CONJUGACY OF PIECEWISE LINEAR LORENZ MAP THAT EXPAND ON AVERAGE

HONG-FEI CUI AND YI-MING DING

ABSTRACT. For piecewise linear Lorenz map that expand on average, we show that it admits a dichotomy: it is either periodic renormalizable or prime. As a result, such a map is conjugate to a β-transformation.

1. INTRODUCTION

A *Lorenz map* on $I = [a,\ b]$ is an interval map $f : I \to I$ such that for some $c \in (a,\ b)$ we have

(i) f is strictly increasing on $[a,\ c)$ and on $(c,\ b]$;

(ii) $f(c-) = \lim_{x \uparrow c} f(x) = b$, $f(c+) = \lim_{x \downarrow c} f(x) = a$.

If, in addition, f satisfies the topological expanding condition

(iii) The pre-images set $C = \cup_{n \geq 0} f^{-n}(c)$ of c is dense in I,

then f is said to be *expanding* [8].

A Lorenz map is said to be *piecewise linear* on $[0,\ 1]$ if it is linear on both intervals $[0, c)$ and $(c, 1]$. Such a map is of the form

$$f_{a,b,c}(x) = \begin{cases} ax + 1 - ac & x \in [0, c) \\ b(x - c) & x \in (c, 1]. \end{cases} \tag{1. 1}$$

The average slope of $f_{a,b,c}$ is $\int f'_{a,b,c}(x)dx = ac + b(1 - c)$. We say that $f_{a,b,c}$ *expand on average* if the average slope $ac + b(1 - c)$ is greater than 1. It is easy to see that the average slope is greater than 1 if and only if $f_{a,b,c}(0) < f_{a,b,c}(1)$. Note that we may have $a < 1 < b$ or $a > 1 > b$ because we only assume $ac + b(1 - c) > 1$. In both cases, $f_{a,b,c}$ is contract on some interval.

The map $T_{\beta,\alpha}$ defined by

$$T_{\beta,\alpha} = \beta x + \alpha \qquad \mod 1$$

is called a β-transformation (see [7]). When $1 < \beta \leq 2$, $0 \leq \alpha < 1$, $T_{\beta,\alpha} = f_{\beta,\beta,c}$ with $c = (1 - \alpha)/\beta$.

The study of β-transformation goes back to Rényi. Based on bounded distortion principe, Rényi proved that β-transformation admits an acip (absolutely

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Corresponding author:Yi-Ming Ding.

continuous invariant probability measure with respect to the Lebesgue measure). Gelfond [6] and Parry [13] [14] obtained the expression of the density of the acip. Flatto and Lagarias [4, 5] studied the lap counting functions. For piecewise linear Lorenz map that expand on average, we proved [3] that such a map admits an ergodic acip because there exists a positive integer n so that $(f^n_{a,b,c})'(x) > \lambda > 1$ for all $x \in I$ except countable points. Such a map is always expanding.

Follows from Milnor and Thurston [11], a Lorenz map f is semi-conjugate to a β-transformation. According to Parry [15], f is conjugate to a β-transformation if f is strongly transitive. Since an expanding Lorenz map is strongly transitive if and only if it is prime [2], it is interesting to know when a renormalizable expanding Lorenz map is conjugate to a β-transformation.

The main purpose of this paper is to prove the following Theorem.

Main Theorem. *If $f_{a,b,c}$ is a piecewise linear Lorenz map that expand on average, then it is conjugate to a β-transformation.*

Our proof is based on the characterization of the renormalizations of piecewise linear Lorenz map.

Definition 1. A Lorenz map $f\colon I \to I$ is said to be *renormalizable* if there is a proper subinterval $[u,\ v] \ni c$ and integers $\ell, r > 1$ such that the map $g\colon [u,\ v] \to [u,\ v]$ defined by

$$g(x) = \begin{cases} f^{\ell}(x) & x \in [u,\ c), \\ f^{r}(x) & x \in (c,\ v], \end{cases} \tag{1. 2}$$

is itself a Lorenz map on $[u,\ v]$. The interval $[u,\ v]$ is called the *renormalization interval.*

If f is not renormalizable, it is said to be *prime.*

The renormalization map g is the first return map of f on the renormalization interval $[u,\ v]$(cf. [10]). Let f be a renormalizable Lorenz map. f may have different renormalizations (cf. [8], [10]). A renormalization $g = (f^{\ell}, f^{r})$ of f is said to be *minimal* if for any other renormalization $(f^{\ell'}, f^{r'})$ of f we have $\ell' \geq \ell$ and $r' \geq r$ (cf. Glendinning and Sparrow [8], Martens and de Melo [10], etc.).

For any nonempty open interval $U \subseteq I$, put

$$N(U) = \min \{n \geq 0 : \exists z \in U \ \text{ such that } f^n(z) = c\} \tag{1. 3}$$

as the index of continuity for the interval U. $|U|$ is the length of U. A subset E of I is completely invariant under f if

$$f(E) = f^{-1}(E) = E,$$

and it is proper if $E \neq I$.

Theorem A. [2] *Let f be an expanding Lorenz map. There is a one-to-one correspondence between the renormalizations and proper completely invariant closed* *sets of f. More precisely, suppose E is a proper completely invariant closed set of f, put*

$$e_- = \sup\{x \in E : x < c\}, \qquad e_+ = \inf\{x \in E : x > c\}, \tag{1. 4}$$

and

$$\ell = N((e_-,\ c)), \qquad r = N((c,\ e_+)).$$

Then

$$f^\ell(e_-) = e_-, \qquad f^r(e_+) = e_+ \tag{1. 5}$$

and the following map

$$R_E f(x) = \begin{cases} f^\ell(x) & x \in [f^r(c+),\ c) \\ f^r(x) & x \in (c,\ f^\ell(c-)] \end{cases} \tag{1. 6}$$

is a renormalization of f.

On the other hand, if g is a renormalization of f, then there exists a unique proper completely invariant closed set B such that $R_B f = g$.

To characterize the minimal renormalization of an expanding Lorenz map, it is necessary to find its minimal completely invariant closed set.

Theorem B. [2] *Let f be an expanding Lorenz map with minimal period κ, $1 < \kappa < \infty$, O be the unique κ-periodic orbit, and $D = \overline{\bigcup_{n \geq 0} f^{-n}(O)}$. Then we have the following statements:*

1. *D is the minimal completely invariant closed set of f.*
2. *f is renormalizable if and only if $D \neq I$. If f is renormalizable, then R_D, the renormalization associated to D, is the minimal renormalization of f.*
3. *We have the following trichotomy: i) $D = I$, ii) $D = O$, iii) D is a Cantor set.*

It is easy to see the cases $\kappa = 1$ and $\kappa = \infty$ are prime.

According to Theorem B, the minimal renormalizaion of renormalizable expanding Lorenz map always exists. We can define a renormalization operator R from the set of renormalizable expanding Lorenz maps to the set of expanding Lorenz maps (cf. [8]). For each renormalizable expanding Lorenz map, we define Rf to be the minimal renormalization map of f. For $n > 1$, $R^n f = R(R^{n-1} f)$ if $R^{n-1} f$ is renormalizable. And f is m $(0 \leq m \leq \infty)$ *times renormalizable* if the renormalization process can proceed m times exactly. For $0 < i \leq m$, $R^i f$ is the ith renormalization of f.

Definition 2. Let f be an expanding Lorenz map. The minimal renormalization is said to be periodic if the minimal completely invariant closed set $D = O$, where O is the periodic orbit with minimal period. Moreover, the ith renormalization $R^i f$ is periodic if it is a periodic renormalization of $R^{i-1} f$.

Remark 1. Let f be an expanding Lorenz map on $[a,\ b]$, κ be the minimal period of f, O be the unique κ-periodic orbit, P_L be the largest $\kappa-$periodic point in $[a,\ c)$ and P_R be the smallest $\kappa-$periodic point in $(c,\ b]$. According to the appendix in [2], we have

(1) The minimal renormalization of f is periodic if and only if

$$[f^\kappa(c+),\ f^\kappa(c-)] \subseteq [f^\kappa(P_L),\ f^\kappa(P_R)]. \tag{1. 7}$$

(2) One can check if the minimal renormalization is periodic or not in following steps:
- Find the minimal period κ of f by considering the preimages of c, see Lemma 1;
- Find the κ-periodic orbit;
- Check if the inclusion (1. 7) holds or not.

The periodic renormalization is interesting because β-transformation can only be renormalized periodically (see [7]). This kind of renormalization was studied by Alsedà and Falcò [1], Malkin [9]. It was called phase locking renormalization in [1] because it appears naturally in Lorenz map whose rotational interval degenerates to a rational point.

Periodic renormalization is relevant to the conjugacy problem. Glendinning [7] showed that an expanding Lorenz map is conjugate to a β-transformation if its renormalizations admit some special forms. In our words, he obtained the following Proposition.

Proposition 1. ([7]) *An expanding Lorenz map f is conjugate to a β-transformation if and only if f is finitely renormalizable and each renormalization of f is periodic.*

In fact, we shall actually prove the following Main Theorem'.

Main Theorem'. *If $f_{a,b,c}$ is a piecewise linear Lorenz map that expand on average, then $f_{a,b,c}$ is finitely renormalizable and each renormalization of $f_{a,b,c}$ is periodic.*

Remark 2. (1) Main Theorem' indicates that the renormalization process of piecewise linear Lorenz map that expand on average is simple: all of the renormalizations are periodic. As a result, one can obtain all the renormalizations of $f_{a,b,c}$ in finite steps.

(2) Suppose $ac+b(1-c) > 1$, and $f_{a,b,c}$ is m-renormalizable, then by Theorem C and Theorem D in [2], $f_{a,b,c}$ admits a cluster of α-limit sets

$$\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I,$$

where m is finite, and E_i is countable with depth i, $i = 1, 2, \ldots, m$.

(3) According to Parry [16], when $a \in (2^{2^{-(m+1)}},\ 2^{2^{-m}}]$, the piecewise linear Lorenz map $f_{a,a,1/2}$ is m-renormalizable, so one can obtain countable set with given finite depth in dynamical way.

Let us point out the main ideas in the proof of our Main Theorem'. Denote by $\mathcal{L}$ the class of all piecewise linear Lorenz maps that expand on average, by $\mathcal{LR}$ the class of maps in $\mathcal{L}$ which are renormalizable, and $\mathcal{L}_2$ be the class of maps in $\mathcal{L}$ and satisfy the additional condition

$$(1.\ 8)\quad (AC)\qquad 1-ac=f(0)<c<f(1)=b(1-c).$$

According to Lemma 1 in Section 2, any map in $\mathcal{L}_2$ admits minimal period $\kappa=2$. Fix $f\in\mathcal{L}$, we denote κ as its the minimal period, O as the unique κ-periodic orbit and D as the minimal completely invariant closed set of f.

Observe that $f\in\mathcal{LR}$ implies the minimal renormalization $Rf\in\mathcal{L}$. So, in order to show each renormalization of f is periodic, it is necessary to show the following

$$(1.\ 9)\qquad \forall f\in\mathcal{LR},\qquad Rf\ \textit{is periodic}.$$

According to Theorem B, (1. 9) is followed from the following dichotomy

(1. 10) **Dichotomy**: *If* $f\in\mathcal{L}$, *then either* $D=O$ *or* $D=I$.

So, our aim is to show the Dichotomy, because, as we shall see, f is finitely renormalizable is a direct consequence of it. This, together with Proposition 1, ensures the conjugacy.

The first step towards the proof of the Dichotomy is to reduce the proof for maps in $\mathcal{L}$ to the maps in $\mathcal{L}_2$ by trivial renormalization (see Section 2 for the details of trivial renormalization). In what follows, we sketch the proof of Dichotomy for $f\in\mathcal{L}_2$.

By Theorem A, any renormalization corresponds two periodic points, e_- and e_+ (see (1. 5)). An m-periodic point is said to be *nice* if f^m is a homeomorphism on the interval between p and the critical point c. $\{p,\ q\}$ is a *nice pair* if both p and q are nice periodic points and $p<c<q$. Let $\{p,\ q\}$ be a nice pair, and the period of p and q be ℓ and r, respectively. Put

$$M_p=\prod_{i=0}^{\ell-1} f'(f^i(p)),\qquad M_q=\prod_{i=0}^{r-1} f'(f^i(q)).$$

Each factor in M_p and M_q is either a or b because f is piecewise linear. The proof of the Dichotomy for $f\in\mathcal{L}_2$ can be divided into two steps:

Step 1: Show that if the nice pair $\{p,\ q\}$ corresponds to a renormalization, then

$$(M_p-1)(M_q-1)\le 1.$$

Step 2: If $D\neq O$, show that for any nice pair $\{p,\ q\}$, we have

$$(1.\ 11)\qquad (M_p-1)(M_q-1)>1.$$

Step 1 is fairly easy, and relies on the properties of renormalization and f is piecewise linear.

Step 2 is more involved. We consider three cases: both $a \geq 1$ and $b \geq 1$, $a < 1 < b$ and $a > 1 > b$. In the first case, all of the factors in the product of M_p and M_q are no less than 1, it is easier to get the lower bound of M_p and M_q. The first case is a direct consequence of some inequalities obtained from the action of f on some interval. The second case and the third case are similar. In order to get a lower bound for M_p and M_q when $a < 1 < b$, we introduce the *first exit decomposition.* Although f is contract on the left side of the critical point, it is possible to find a set A so that $M_A(x) \geq 1$ for many initial x, where

$$M_A(x) = \prod_{i=0}^{n_A(x)-1} f'(f^i(x)),$$

and $n_A(x)$ is the first exit time of the orbit $O(x)$ from A.

Suppose the orbit $O(c-)$ leave A exact s times, and the orbit $O(c+)$ leaves A exact t times, using the first exit decomposition, we can obtain

$$M_p = M_A(c+)M_A(y_1) \cdots M_A(y_{t-1})W(y_t),$$
$$M_q = M_A(c-)M_A(x_1) \cdots M_A(x_{s-1})W(x_s).$$

Depending on the position of $f(0) = 1 - ac$, we have three cases. In each case, we can obtain lower bound of M_p and M_q to ensure (1. 11).

The remain parts of the paper is organized as follows. We describe trivial renormalization in Section 2, set up the *expansion of nice pair* (1. 11) in Section 3, and prove Main Theorem' in the last section.

2. Trivial renormalization

In the definition of renormalization, we assume that both $\ell > 1$ and $r > 1$. And we have a one-to-one correspondence between such kind of renormalizations and proper completely invariant closed sets.

Definition 3. ([8]) A Lorenz map f is said to be *trivially renormalizable* if we have $(\ell,\ r) = (1,\ 2)$ or $(\ell,\ r) = (2,\ 1)$ in Definition 1, and such a g is called a trivial renormalization of f.

Lemma 1. *([2]) Suppose f is an expanding Lorenz map on $[a,\ b]$ without fixed point. Then the minimal period of f is equal to $\kappa = m + 2$, where*

$$m = \min\{i \geq 0 : f^{-i}(c) \in [f(a), f(b)]\}. \tag{2. 1}$$

Proposition 2. *Let f be an expanding Lorenz map with minimal period κ. If $c \notin (f(a),\ f(b))$, then there exists a Lorenz map g with minimal period less than κ, such that f is renormalizable if and only if g is renormalizable. Moreover, if f is renormalizable, then the minimal renormalization of f is periodic if and only if the minimal renormalization of g is periodic.*

Proof. Since $c \notin (f(a),\ f(b))$, we have two cases: $c \leq f(a)$ or $c \geq f(b)$.

For the case $c \leq f(a)$, the following map

$$g(x) = \begin{cases} f^2(x) & x \in [a,\ c) \\ f(x) & x \in (c,\ f(b)]. \end{cases}$$

is an expanding Lorenz map with minimal period less than κ, and

$$orb(x, g) = orb(x, f) \cap [a,\ f(b)]. \tag{2. 2}$$

If $c \geq f(b)$, the following

$$g(x) = \begin{cases} f(x) & x \in [f(a),\ c) \\ f^2(x) & x \in (c,\ b]. \end{cases}$$

is also an expanding Lorenz map with minimal period less than κ, and

$$orb(x, g) = orb(x, f) \cap [f(a),\ b]. \tag{2. 3}$$

See Figure 2 (Heavy Lines) for the intuitive pictures of g.

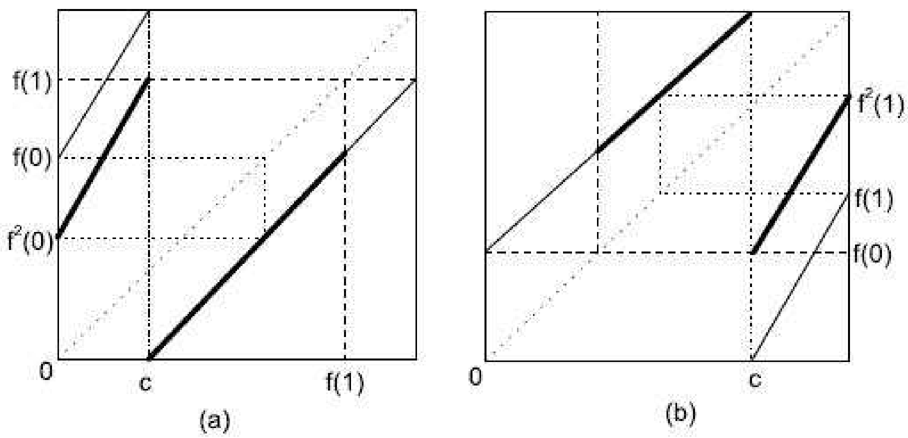

FIGURE 1. Trivial renormalization of a map on [0, 1], the pictures of g: (a) $c \leq f(0)$, (b) $f(1) \leq c$.

Denote O_f and O_g as the periodic orbit with minimal period of f and g, and $D(f)$ and $D(g)$ as the minimal completely invariant closed set of f and g, respectively.

If $c \leq f(a)$, by (2. 2), we get $O_g = O_f \cap [a,\ f(b)]$, and $D(g) = D(f) \cap [a,\ f(b)]$. It follows that $D(f) = I$ is if and only if $D(g) = [a,\ f(b)]$, and $D(f) = O_f$ if any only if $D(g) = O_g$.

If $c \geq f(b)$, by (2. 3), we obtain $O_g = O_f \cap [f(a),\ b]$, and $D(g) = D(f) \cap [f(a),\ b]$. It follows that $D(f) = I$ is if and only if $D(g) = [f(a),\ b]$, and $D(f) = O_f$ if any only if $D(g) = O_g$.

In both cases, according to Theorem B, we know that f is renormalizable if and only if g is renormalizable. Moreover, if f is renormalizable, the minimal renormalization of f is periodic if and only if the minimal renormalization of g is periodic. □

It is easy to see that a Lorenz map with $c \in (f(a),\ f(b))$ can not be trivially renormalizable, so the statement in Proposition 2 is just the the fact that an expanding Lorenz map f is *trivially renormalizable* if and only if $c \notin (f(a),\ f(b))$.

Applying trivial renormalization (see Proposition 2, (2. 2) and (2. 3)) consecutively if possible, we get the following Corollary.

Corollary 1. *Let f be an expanding Lorenz map with minimal period κ. If $\kappa < \infty$, then f can be trivially renormalized finite times to be an expanding Lorenz map g with $\kappa(g) \le 2$.*

3. Expansion of nice pair

Suppose p is a periodic point with period m. p is called a *nice periodic point* if f^m is continuous on the interval between p and the critical point c. $\{p,\ q\}$ is called a *nice pair* if $p < c < q$, and both p and q are nice periodic points. If E is a proper completely invariant closed set of f, e_- and e_+ are defined by (1. 4) in Theorem A, then $\{e_-,\ e_+\}$ is a nice pair. A nice pair $\{p,\ q\}$ corresponds to a renormalization if and only if $[f^r(c+),\ f^\ell(c-)] \subseteq [p,\ q]$, where ℓ and r are the periods of p and q, respectively.

Let $f \in \mathcal{L}_2$, $\{p,\ q\}$ be a nice pair of f, ℓ and r be the period of p and q, respectively. f^ℓ is linear on $[p,\ c-]$, and f^r is linear on $[c+,\ q]$. Put

$$M_p := (f^\ell)'(p) = \prod_{i=0}^{\ell-1} f'(f^i(p)) \qquad M_q := (f^r)'(q) = \prod_{i=0}^{r-1} f'(f^i(q)).$$

Theorem 1. *Suppose $f \in \mathcal{L}_2$, $\{p,\ q\}$ is a nice pair of f, and M_p and M_q are defined as above. If $[f(0),\ f(1)] \nsubseteq [P_L,\ P_R]$, then*

$$(M_p - 1)(M_q - 1) > 1. \tag{3. 1}$$

Remark 3. By (1. 7) and the trichotomy in Theorem B, $[f(0),\ f(1)] \nsubseteq [P_L,\ P_R]$ is equivalent to $D \neq O$.

The proof of Theorem 1 is technical. Let $f \in \mathcal{L}_2$ such that $D \neq O$, we divide the proof into three cases: both $a \ge 1$ and $b \ge 1$, $a < 1 < b$ and $a > 1 > b$. In the first case, all of the factors in the product of M_p and M_q are no less than 1, it is easier to get the lower bound of M_p and M_q. In the first case, the lower bound (3. 1) can be achieved by Lemma 4. The first case is a direct consequence of some inequalities obtained from the action of f on some interval. The second case and the third case are similar. In order to get a lower bound for M_p and M_q when $a < 1 < b$, we introduce the *first exit decomposition.* Although f is contract on the left side of the critical point, we try to decompose M_p and M_q into parts so that each part is no less than 1. Depending on the position of

$f(0) = 1 - ac$, we have three cases. In each case, we can obtain lower bound of M_p and M_q to ensure (3. 1). In the remain parts of this section, we introduce the *first exit decomposition* firstly, then we prove some technical Lemmas based on the detailed dynamics of f, and prove Theorem 1 finally.

3.1. **First exit decomposition.** Let A be a given set, $O(x) = \{f^j(x); j \geq 0\}$ be the orbit with initial x. If $O(x)$ visits A, denote

$$n_A(x) = \min\{k : f^{k-1}(x) \in A,\ f^k(x) \notin A\}$$

as the first exit time of $O(x)$ from A, and the sth ($s \geq 1$) exit time $n_s(x)$ from A are defined inductively by

$$n_1(x) := n_A(x), \quad n_s(x) := \min\{k > n_{s-1} : f^{k-1}(x) \in A,\ f^k(x) \notin A\}.$$

If $O(x)$ does not visit A, $n(x) = \infty$.

Denote $x_s := f^{n_s}(x), \quad s = 1, 2, \ldots$. Put

$$M_A(x) = \prod_{j=0}^{n_A(x)-1} f'(f^j(x)).$$

Using above notations, the following *first exit decomposition* is trivial.

Lemma 2. *$x \in I$, and $n_s(x) \leq n < n_{s+1}$,*

$$(f^n)'(x) = \prod_{j=0}^{n-1} f'(f^j(x)) = M_A(x)M_A(x_1)\cdots M_A(x_{s-1})W(x_s), \tag{3. 2}$$

where

$$W(x_s) = f'(x_s)f'(f(x_s))\cdots f'(f^{n-1}(x)),$$

and $W(x) = 1$ if and only if $x_s = f^n(x)$.

3.2. **Technical Lemmas.** Suppose $f := f_{a,b,c} \in \mathcal{L}$ and $\kappa = 2$. Denote the 2-periodic points are P_L and P_R, $P_L < c < P_R$, and c_* and c^* are the preimages of c. By direct calculations, we get

$$\begin{aligned} P_L &= \frac{b(c-(1-ac))}{ab-1}, \qquad & P_R &= \frac{abc-(1-ac)}{ab-1} \\ c_* &= \frac{c-(1-ac)}{a}, \qquad & c^* &= \frac{c+bc}{b}. \end{aligned} \tag{3. 3}$$

Observe that f^2 is linear (with slope $ab = f^2(P_L) > 1$) on $[c_*,\ P_L]$, and $f^2(P_L) = P_L$. Track the preimages of c_* on $[c_*,\ P_L]$, one can get an increasing sequence $\{c_n\} \subset [c_*, P_L]$,

$$c_0 := c_*, \quad f^2(c_1) = c_0, \ \cdots, \ f^2(c_n) = c_{n-1}, \ \cdots \tag{3. 4}$$

and $c_n \uparrow P_L$. $(c_*, P_L) = \bigcup_{k\geq 1}(c_{k-1},\ c_k]$. Similarly, there exists a decreasing sequence $\{c'_n\}$ approaches to P_R so that

$$c'_0 := c^*,\quad f^2(c'_1) = c'_0,\ \cdots,\ f^2(c'_n) = c'_{n-1},\ \cdots. \tag{3. 5}$$

Lemma 3. *Let $\{c_n\}$ and $\{c'_n\}$ are defined as (3. 4) and (3. 5), we have*

$$|(c_{n-1},\ c_n)| \leq |(c_n,\ c)|, \tag{3. 6}$$

$$|(c'_n,\ c_{n-1})| \leq |(c,\ c'_n)|. \tag{3. 7}$$

Proof. At first, we prove (3. 6). Using (3. 3),

$$|(c_*,\ P_L)| = \frac{c-(1-ac)}{a(ab-1)},\quad |(P_L,\ c)| = \frac{a(b(1-c)-c)}{c-(1-ac)}|(c_*,\ P_L)|.$$

Since f^{2k} maps $(c_k,\ P_L)$ homeomorphically to $(c_*,\ P_L)$,

$$|(c_n,\ P_L)| = \frac{1}{a^nb^n}|(c_*,\ P_L)|,\qquad |(c_{n-1},\ P_L)| = \frac{1}{a^{n-1}b^{n-1}}|(c_*,\ P_L)|.$$

It follows

$$|(c_{n-1},\ c_n)| = |(c_{n-1},\ P_L)| - |(c_n,\ P_L)| = (\frac{1}{a^{n-1}b^{n-1}} - \frac{1}{a^nb^n})|(c_*,\ P_L)|,$$

and

$$|(c_n,\ c)| = |(c_n,\ P_L)| + |(P_L,\ c)| = (\frac{1}{a^nb^n} + \frac{a(b(1-c)-c)}{c-(1-ac)})|(c_*,\ P_L)|.$$

Hence, (3. 6) is equivalent to

$$\frac{2}{ab} + (ab)^{n-1}\frac{a(b(1-c)-c)}{c-(1-ac)} \geq 1. \tag{3. 8}$$

Remember that $\kappa = 2$ implies that $f(1) = b(1-c) > c$ and $f(0) = 1-ac < c$, $\frac{a(b(1-c)-c)}{c-(1-ac)}$ is positive.

Since $ab > 1$, it is enough to prove (3. 8) with $n = 1$, i.e.,

$$F(b) := \frac{2}{ab} + \frac{a(b(1-c)-c)}{c-(1-ac)} \geq 1. \tag{3. 9}$$

If $ab \leq 2$, then $F(b) \geq \frac{2}{ab} \geq 1$. For the case $ab > 2$, a is fixed,

$$F'(b) = -\frac{2}{ab^2} + \frac{a(1-c)}{c-(1-ac)} = \frac{a^2b^2(1-c) - 2(c-(1-ac))}{ab^2(c-(1-ac))}.$$

Using $ab > 2$ and $f(1) = b(1-c) > c$,

$$a^2b^2(1-c) - 2(c-(1-ac)) > a^2bc - 2c + 2 - 2ac > ac(ab-2) + 2(1-c) > 0.$$

So $F'(b) > 0$ when $ab > 2$. It follows that $F(b) > F(\frac{2}{a}) = 1$. (3. 9) holds.

For the second inequality, by similar calculations, one can see that (3. 7) is equivalent to

$$\frac{2}{ab} + (ab)^{n-1}\frac{b(c-(1-ac))}{b(1-c)-c} \geq 1. \tag{3. 10}$$

We shall prove (3. 10) with $n = 1$, i.e.,

$$G(a) := \frac{2}{ab} + \frac{b(c-(1-ac))}{b(1-c)-c} \geq 1. \tag{3. 11}$$

If $ab \leq 2$, then $G(a) \geq \frac{2}{ab} \geq 1$. When $ab > 2$,

$$G'(a) = -\frac{2}{a^2b} + \frac{bc}{b(1-c)-c} = \frac{a^2b^2c - 2(b(1-c)-c)}{a^2b(b(1-c)-c)}.$$

Using $ab > 2$ and $f(0) = 1 - ac < c$, one obtains

$$a^2b^2c - 2(b(1-c)-c) > 2abc - 2b(1-c) + 2c > 2b(c-(1-ac)) + 2c > 0.$$

So $G'(a) > 0$ when $ab > 2$. It follows that $G(a) > G(\frac{2}{b}) = 1$. (3. 11) holds. □

Lemma 4. *Let* $\{c_n\}$ *and* $\{c'_n\}$ *be defined as (3. 4) and (3. 5).*

(1) *Suppose* $f(0) \in (c_{k-1},\ c_k]$, *we have*

$$aba^{k+1}b^k > 1 + a^{k+1}b^k \qquad \textit{and} \qquad a^{k+1}b^k > 1.$$

(2) *Suppose* $f(1) \in [c'_k,\ c'_{k-1})$, *we have*

$$aba^kb^{k+1} > 1 + a^kb^{k+1} \qquad \textit{and} \qquad a^kb^{k+1} > 1.$$

Proof. It is necessary to prove (1), (2) can be proved similarly.

Consider the interval $(c_k,\ \ P_L)$, we have

$$(c_k, P_L) \xrightarrow[(ab)^k]{f^{2k}} (c_*, P_L) \xrightarrow[a]{f} (c, P_R) \xrightarrow[ab]{f^2} (f(0), P_R) \supset (c_k, P_L) \cup (c, P_R).$$

It follows that

$$a^{k+2}b^{k+1}|(c_k,\ \ P_L)| > |(c_k,\ P_L)| + |(c,\ P_R)|.$$

Since $|(c,\ P_R)| = a^{k+1}b^k|(c_k,\ P_L)|$, we obtain $a^{k+2}b^{k+1} > 1 + a^{k+1}b^k$.

Consider the interval $(c_{k-1},\ \ c_k)$, we obtain

$$(c_{k-1},\ \ c_k) \xrightarrow[(ab)^{k-1}]{f^{2(k-1)}} (c_*,\ c_1) \xrightarrow[ab]{f^2} (0,\ c_*) \xrightarrow[a]{f} (f(0),\ c).$$

It follows

$$a^{k+1}b^k|(c_{k-1},\ \ c_k)| = |(f(0),\ c)|.$$

By Lemma 3 and the condition that $f(0) \in (c_{k-1},\ c_k]$,

$$a^{k+1}b^k = \frac{|(f(0),\ \ c)|}{|(c_{k-1},\ \ c_k)|} > \frac{|(c_k,\ c)|}{|(c_{k-1},\ c_k)|} \geq 1.$$

□

Lemma 5. *Suppose* $a < 1 < b$, $A = [0,\ c_*]$,

$$M(x) := M_A(x) = \prod_{i=0}^{n_A(x)-1} f'(f^i(x)),$$

where $n_A(x)$ *is the first exit time of the orbit* $O(x)$ *from* A. *If* $f(0) \in (c_*,\ c)$, *then*

$$M(x) := M_A(x) > 1, \qquad \forall\ x \geq f(0). \tag{3. 12}$$

Similarly, suppose $a > 1 > b$, $B = [c^*,\ 1]$, $M_B(x) = \prod_{i=0}^{n_B(x)-1} f'(f^i(x))$, *where* $n_B(x)$ *is the first exit time of the orbit* $O(x)$ *from* B. *If* $f(1) \in (c,\ c^*)$, *then*

$$M_B(x) > 1, \qquad \forall\ x \leq f(1). \tag{3. 13}$$

Proof. We only prove the Lemma for case $a < 1 < b$, because the proof can adapt to the case $a > 1 > b$ easily.

Observe that $f(x) > c$ for all $x \in (c_*,\ c)$ and $ab > 1$. It follows that $M(x) = \infty$ when $n_A(x) = \infty$. In what follows, we show that $M(x) > 1$ for $x \in I$ with $n_A(x) < \infty$. The remain case can be proved similarly.

Observe that when $f(0) > c_*$, each orbit of f can stay on the left of c at most two consecutive times. Furthermore, any orbit can not stay on the left of c two times before it visits A, because f maps $(c_*,\ c)$ homeomorphically to $(c,\ 1)$.

If $x \geq c+$, the product $M(x)$ begin with b and end with only one a, and it can not have two consecutive a. So $M(x) > 1$ because $ab > 1$.

If $x \in (P_L,\ c-]$, then $f(x) \in (P_R,\ 1]$. There is a nonnegative integer m such that $f^{2m}(f(x)) \geq c^*$. So $f^{2m+2}(x) \geq c$ and $M(f^{2m+2}(x)) > 1$. It follows

$$M(x) = (ab)^{m+1} M(f^{2m+2}(x)) > 1.$$

Suppose $f(0) \in (c_*,\ P_L)$, there exists positive integer k so that $f(0) \in (c_{k-1},\ P_L)$. For $x \in (f(0),\ P_L)$, one can see that $M(x) = (ab)^m a^{k+1} b^k$ for some $m \geq 0$. By Lemma 4,

$$M(x) \geq a^{k+1} b^k > 1.$$ □

Let $i = \min\{k : f^k(0) > c\}$ be the least integer so that $f^i(0) > c$. Each orbit of f can stay consecutively on the left of c at most i times. $f(0) \leq c_*$ implies $i \geq 3$.

Let $j = \min\{k : f^k(1) < c\}$ be the least integer so that $f^i(1) < c$. $f(1) \geq c^*$ implies $j \geq 3$.

Lemma 6. *Let* i *and* j *be defined as above, we have*

$$ba^{i-1} > 1 + a + \cdots + a^{i-2}, \tag{3. 14}$$

$$ab^{j-1} > 1 + b + \cdots + b^{j-2}. \tag{3. 15}$$

Proof. Since i is the least positive integer such that $f^{i-1}(0) < c < f^i(0)$, by direct calculation,

$$f(0) = 1 - ac, f^2(0) = (1-ac)(1+a),\ \ldots, f^{i-1}(0) = (1-ac)(1+a+\cdots+a^{i-2}) < c.$$

It follows

$$c > \frac{1 + a + \cdots + a^{i-2}}{1 + a + \cdots + a^{i-1}}.$$

On the other hand, by assumption (1. 8), $c < f(1) = b(1 - c)$ implies $c < \frac{b}{1+b}$. Hence

$$\frac{1 + a + \cdots + a^{i-2}}{1 + a + \cdots + a^{i-1}} < \frac{b}{1 + b},$$

which is equivalent to (3. 14).

(3. 15) can be proved by similar calculations. □

Remember c_1 and c_1' are defined by (3. 4) and (3. 5).

Lemma 7. *Let i and j be defined as above, we have*

$$ba^i < 1 \quad \text{implies} \quad f^{i-1}(0) \in (c_1,\ c), \tag{3. 16}$$

$$ab^j < 1 \quad \text{implies} \quad f^{j-1}(1) \in (c,\ c_1'). \tag{3. 17}$$

Proof. We only show (3. 16). By the definition of i,

$$0 < f(0) < f^2(0) < \cdots < f^{i-1}(0) < c < f^i(0).$$

Since f^{i-1} maps $(0,\ f(0))$ to $(f^{i-1}(0),\ f^i(0)) \ni c$ homeomorphically, there exists $y \in (0,\ f(0))$ so that $f^{i-1}(y) = c$.

Observe

$$(c_*,\ \ c_1) \xrightarrow[ab]{f^2} (0,\ c_*),$$

there exists $z \in (c_*,\ \ c_1)$ such that $f^2(z) = y$.

Consider the interval $(c_*,\ \ z)$, we have

$$(c_*,\ \ z) \xrightarrow[ab]{f^2} (0,\ y) \xrightarrow[a^{i-1}]{f^{i-1}} (f^{i-1}(0),\ c).$$

It follows that

$$ba^i|(c_*,\ \ z)| = |(f^{i-1}(0),\ \ c)|.$$

If $f^{i-1}(0) < c_1$, by Lemma 3,

$$ba^i = \frac{|(f^{i-1}(0),\ \ c)|}{|(c_*,\ \ z)|} > \frac{|(c_1,\ c)|}{|(c_*,\ c_1)|} \geq 1.$$

We obtain a contradiction. Hence, (3. 16) is true. □

Lemma 8. *Suppose $a < 1 < b$, $A = [0,\ c_*]$, $M(x) := M_A(x)$ is defined as in Lemma 5. If $f(0) < c_*$, then*

$$M(x) := M_A(x) > 1, \qquad \forall x \geq c_1. \tag{3. 18}$$

Similarly, Suppose $a > 1 > b$, $B = [c^*,\ 1]$, $M_B(x)$ is defined as in Lemma 5. If $f(1) > c^*$, then

$$M_B(x) > 1, \qquad \forall x \leq c_1'. \tag{3. 19}$$

Proof. Let i be defined as above. If $x \in (c_1,\ c_2]$, then $M(x) = ababa^m$ for some $0 < m \le i-1$. Since $a < 1 < b$, we have $M(x) \ge ababa^{i-1} \ge (ba^2)(ba^{i-1}) \ge 1$. In fact, Lemma 7, together with $i \ge 3$, implies both ba^{i-1} and ba^2 are no less than 1. The remain cases can be shown by similar arguments in the proof of Lemma 5. □

3.3. **Proof of Theorem 1.** It is time to present the proof of Theorem 1.

The proof can be divided into three cases: both $a \ge 1$ and $b \ge 1$, $a < 1 < b$, and $a > 1 > b$.

Case A: $a \ge 1$ and $b \ge 1$.

Since $[f(0),\ f(1)] \not\subseteq [P_L,\ P_R]$, we have $f(0) < P_L$ or $f(1) > P_R$. Without loss of generality, we assume $f(1) > P_R$. It follows either $f(1) \in (P_R, c^*)$ or $f(1) \ge c^*$.

When $f(1) \in (P_R, c^*)$, there exists k so that $f(1) \in [c'_k,\ c'_{k-1})$, by Lemma 4, we have $aba^kb^{k+1} > 1 + a^kb^{k+1}$. p is a nice ℓ-periodic point indicates $\ell \ge 2k+3$. In fact, in this case, when $m < 2k+2$, the interval $(f^m(p),\ f^m(c-))$ does not contain c_* and c^*, so $N((f^{2k+2}(p),\ f^{2k+2}(c-))) \ge 1$.

Since $a \ge 1$ and $b \ge 1$, $M_p \ge aba^kb^{k+1}$ and $M_q \ge ab$. Therefore,

$$(M_p - 1)(M_q - 1) \ge (aba^kb^{k+1} - 1)(ab - 1) > a^kb^{k+1}(ab - 1) > 1.$$

When $f(1) \ge c^*$, by similar arguments as above, we get $M_p \ge ab^2 > 1 + b$ by Lemma 6 and $M_q \ge ab$. Hence

$$(M_p - 1)(M_q - 1) \ge (ab^2 - 1)(ab - 1) > b(ab - 1) > 1.$$

(3. 1) is proved when both a and b are no less than 1.

Case B: $a < 1 < b$.

Put $A = [0,\ c_*]$. Suppose $O_r(c+) := \{c+,\ f(c+),\ \ldots, f^{r-1}(c+)\}$ exits $A = [0,\ c_*]$ exact s ($s \ge 1$) times. Put $x_j := f^{n_j}(c+)$, where n_j is the jth exit time for the finite orbit $O_r(c+)$ with respect to A. According to the first exit Lemma 2,

$$M_q = (f^r)'(c+) = \prod_{k=0}^{r-1} f'(f^k(c+)) = M(c+)M(x_1)\cdots M(x_{s-1})W(x_s),$$

where $W(x_s) = f'(x_s)f'(f(x_s))\cdots f'(f^{r-1}(c+))$. $W(x_s) \ge 1$ because it can not contain two consecutive a, the last factor is b, and $ab > 1$.

Similarly, suppose $O_\ell(c-)$ exits A exact t times. Denote $y_j := f^{n_j}(c-)$, one gets

$$M_p = (f^\ell)'(c-) = \prod_{k=0}^{\ell-1} f'(f^k(c-)) = M(c-)M(y_1)\cdots M(y_{t-1})W(y_t),$$

and $W(y_t) = f'(y_t)f'(f(y_t))\cdots f^{\ell-1}(c-) \ge 1$.

Depending on the position of $f(0)$, we distinguish three subcases: $P_L \leq f(0) < c$, $c_* < f(0) < P_L$ and $f(0) \leq c_*$. We shall show that the lower bound (3. 1) holds in each subcase.

(i) **Subcase** $P_L < f(0) < c$**.**

By the definition of x_j, we know that $x_j \geq f(0)$ and $y_j \geq f(0)$. By Lemma 5 we get $M_q \geq M(c+) \geq ab$ and $M_p \geq M(c-)$. Since $[f(0),\ f(1)]$ does not contained in $[P_L,\ P_R]$, we conclude $f(1) > P_R$.

By Lemma 5, $M_p \geq M(c-)$.

If $f(1) \in (P_R,\ c^*)$, by Lemma 4, there exists $k > 0$ so that $M(c-) \geq aba^k b^{k+1}$. We obtain

$$(M_p - 1)(M_q - 1) \geq (aba^k b^{k+1} - 1)(ab - 1) > a^k b^{k+1}(ab - 1) > 1.$$

If $f(1) \geq c^*$, by Lemma 6, $ab^2 > 1+b$. It follows $M_p \geq M(c-) \geq abbM(f^2(1)) \geq ab^2$. As a result,

$$(M_p - 1)(M_q - 1) \geq (ab^2 - 1)(ab - 1) > b(ab - 1) > 1.$$

(ii) **Subcase** $c_* < f(0) < P_L$**.**

There exist $k \geq 1$ so that $f(0) \in (c_{k-1},\ c_k]$. Since $f(x) > f(0)$ for each $x \in A$, by Lemma 5, we know that $M(y_j) \geq 1$ for $j = 1, 2, \dots, s-1$, and $M(x_j) \geq 1$ for $j = 1, 2, \dots, t-1$. We have

$$M_q = M(c+)M(x_1)\cdots M(x_{s-1})W(x_s) \geq M(c+)M(x_1) = aba^{k+1}b^k.$$

$$M_p \geq M(c-) = abM(f(1)) \geq ab.$$

It follows that

$$(M_p - 1)(M_q - 1) \geq (aba^{k+1}b^k - 1)(ab - 1) > a^{k+1}b^k(ab - 1) > 1.$$

(iii) **Subcase** $f(0) \leq c_*$**.**

Let i be the minimal positive integer so that $f^i(0) > c$. Each orbit can stay on the left of c at most i consecutive times. In what follows, we shall prove

$$M_q \geq ba^{i-1}, \qquad M_p \geq ba. \tag{3. 20}$$

Claim 1: $M_q \geq ba^{i-1}$.

Claim 1 will be proved in two separated cases: $ba^i \geq 1$ and $ba^i < 1$.

Suppose $ba^i \geq 1$. By Lemma 2 and Lemma 5,

$$M_q = \prod_{m=0}^{r-1} f'(f^m(c+)) = M(c+)M(x_1)\cdots M(x_{s-1})W(x_s) \geq M(c+) = ba^{i-1}$$

because $x_j > c_*$ for $j = 1, 2, \cdots, s-1$.

Now we consider the case $ba^i < 1$. By Lemma 6, we know that $ba^{i-1} > 1 + a + \cdots + a^{i-2} > 1$. By Lemma 2,

$$M_q = \prod_{m=0}^{r-1} f'(f^m(c+)) = M(c+)M(x_1)\cdots M(x_{s-1})W(x_s),$$

where $E = f'(x_s)f'(f(x_s))\cdots f'(f^{r-1}(c+)) \geq 1$.

In what follows we show that $M = M(x_1)M(x_2)\cdots M(x_{s-1}) \geq 1$, which implies $M_q \geq M(c+) = ba^{i-1}$.

By Lemma 5, $M(x_j) \geq 1$ for all $x_j > c_1$. So $M \geq 1$ if all of the x_j are greater than c_1.

Suppose there are some j so that $M(x_j) < 1$. We denote them as $j_1 < j_2 < \cdots$. According to Lemma 7, $ba^i < 1$ implies $c_1 < f^{i-1}(0) < c$. Using Lemma 8 we get $M(x_1) > 1$. As a result, we have $j_1 > 1$. By Lemma 7, we know that $x_{j_1} \in (c_*,\ c_1]$, and $M(x_{j_1}) = ba^i$ because each orbit can stay on the left of c at most i consecutive times and $ba^{i-1} > 1$.

Let $k_1 = max\{t : x_t > c_1, t < j_1\}$. It follows from Lemma 7 and Lemma 8 that $1 \leq k_1 < j_1$ and $x_{k_1} > c_1$, which, together with $ab > 1$, implies $M(x_{k_1}) \geq ababa^m$. Moreover, we conclude that $m < i-1$, because $m = i-1$ implies $x_{k_1} > c_1$ by Lemma 7. We obtain $M(x_{k_1}) \geq ababa^{i-2}$. Therefore, $M(x_{k_1})M(x_{j_1}) \geq ababa^{i-2}ba^i = (ba^2)(ba^{i-1})(ba^{i-1}) \geq 1$.

By similar arguments, one can find $j_1 < k_2 < j_2$ so that $M(x_{k_2})M(x_{j_2}) \geq 1$. Repeat the above procedures several times if possible, we conclude that $M \geq 1$. Therefore, $M_q \geq M(c+) = ba^{i-1}$.

Claim 1 is true.

Claim 2: *$M_p \geq ab$.*

Since the orbit

$$O_\ell(c-) = \{c-,\ 1,\ f(1),\ \ldots,\ f^{\ell-1}(c-)\}$$

exits A exact $s(\geq 0)$ times, and the first point after jth exit is y_j, we conclude that

$$\begin{aligned} M_p = (f^\ell)'(c-) &= M(c-)M(y_1)\cdots M(y_{t-1})W \\ &= abM(f(1))M(y_1)\cdots M(y_{t-1})W \end{aligned} \tag{3. 21}$$

where $W = f'(y_t)f'(f(y_t))\cdots f'(f^{\ell-1}(c-))$.

Using the same arguments in the proof of Claim 1, one can show that both $M(f(1))M(y_1)\cdots M(y_{t-1})$ and W are greater than 1. Claim 2 holds.

Using (3. 20) and Lemma 6,

$$(M_p - 1)(M_q - 1) \geq (ab-1)(ba^{i-1} - 1) > (ab-1)a^{i-2} > 1.$$

So (3. 1) is proved when $a < 1 < b$.

Case C: $a > 1 > b$.

One can adapt the proof of the case $a < 1 < b$ to this case by using the first exit decomposition of M_p and M_q with respect to the set $B = [c^*,\ 1]$. □

4. Proof of Main Theorem'

Now we are ready to prove the Main Theorem', which, together with Proposition 1, implies our Main Theorem.

Proof. It is proved in [3] that a piecewise linear Lorenz map that expand on average is always expanding. During the proof, we denote the piecewise linear Lorenz map $f_{a,b,c}$ by $f \in \mathcal{L}$.

Step 1. Since the renormalization of piecewise linear Lorenz map is still piecewise linear, in order to prove each renormalization of f is periodic, it is necessary to show that the minimal renormalization of any renormalizable piecewise linear Lorenz map is always periodic.

If f does not satisfy the additional condition $1-ac < c < b(1-c)$, by Proposition 2, there is an expanding Lorenz map g with minimal period $\kappa(g) < \kappa(f)$, such that f is renormalizable if and only if g is renormalizable, and if f is renormalizable, then minimal renormalization of f is periodic if and only if the minimal renormalization of g is periodic. Furthermore, since f is piecewise linear with $ac + b(1-c) > 1$, g is also piecewise linear that expand on average.

Applying Proposition 2 several times if necessary, we can assume that $\kappa(f) \leq 2$. It follows from Proposition 2 that $f \in \mathcal{L}$ can not be renormalized trivially if and only if either $\kappa(f) = 1$ or $f \in \mathcal{L}_2$. Since any expanding Lorenz map with $\kappa(f) = 1$ is prime, we only need to consider the case $f \in \mathcal{L}_2$.

Step 2. Suppose that $f \in \mathcal{L}_2$. Let P_L and P_R be the 2-periodic points of f, and $P_L < c < P_R$. $D = \alpha(P_L)$ is the minimal completely invariant closed set of f. We shall prove f is prime if $D \neq O$ by contradiction.

Now suppose f is not prime, according to Theorem A, the minimal renormalization map of f is Rf,

$$Rf(x) = \begin{cases} f^{\ell}(x) & x \in [f^r(c+), c) \\ f^r(x) & x \in (c, f^{\ell}(c-)], \end{cases}$$

where

$$\ell = N([p,c)) \qquad p = \sup\{x < c : x \in D\},$$
$$r = N((c,q)) \qquad q = \inf\{x > c : x \in D\},$$

and

$$f^{\ell}(p) = p, \qquad f^r(q) = q. \tag{4. 1}$$

Obviously, $\{p,\ q\}$ is a nice pair. Put $L = (p,\ c)$, $R = (c,\ q)$, $M_p = (f^\ell)'(p)$ and $M_q = (f^r)'(q)$. Since Rf is a piecewise linear Lorenz map, we have

$$|f^\ell(L)| = |f^\ell((p,\ c))| = |(p,\ f^\ell(c-))| = M_p|L| \leq |L|\ +\ |R|$$

$$|f^r(R)| = |f^r((c,\ q))| = |(f^r(c+),\ q)| = M_q|R| \leq |L|\ +\ |R|,$$

which implies

$$(M_p - 1)(M_q - 1) \leq 1. \tag{4. 2}$$

On the other hand, if $D \neq O$, then $[f(0),\ f(1)] \nsubseteq [P_L,\ P_R]$ by (1. 7). According to Theorem 1, we have

$$(M_p - 1)(M_q - 1) > 1$$

because $\{p,\ q\}$ is a nice pair. We obtain a contradiction.

It follows that f is prime if $D \neq O$, where O is the periodic orbit with minimal period. So we conclude that the minimal renormalization of f is periodic. As a result, each renormalization of f is periodic.

Step 3. Now we show that f can only be renormalized finite times. If f is renormalizable, then the minimal renormalization Rf is a β-transformation because Rf is a periodic renormalization indicates $M_p = M_q$. So $g := Rf$ is a β-transformation with slope M_p, which can be renormalized at most finite times. As a result, f can be renormalized at most finite times. □

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Hong-Fei Cui
Wuhan Institute of Physics and Mathematics
The Chinese Academy of Sciences
Wuhan 430071, P. R. China
E-mail address: cuihongfei05@mails.gucas.ac.cn

Yi-Ming Ding
Wuhan Institute of Physics and Mathematics
The Chinese Academy of Sciences
Wuhan 430071, P. R. China
E-mail address: ding@wipm.ac.cn