A_0 is the second fundamental form of Σ_0.

Let N_0 be the normal vector field over Σ_0. Let $f : \Sigma_0 \to \mathbb{R}$ be a smooth function representation an infinitesimal normal deformation of Σ_0. Then by Lemma 3.1 of [25]:

$$D\mathrm{SL}_\theta \cdot f = -\Delta^B f + gf,$$

where g is a bounded function. For $\epsilon > 0$ sufficiently small, A_0 is bounded above and below over $B_\delta(p)$. Moreover by Lemma 3.1 of [25], since the sectional curvature of M is bounded above by -1, for ϵ sufficiently small $g > 0$. There thus exists $K > 0$ such that, for all smooth f of compact support:

$$\langle D\mathrm{SL}_\theta \cdot f, f \rangle_{L^2} \geq K \|f\|^2_{L^2}.$$

Thus, if $G : C^\infty(B_\epsilon(p)) \to C^\infty_0(B_\epsilon(p))$ is the Green's operator of $D\mathrm{SL}_\theta$, then:

$$\|G\| \leq K.$$

The radius in $W_{2,p}$ over which the inverse of SL_θ is defined is determined by the norms of G, $D\mathrm{SL}_\theta$ and $D^2\mathrm{SL}_\theta$. It is thus uniformly bounded below as the radius, δ, tends to 0. However, the $W_{2,p}$ distance between $\mathrm{SL}_\theta(\Sigma_0)$ and the constant function tends to 0 as δ tends to 0. Thus, for δ sufficiently small, the Implicit Function Theorem yields an immersed hypersurface of constant special Lagrangian curvature. This reasoning can be adapted to ensure that the resulting hypersurface passes through p, is normal to N_p at p and has second fundamental form colinear to Id at p. The result follows by reducing δ further if necessary. $\square$

We now prove Theorem 1.2:

Proof of Theorem 1.2: We suppose that $\theta > (n-1)\pi/2$. The case $\theta = (n-1)\pi/2$ follows by approximation. We consider first the case where $r > R_0$. By Lemma 2.2, there exists $\epsilon_0 > 0$ which only depends on r and θ such that $\hat{\Sigma}$ is ϵ_0-convex. Let Σ' be an ϵ-convex immersion in M which is a graph over $\hat{\Omega}$ such that $\Sigma' \leq \hat{\Sigma}$ and $R_\theta(\Sigma') \geq r$ in the weak sense. Let $p \in \Sigma'$ be an interior point. Let N_p be a supporting normal to Σ' at p. By Lemma 4.7 (fatness), N_p may be chosen such that for any other supporting normal N'_p to Σ' at p:

$$\langle \mathsf{N}_p, \mathsf{N}'_p \rangle \geq \epsilon_1,$$

for some $\epsilon_1 > 0$. Let $0 < \delta \ll \epsilon_0$ be small. Let $(\Sigma_\delta, \partial\Sigma_\delta)$ be a δ-adapted disk with normal N_p. By ϵ_0-convexity, $\partial\Sigma_\delta$ lies above Σ' for δ sufficiently small. Let $(\Sigma_{\delta,t})$ be a family of inward deformations of Σ_δ (in the direction opposite to N_p) such that $\partial\Sigma_{\delta,t}$ lies above Σ' for all t. By making Σ_δ smaller if necessary, we may assume that, for sufficiently small t, $\Sigma_{\delta,t}$ still has constant θ-special Lagrangian curvature.

Since Σ' is strictly convex, for sufficiently small t, there exists a non-trivial open subset $\Omega_t \subseteq \Sigma_{\delta,t}$ which is relatively compact with respect to $\Sigma_{\delta,t}$ and such that a portion of Σ' is a graph over Ω. We denote this portion by $\Sigma'_{\delta,t}$.

By continuity of the supporting normal to a convex set, there exists $t_0 > 0$ such that, for $t < t_0$, the pair $(\Omega_t, \Sigma'_{\delta,t})$ satifies the hypotheses of Theorem 3.22. There thus exists a convex immersion $\Sigma^r_{\delta,t}$ such that:

(i) $\Sigma^r_{\delta,t}$ is a graph over Ω_t;
(ii) $\Sigma^r_{\delta,t}$ lies beneath $\Sigma'_{\delta,t}$ as a graph over Ω_t;
(iii) $\Sigma^r_{\delta,t}$ is smooth away from the boundary; and
(iv) $R_\theta(\Sigma^r_{\delta,t}) = r$.

We define Σ''_t by replacing the portion $\Sigma'_{\delta,t}$ of Σ' with $\Sigma^r_{\delta,t}$. By Lemma 2.4, $R_\theta(\Sigma'') \geq r$ in the weak sense. Moreover, Σ''_t can be chosen to vary continuously with t.

Suppose that, for $t < t_0$, $\partial\Omega_t$ does not intersect $\partial\Sigma' = \partial\Omega$. Then, for all t, $\Sigma'_{\delta,t}$ is a strict graph over $\hat{N}\Omega$ which lies below $\hat{\Sigma}$. Indeed, suppose first that there exists $t_1 < t_0$ and $p \in \Sigma^r_{\delta,t}$ which also lies in Ω. Let t_1 be the first such time. Since $\partial\Omega_t$ does not intersect $\partial\Omega$, p is an interior point. Since t_1 is the first intersection time, $\Sigma^r_{\delta,t}$ is an exterior tangent to Ω at this point. However, at p:

$$R_\theta(\Sigma^r_{\delta,t}) = r > R_0.$$

This is absurd, by the Geometric Maximum Principal.

Suppose now that $\Sigma'_{\delta,t}$ is not a graph over $\hat{N}\Omega$. By continuity, there exists in interior point $p \in \Sigma'_{\delta,t}$ such that the vertical vector at p is tangent to $\Sigma'_{\delta,t}$ at p. By continuity, the vertical geodesic segment joinging $\hat{N}\Omega$ to p lies below $\Sigma'_{\delta,t}$. It follows that the vertical vector at p is an interior tangent to $\Sigma'_{\delta,t}$ at p. This is impossible by strict convexity.

We denote by A the above described operation for obtaining new convex immersions out of old ones. Let Σ_1 and Σ_2 be two convex immersions which are graphs over $\hat{N}\Omega$ such that $\Sigma_1, \Sigma_2 \leq \hat{\Sigma}$ and $R_\theta(\Sigma_1), R_\theta(\Sigma_2) \geq r$ in the weak sense. Let f_1 and f_2 be their respective graph functions. Define $f_{1,2}$ by:

$$f_{1,2} = \mathrm{Min}(f_1, f_2),$$

Let $\Sigma_{1,2}$ be the graph of $f_{1,2}$. Trivially, $\Sigma_{1,2} \leq \hat{\Sigma}$. Moreover, by Lemma 2.4, $R_\theta(\Sigma_{1,2}) \geq r$ in the weak sense. We denote this operation for obtaining new convex immersions out of old ones by B.

Let $\mathcal{F}$ be the family of all convex immersions which may be obtained from $\hat{\Sigma}$ by a finite combination of the operations A and B. Define $V_0 \geq 0$ by:

$$V_0 = \mathrm{Inf}\left\{\mathrm{Vol}(\Sigma) \text{ s.t. } \Sigma \in \mathcal{F}\right\}.$$

There exists a sequence $\Sigma_1 > \Sigma_2 > \dots$ of strictly convex immersions in $\mathcal{F}$ such that:

$$\mathrm{Vol}(\Sigma_n)_{n\in\mathbb{N}} \to V_0.$$

For all $n \in \mathbb{N}$, let f_n and $\hat{\Omega}_n$ be the graph function and graph domain of Σ_n respectively. $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence. By Proposition 4.6, there exists $f_0 : \hat{\Omega}_0 \to [0, \infty[$ such that:

(i) f_0 is continuous over the closure of $\hat{\Omega}_0$;
(ii) f_0 vanishes along $\partial\hat{\Omega}_0$;
(iii) f_0 is $C^{0,1}_{\mathrm{loc}}$ inside $\hat{\Omega}_0$;
(iv) $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the $C^{0,\alpha}_{\mathrm{loc}}$ sense over $\hat{\Omega}_0$ for all α; and
(v) if Σ_0 is the graph of f_0, then Σ_0 is ϵ-convex.

Let $p \in \Omega_0$ be an interior point. Let N_p be a supporting normal to Σ_0 at $\hat{p}$ chosen such that, for any other supporting normal N'_p at $\hat{p}$:

$$\langle \mathsf{N}_p, \mathsf{N}'_p \rangle \geq \epsilon_1,$$

for some $\epsilon_1 > 0$. For all n, let d_n be the restriction to Σ_n of the length metric of $\mathcal{E}(\Omega)$. The construction outlined at the beginning of the proof may be carried out uniformly near $\hat{p}$ for all n. We thus obtain $t > 0$ and for all n:

(i) $\Omega_{t,n}$;
(ii) $\Sigma_{n,\delta,t}$; and
(iii) $\Sigma^r_{n,\delta,t}$,

such that, for all n:

(i) $\Sigma_{n,\delta,t}$ and $\Sigma^r_{n,\delta,t}$ are graphs above $\Omega_{t,n}$;
(ii) $\Sigma^r_{n,\delta,t}$ lies below $\Sigma_{n,\delta,t}$; and

(iii) $\Sigma^r_{n,\delta,t}$ has radius at least ϵ_2 about $\hat{p}_n$ with respect to d_n for some fixed $\epsilon_2 > 0$, where $\hat{p}_n$ is the point in $\hat{\Sigma}_{n,\delta,t}$ lying above p_0.

For all n, we define Σ'_n by replacing the portion $\Sigma_{n,\delta,t}$ of Σ_n with $\Sigma^r_{n,\delta,t}$. For all n, $\Sigma'_n \in \mathcal{F}$ and $\Sigma'_n \leq \Sigma_n$. Let Σ'_0 be the limit of $(\Sigma'_n)_{n\in\mathbb{N}}$. Trivially:

$$\Sigma'_0 \leq \Sigma_0.$$

We assert that $\Sigma'_0 = \Sigma_0$. Indeed, otherwise, $\Sigma'_0 \neq \Sigma_0$, in which case:

$$\mathrm{Vol}(\Sigma'_0) < \mathrm{Vol}(\Sigma_0),$$

which is absurd. By Theorem 1.4 of [25], it follows that Σ_0 is smooth over a radius of ϵ_2 about $\hat{p}_0$. Since $p \in \hat{\Omega}_0$ is abitrary, it follows that Σ_0 is smooth over the whole of $\hat{\Omega}_0$. Moreover, $R_\theta(\Sigma_0) = r$, and the result follows for $r > R_0$.

Let Σ^r be the hypersurface obtained in this manner such that $R_\theta(\Sigma^r) = r$. Then, for all $r > r'$:

$$\Sigma^r > \Sigma^{r'}.$$

Thus, taking the limit as r tends to R_0 yields the desired solution when $r = R_0$. The result follows. □

5. The Perron Method II

5.1. **Pseudo-Immersions.** In order to prove Theorem 1.1, we require a compactification of the space of convex immersions when there is no ambient end. To this end, we define pseudo-immersions.

Let M be an $(n+1)$-dimensional Hadamard manifold. Let TM and $UM \subseteq TM$ be the tangent and unitary bundles respectively over M. Let $\pi : TM \to M$ be the canonical projection. Let N be a compact n-dimensional manifold without boundary. A pseudo-immersion of N into M is a pair $(\varphi, \hat{\varphi})$ where:

(i) $\varphi : N \to M$ is a $C^{0,1}$ mapping; and
(ii) $\hat{\varphi} : N \to UM$ is an injective $C^{0,1}$ mapping,

such that:

$$\pi \circ \hat{\varphi} = \varphi.$$

In the sequel, we will denote such a pair simply by φ. Since φ is Lipschitz, the path metric and the volume of M pull back to a (possibly degenerate) path metric and volume form over N, which we denote by d_φ and Vol_φ respectively. Likewise, the path metric of UM pulls back to a path metric over N, which we denote by $\hat{d}_\varphi$. Since $\hat{\varphi}$ is injective, $\hat{d}_\varphi$ is non-degenerate. For $p \in N$, we denote the balls of radius r in N about p with respect to d_φ and $\hat{d}_\varphi$ by $B_r(p; N)$ and $\hat{B}_r(p; N)$ respectively. We denote these simply by $B_r(p)$ and $\hat{B}_r(p)$ respectively when there is no ambiguity concerning the ambient manifold.

We say that a sequence $(\varphi_n, \hat{\varphi}_n)_{n\in\mathbb{N}}$ converges to $(\varphi_0, \hat{\varphi}_0)$ if and only if $(\varphi_n)_{n\in\mathbb{N}}$ and $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ converge to φ_0 and $\hat{\varphi}_0$ respectively in the $C^{0,\alpha}$ sense over N for all α.

For $r > 0$, if φ is a pseudo-immersion and $p \in N$, we say that φ is convex over a radius of $r > 0$ at p, if and only if there exists a convex set $K \subseteq M$ such that:

(i) $\varphi(p) \in \partial K$;
(ii) $\hat{\varphi}(p)$ is normal to K at $\varphi(p)$; and
(iii) $\varphi(\hat{B}_r(p)) \subseteq K$.

We say that a pseudo-immersion, φ, is convex if and only if there exists $r > 0$ such that φ is convex over a radius r at every point of N. For $\epsilon > 0$, we define ϵ-convexity in an analogous manner. Trivially, every convex immersion is also a convex pseudo-immersion and every ϵ-convex immersion is also an ϵ-convex pseudo-immersion.

For a convex pseudo-immersion φ, we define the mapping $\Phi : N \times [0, \infty[\to M$ by:

$$\Phi(p, t) = \mathrm{Exp}(t\hat{\varphi}(p)).$$

Proposition 5.1. *For every $p \in N$, there exists a neighbourhood $p \in U \subset N$ such that the restriction of Φ to $U \times [0, \infty[$ is injective.*

Remark. By conservation of the domain, the restriction of Φ to this set is then a homeomorphism onto its image.

Remark. In this case, we refer to Φ as the end of φ. We furnish the manifold $N \times]0, \infty[$ with the differential structure of M pulled back through Φ. We also refer to the resulting manifold as the end of ϕ, and we denote it by $\mathcal{E}(\varphi)$.

Proof. Let $r > 0$ be such that φ is convex over a radius r about every point in N. Choose $q \in \hat{B}_{r/2}(p)$. Let K_p and K_q be convex sets as in the definition of convexity at p and q. Let $K = K_p \cap K_q$. K is also convex. Thus, if γ is the geodesic segment joining p to q, then γ lies in K and thus makes an optuse angle with any normal vector to K at p and q. Consequently, the half geodesics leaving $\varphi(p)$ and $\varphi(q)$ in the respective directions of $\hat{\varphi}(p)$ and $\hat{\varphi}(q)$ never intersect. Since $q \in \hat{B}_{r/2}(p)$ is arbitrary, the result follows. □

Let $\varphi, \varphi' : N \to M$ be two convex pseudo-immersions. We say that φ' is a graph over φ if and only if there exists a $C^{0,1}$ function $f : N \to [0, \infty[$ such that for all $p \in N$:

$$\varphi'(p) = \mathrm{Exp}_{\varphi(p)}(f(p)\hat{\varphi}(p)).$$

We observe that if φ' is a graph over φ, then $\mathrm{Vol}_{\varphi'} \geq \mathrm{Vol}_{\varphi}$, with equality if and only if $\varphi' = \varphi$. We thus define the partial order "$\leq$" on the set of convex pseudo-immersions such that $\varphi \leq \varphi'$ if and only if φ' is a graph over φ.

Proposition 5.2. *Let $\varphi : N \to M$ be an ϵ-convex pseudo-immersion. Let $\Sigma \subseteq M$ be a convex immersed hypersurface such that:*

(i) *the second fundamental form of* Σ *is bounded above by* ϵ *in the weak sense;*
(ii) $\varphi(p) \in \Sigma$ *and* $\hat{\varphi}(p)$ *is normal to* Σ *at* $\varphi(p)$*; and*
(iii) Σ *has radius at most* ϵ^{-1} *about* $\varphi(p)$*.*

Then Σ *lifts to an immersed hypersurface in* $\mathcal{E}(\varphi)$*.*

Proof. This follows from Lemma 4.4. □

Proposition 5.3. *Let* $K \subseteq M$ *be compact. Choose* $\epsilon > 0$ *and let* $\varphi : N \to M$ *be an* ϵ*-convex pseudo-immersion such that* $\varphi(N) \subseteq K$*. There exists* r*, which only depends on* K *and* ϵ *such that* φ *is* ϵ*-convex over a radius of* r*.*

Remark. This lemma makes ϵ-convexity uniform over sequences ensuring that this property is preserved when limits are taken.

Proof. Choose $p \in N$. Let $\Sigma \subseteq M$ be a convex immersed hypersurface normal to $\hat{\varphi}(p)$ at p such that the norm of its second fundamental form is bounded above by ϵ. For $r > 0$, let Σ_r be the ball of radius r in Σ about p. By Proposition 5.2, Σ_{r_1} lifts to an immersed hypersurface $\hat{\Sigma}$ in $\mathcal{E}(\varphi)$ for some $r_1 > 0$. There exists $r_2 > 0$ be such that $\hat{B}_{r_2}(p, \Sigma) \subseteq \Sigma_{r_1}$. There exists a neighbourhood, U of $\hat{\varphi}(p) \in UM$ such that every geodesic passing through U intersects Σ_{r_1} transversely. Consequently, the slope of $\hat{\Sigma}_{r_1}$ as a graph over φ is uniformly bounded for $\varphi(q) \in U$. There therefore exists $K_1, r_3 > 0$ such that, if γ is a curve in $\hat{B}_{r_3}(p, N)$ and $\hat{\gamma}$ is the curve in $\hat{\Sigma}$ lying above γ, then:

$$\hat{l}(\gamma)/K_1 \leq \hat{l}(\hat{\gamma}) \leq K_1 \hat{l}(\gamma),$$

where $\hat{l}$ denotes length with respect to $\hat{d}$. Thus there exists $r_4 > 0$ such that a subset of $\hat{\Sigma}_r$ is a graph over $\hat{B}_{r_4}(p, N)$. In otherwords, for all $q \in \hat{B}_{r_4}(p, N)$, every half-geodesic leaving $\varphi(q)$ in the direction of $\hat{\varphi}(q)$ intersects $\hat{\Sigma}_{r_1}$ non-trivially.

Let Ω be a convex set such that $\Sigma_{r_1} \subseteq \partial\Omega$. Define $d_\Omega : M \to [0, \infty[$ by:

$$d_\Omega(q) = d(q, \Omega).$$

d_Ω is a convex function over M. Choose $q \in B_{r_4}(p; N)$ and suppose that $\varphi(q) \notin \Omega$. Since the half-geodesic leaving $\varphi(q)$ in the direction of $\hat{\varphi}(q)$ intersects Ω non-trivially, and since d_Ω is a convex function, at q:

$$\langle \hat{\varphi}(q), \nabla d_\Omega(q) \rangle < 0.$$

For sufficiently small r_4, this is not possible and we therefore obtain the desired value for r. Since this construction may be carried out uniformly for $\varphi(p) \in K$, the result follows. □

Lemma 5.4. *Let* $\varphi : N \to M$ *be a smooth strictly convex immersion. Let* $(\varphi_n)_{n \in \mathbb{N}} : N \to M$ *be* ϵ*-convex pseudo-immersions such that:*

(i) *for all* n*,* $\varphi_n \leq \varphi$*;*
(ii) *there exists* $p \in N$ *such that* $(f_n(p))_{n \in \mathbb{N}}$ *is bounded; and*

(iii) *there exists* $\delta > 0$ *and* $N \in \mathbb{N}$ *such that for* $n \geq N$*:*

$$f_n \geq \delta.$$

Then, there exists an ϵ*-convex immersion* $\varphi_0 : N \to M$ *such that:*

(i) $\varphi_0 \leq \varphi$*; and*

(ii) $(\varphi_n)_{n\in\mathbb{N}}$ *subconverges to* φ_0*.*

Moreover, $(\hat{d}_n)_{n\in\mathbb{N}\cup\{0\}}$ *is uniformly equivalent to* d *over* N*.*

Proof. Since φ is smooth, d_φ and $\hat{d}_\varphi$ are equivalent. Let $\gamma : I \to N$ be a curve. By convexity, for all n:

$$\text{Length}(\varphi \circ \gamma) \geq \text{Length}(\varphi_n \circ \gamma).$$

Thus, for all n, φ_n is 1-Lipschitz. Moreover, for all $p \in N$:

$$f_n(p) = d(p, \varphi_n(p)).$$

Thus, for all n and for all $p, q \in N$:

$$|f_n(p) - f_n(q)| \leq d(p, q) + d(\varphi_n(p), \varphi_n(q)) \leq 2d(p, q).$$

Thus f_n is 2-Lipschitz for all n. Since $(f_n)_{n\in\mathbb{N}} = (d(\varphi_n(p), \varphi(p)))_{n\in\mathbb{N}}$ is uniformly bounded at one point, there exist $C^{0,1}$ functions $\varphi_0 : N \to M$ and $f_0 : N \to [0, \infty[$ to which $(\varphi_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ respectively converge in the $C^{0,\alpha}$ sense over N for all α. By condition (iv), $f_0 > 0$ over N.

For all n, and for all $p \in N$:

$$\hat{\varphi}_n(p) = \frac{1}{f_n(p)} \text{Exp}^{-1}_{\varphi_n(p)}(\varphi(p)).$$

There thus exists $\hat{\varphi}_0$ to which $(\hat{\varphi}_n)_{n\in\mathbb{N}}$ converges in the $C^{0,\alpha}$ sense over N for all α. By Proposition 5.3, φ_0 is ϵ-convex.

Let γ be a curve in N. Define $\tilde{\gamma}_n$ by:

$$\tilde{\gamma}_n = (f_n\hat{\varphi}_n)(\gamma_n(t)).$$

Since f_n and $\hat{\varphi}_n$ are uniformly bounded in the $C^{0,1}$ sense, there exists $B > 0$ such that, for all n:

$$\text{Length}(\tilde{\gamma}_n) \leq B\text{Length}(\gamma)_n.$$

Conversely, since the derivative of the exponential mapping is bounded over any compact subset of TM, and since $\text{Exp}(\tilde{\gamma}_n) = \gamma_n$, by increasing B if necessary, we obtain, for all n:

$$\text{Length}(\gamma_n) \leq B\text{Length}(\tilde{\gamma}_n).$$

For all n, and for all $p \in N$, define $\eta_{n,p}$ to be the geodesic leaving $\varphi_n(p)$ in the direction of $\hat{\varphi}_n(p)$. Since φ is strictly convex and is a graph over φ_n, there exists $\epsilon > 0$ such that, for all $p \in N$ and for all n:

$$\langle \partial_t \eta_{n,p}, \hat{\varphi}(p) \rangle \geq \epsilon.$$

Indeed, let B be a small ball lying on the outside of φ and tangent to φ at p. Let N be such that, for $n \geq N$, $f_n(p) > 0$. Since φ is smooth, moving B inwards slightly and intersecting with the interior of φ yields a convex set K_p lying in the end of φ_n for all $n \geq N$. If $\gamma_n(p)$ is the geodesic leaving $\varphi_n(p)$ in the direction of $\hat{\varphi}_n(p)$, then there exists $\epsilon_1 > 0$ such that:

$$\gamma_n(p)([f_n(p) - \epsilon, f_n(p)]) \subseteq K.$$

This yields a sequence of geodesic segments with length uniformly bounded below. The assertion now follows, since otherwise, these segments would converge to a segment tangent to φ at p, which is impossible, by the strict convexity of φ.

Thus, the derivative of the projection onto UM is uniformly bounded below along $\tilde{\gamma}$. So, by increasing B again if necessary, we obtain, for all n:

$$\frac{1}{B}\text{Length}(\gamma; \hat{d}_{\varphi_n}) \leq \text{Length}(\gamma; \hat{d}_{\varphi}) \leq B\text{Length}(\gamma; \hat{d}_{\varphi_n}).$$

The result follows. □

5.2. **The Isotopy Problem.** We now prove Theorem 1.1:

Proof of Theorem 1.1: Suppose first that $\theta > (n-1)\pi/2$. Let $\varphi : N \to M$ be the immersion. We may assume that $R_\theta(\varphi) \geq r$ in the weak sense. Indeed, let $\hat{\varphi} : N \to UM$ be the exterior normal over N. For $t \geq 0$, define φ_t by:

$$\varphi_t(p) = \text{Exp}_{\varphi(p)}(t\hat{\varphi}(p)).$$

Since the sectional curvature of M is bounded above by -1, for all $\epsilon > 0$, there exists $T > 0$ such that for $t \geq T$, φ_t is $(1-\epsilon)$-convex. In particular, for ϵ sufficiently small, $R_\theta(\varphi_t) \geq r$. We may thus replace φ with φ_t for sufficiently large t.

By Lemma 2.13, we may assume that φ is smooth. Let $\varphi' : N \to M$ be a convex immersion such that $\varphi \geq \varphi'$, and $R_\theta(\varphi') \geq r$ in the weak sense. Choose $p \in N$. By Propositions 5.2 and 4.8, we may construct an adapted disk $(\Sigma, \partial\Sigma)$ at p which is normal to $\hat{\varphi}(p)$ and which lifts to $\mathcal{E}(\varphi')$. By choosing the norm of the second fundamental form of Σ sufficiently small, we may assume that Σ has negative curvature.

Let $(\Sigma_t)_{t\in[0,\epsilon[}$ be a family obtained by moving Σ downwards (in the direction opposite to $\hat{\varphi}'(p)$). For sufficiently small ϵ, Σ_t can be chosen to be adapted for all t. Moreover, the norm of the second fundamental form of Σ may be chosen sufficiently small so that the intersection of $\varphi'(N)$ with Σ is η-convex, for some $\eta > 0$. Finally, we assume that $\partial\Sigma_t$ lies in $\mathcal{E}(\varphi')$ for all t.

Let $(\Omega_t)_{t\in[0,\epsilon[}$ be the continuous family of connected open subsets of Σ_t defined such that $\Omega_0 = \{p\}$ and $\partial\Omega_t = \varphi'(N) \cap \Sigma_t$. Let Σ'_t be the portion of $\varphi'(N)$ lying above Ω_t. We claim that, for all t, Ω_t is a convex open set with non-trivial interior. Indeed, suppose that Ω_t degenerates. By strict convexity, this is only possible if Ω_{t_0} is a single point for some $t_0 > 0$. By Lemma 2.5, $\varphi'(N)$ is the boundary of a convex set, and is therefore homotopically trivial, which contradicts the hypotheses. The assertion follows.

We now claim that, for all t, Σ'_t is a graph over the extended normal of Ω_t. Indeed, suppose the contrary. By continuity and strict convexity, there exists $t_0 > 0$ such that, either, the graph of Σ'_{t_0} is vertical over Ω_{t_0} at some interior point, or the outward normal of Σ'_{t_0} points vertically downwards at some point on the boundary. The former case is excluded by strict convexity of Σ'_t. In the latter case, Ω_{t_0} is a single point, in which case $\varphi'(N)$ is the boundary of a convex set, which contradicts the hypotheses as before. The assertion follows.

Choose $0 < t < \epsilon$. By Theorem 1.2, there exists $\Sigma'_{t,r}$ which is smooth up to the boundary, and which is a graph over the extended normal of Ω_t lying between Ω_t and Σ'_t such that:

$$R_\theta(\Sigma'_{t,r}) = r.$$

We define φ'' by replacing Σ'_t in φ' with $\Sigma'_{t,r}$. φ'' is a convex immersion and $\varphi'' \leq \varphi'$. By Lemma 2.5, $R_\theta(\varphi'') \geq r$ in the weak sense. Moreover, by examening the proof of Theorem 1.2, if $\Sigma'_{t,r}$ is chosen to be the maximal solution (in the sense that its graph function is maximal), then φ'' is obtained from φ' by isotopic deformation. In particular, this implies as before that φ is a graph over φ''.

Let $\mathcal{F}$ be the family of convex immersions in M obtained by a finite number of iterations of the operation described above. By Lemma 4.1 of [11], if φ_1 and φ_2 are two convex immersions in $\mathcal{F}$, then there exists a third convex immersion $\varphi_{1,2}$ in $\mathcal{F}$ such that $\varphi_1, \varphi_2 \geq \varphi_{1,2}$. For $\varphi' \in \mathcal{F}$, let $\mathrm{Vol}(\varphi')$ denote the volume between φ' and φ in the end of φ'. Define V_0 by:

$$V_0 = \mathrm{Sup}\left\{\mathrm{Vol}(\varphi') \text{ s.t. } \varphi' \in \mathcal{F}\right\}.$$

There exists a sequence $\varphi_1 \geq \varphi_2 \geq \ldots$ in $\mathcal{F}$ such that:

$$(\mathrm{Vol}(\varphi_n))_{n\in\mathbb{N}} \to V_0.$$

For all n, define d_n by:

$$d_n = \mathrm{Inf}\left\{f_n(p) \text{ s.t. } p \in N\right\}.$$

We claim that $(d_n)_{n\in\mathbb{N}}$ is bounded. Indeed, suppose the contrary. Since the sectional curvature of M is bounded above by -1, by convexity:

$$\mathrm{Diam}(\varphi_n) \leq \mathrm{Log}(\sinh(d_n)))^{-1}\mathrm{Diam}(\varphi).$$

Thus, as $(d_n)_{n\in\mathbb{N}} \to \infty$, $\mathrm{Diam}(\varphi_n)_{n\in\mathbb{N}} \to 0$. This contradicts the hypotheses on N, and the assertion follows. In particular, V_0 is finite.

Thus, by Lemma 5.4, there exists an ϵ-convex pseudo-immersion $\varphi_0 : N \to M$ such that $\varphi_0 < \varphi$ to which $(\varphi_n)_{n\in\mathbb{N}}$ subconverges. Since φ_0 maximises volume, by an analogous reasoning to that used in the proof of Theorem 1.2, φ_0 is smooth and:

$$R_\theta(\varphi_0) = r.$$

By construction, φ_0 is isotopic to φ and (i) follows.

Suppose now that $\theta = (n-1)\pi/2$. Let $(\theta_n)_{n\in\mathbb{N}}$ be a decreasing sequence converging to θ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence converging to r. For all n, let $\varphi_n : N \rightarrow M$ be a smooth immersion such that $\varphi_n \leq \varphi$ and:

$$R_{\theta_n}(\varphi_n) = r_n.$$

By Theorem 1.4 of [25], there exists a (possibly degenerate) immersion $\varphi_0 : N \rightarrow M$ to which $(\varphi_n)_{n\in\mathbb{N}}$ subconverges. In the degenerate case, the image of φ_0 is a bundle of $(n-1)$-dimensional spheres over a complete geodesic. By compactness, it follows that $N = S^{n-1}\times S^1$, which contradicts the hypotheses. φ_0 is therefore not degenerate, and so:

$$R_\theta(\varphi_0) = r.$$

(ii) follows, and this concludes the proof. □

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