

COMPUTATION OF AN INTEGRAL BASIS OF QUARTIC NUMBER FIELD

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ABSTRACT. In this paper, based on techniques of Newton polygons, a result which allows the computation of a p integral basis of every quartic number field is given. For each prime integer p , this result allows to compute a p -integral basis of a quartic number field K defined by an irreducible polynomial $P(X) = X^4 + aX + b \in \mathbb{Z}[X]$ in methodical and complete generality.

Key words : p -integral basis, Newton polygon.

INTRODUCTION

Let K be a quartic number field defined by an irreducible polynomial $P(X) = X^4 + mX^2 + aX + b \in \mathbb{Z}[X]$, α a complex root of P , \mathbb{Z}_K the ring of integers of K , d_K its discriminant and $ind(P) = [\mathbb{Z}_K : \mathbb{Z}[\alpha]]$ the index of $\mathbb{Z}[\alpha]$ in \mathbb{Z}_K . It is well known that: $\Delta = N_{K/\mathbb{Q}}(P')(\alpha)$ and $\Delta = (ind(P))^2 d_K$, where Δ is the discriminant of P and we can assume that for every prime p , $v_p(a) \leq 2$ or $v_p(b) \leq 3$. Let p be a prime integer. A p -integral basis of K is a set of integral elements $\{w_1, \dots, w_4\}$ such that p does not divide the index $[\mathbb{Z}_K : \Lambda]$, where $\Lambda = \sum_{i=1}^4 \mathbb{Z}w_i$. In that case, we said that Λ is a p -maximal order of K . A triangular p -integral basis of K is a p -integral basis of K $\{1, w_2, w_3, w_4\}$ such that $w_1 = \frac{\alpha+x_1}{p^{r_1}}$, $w_2 = \frac{\alpha^2+y_2\alpha+x_2}{p^{r_2}}$ and $w_3 = \frac{\alpha^3+z_3\alpha^2+y_3\alpha+x_3}{p^{r_3}}$. In Theorem 1.2, for every prime p , a triangular p -integral basis of K is given. For every prime p and $(x, m) \in \mathbb{Z}^2$, denote $x_p = \frac{x}{p^{v_p(x)}}$ and $x[m]$: the remainder of the Euclidean division of x by m .

In this paper, based on techniques of Newton polygons, a result which allows the computation of a p integral basis of every quartic number field is given (Theorem 1.2). Theorem 1.2 allows to find again the results given in [1] in methodical and complete generality.

NEWTON POLYGON

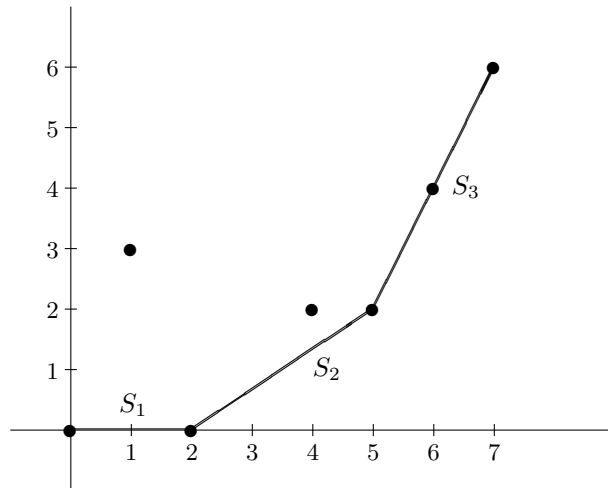
Let p be a prime integer such that p^2 divides Δ and $\phi(X)$ is an irreducible divisor of $P(X)$ modulo p . Set $m = deg(\phi(X))$ and let

$$P(X) = a_0(X)\phi(X)^t + a_1(X)\phi(X)^{t-1} + \dots + a_t(X),$$

be the $\phi(X)$ -adic development of $P(X)$ (every $a_i(X) \in \mathbb{Z}[X]$ and $\deg a_i(X) < m$). To any coefficient $a_i(X)$ we attach the integer $u_i = v_p(a_i(X))$ and the point of the plane $P_i = (i, u_i)$, if $u_i < \infty$.

The ϕ -Newton polygon of $P(X)$ is the lower convex envelope of the set of points $P_i = (i, u_i)$, $u_i < \infty$, in the cartesian plane. This (open) polygon is denoted by $N_\phi(P)$.

For instance, for a ϕ -development of degree 7 with $u_i = 0, 3, 0, \infty, 2, 2, 4, 6$ for $i = 0, 1, \dots, 7$, the polygon is let N be the $\phi(X)$ -Newton polygon of $P(X)$.



The *length* $\ell(N_\phi(P))$ and the *height* $h(N_\phi(P))$ of the polygon are the respective lengths of the projection to the horizontal and vertical axis. Clearly, $\deg P(X) = m\ell(N_\phi(P)) + \deg a_0(X)$, where $m = \deg \phi$.

The ϕ -Newton polygon is the union of different adjacent *sides* S_1, \dots, S_t with increasing slope $\lambda_1 < \lambda_2 < \dots < \lambda_t$. We shall write $N_\phi(P) = S_1 + \dots + S_t$. The points joining two different sides are called the *vertices* of the polygon. The polygon determined by the sides of positive slopes of $N_\phi(P)$ is called the *principal ϕ -polygon* of $P(X)$ and denoted by $N_\phi^+(P)$. The length and the height of $N_\phi^+(P)$ are the respective lengths of the projection to the horizontal and vertical axis.

For instance, the polygon of the figure has three sides S_1, S_2, S_3 with slopes $0 < 2/3 < 2$ and $N_\phi^+(P) = S_2 + S_3$. For every side S of the principal part $N_\phi^+(P)$, the *length* $\ell(S)$ and the *height* $h(S)$, of S , are the respective lengths of the projection to the horizontal and vertical axis. The *slope* of S is the quotient $h(S)/\ell(S)$. The positive integer $d(S) := \gcd(h(S), \ell(S))$ is called the *degree* of S . Denote $d := d(S)$ the degree of S , $h := h(S)/d$ and $e := \ell(S)/d$ positive coprime integers such that h/e is the slope of S . Let $s = \lfloor \frac{n}{m} \rfloor$ be the integral part of $\frac{n}{m}$, where $n = \deg(P)$ and $m = \deg(\phi)$. For every $1 \leq j \leq s$, let H_j be the length of the projection of $P_j(j, u_j)$ to the horizontal axis, h_j its integral part and $t_j = \text{red} \left(\frac{a_j(X)}{p^{h_j}} \right)$, where *red* is the canonical map defined on $\mathbb{Z}[X]$ by reduction

modulo p . If $P_j \notin S$, then $t_j = 0$ and if $P_j \in S$, then $t_j \neq 0$. If i is the abscissa of the initial point of S , let $P_S(Y)$ be the *residual polynomial* attached to S :

$$P_S(Y) := t_i Y^d + t_{i+e} Y^{d-1} + \cdots + t_{i+(d-1)e} Y + t_{i+de} \in \mathbb{F}_\phi[Y].$$

Let $\text{ind}_N(P) := \sum_{j=1}^s h_j$ the number of points with integer coordinates that lie below the polygon N , strictly above the horizontal axis and whose abscissas satisfy $1 \leq j < l - 1$, where l is the length of N , $s = \lfloor \frac{n}{m} \rfloor$, $n = \text{deg}(P)$ and $m = \text{deg}(\phi)$.

Let $\bar{P}(X) = (\bar{\phi}_1(X))^{l_1} \bar{P}_2$ such that $\bar{\phi}_1(X)$ is irreducible modulo p , $\bar{\phi}_1(X)$ does not divide $P_2(X)$ and $N_1^+ = S_1 + \dots + S_s$ the principal part of $N_{\phi_1}(P)$. P is said to be ϕ_1 -regular if for every $1 \leq i \leq s$, $P_{S_i}(Y)$ is square free. P is said to be p -regular if $\bar{P}(X) = \prod_{i=1}^r \bar{\phi}_i(X)^{l_i}$ the factorization of $\bar{P}(X)$ modulo p of irreducible polynomials and for every $1 \leq i \leq r$, P is ϕ_i -regular. Since the p -regularity of P depends on the choice of $\phi_1(X), \dots, \phi_r(X)$. Then P is said to be p -regular if there exist $\phi_1(X), \dots, \phi_r(X)$ monic polynomial in $\mathbb{Z}[X]$ such that $\bar{P}(X) = \prod_{i=1}^r \bar{\phi}_i(X)^{l_i}$ is the factorization of $\bar{P}(X)$ modulo p and for every $1 \leq i \leq r$, P is ϕ_i -regular.

Theorem of index: $v_p(\text{ind}(P)) \geq \sum_{i=1}^r m_i \text{ind}_{N_i}(P)$, where $m_i = \text{deg}(\phi_i(X))$ for every i . With equality, if $P(X)$ is a p -regular polynomial. (cf. [4, p 326]).

1. p -INTEGRAL BASIS OF QUARTIC NUMBER FIELD DEFINED BY $X^4 + aX + b$

In this section, $K = \mathbb{Q}[\alpha]$, where α is a complex root of an irreducible trinomial $P(X) = X^4 + aX + b \in \mathbb{Z}[X]$ such that for every prime p , $v_p(a) \leq 2$ or $v_p(b) \leq 3$.

Lemma 1.1. *Let p be a prime integer and $w = \frac{t\alpha^3 + z\alpha^2 + y\alpha + x}{p^i} \in K$. Then $\text{ch}_w = X^4 + \frac{A_3}{p^i} X^3 + \frac{A_2}{p^{2i}} X^2 + \frac{A_1}{p^{3i}} X + \frac{A_0}{p^{4i}}$ is the characteristic polynomial of l_w the endomorphism of K defined by $l_w(x) = wx$, where*

$$A_0 = x^4 + 3ax^2yz + 2bx^2z^2 - axy^3 - 4bxy^2z - 3ax^3t + by^4 + b^2z^4 + b^3t^4 + 3a^2x^2t^2 - 3a^2xyzt + a^2xz^3 - 5abxyt^2 + abxz^2t + 4b^2xzt^2 - a^3xt^3 + 4bx^2yt + 3aby^2zt + 2b^2y^2t^2 - aby^3z - 4b^2yz^2t + a^2byt^3 - ab^2zt^3,$$

$$A_1 = -(4x^3 - 9ax^2t + 4bxz^2 + 8bxyt + 6axyz + 6a^2xt^2 - ay^3 - 4by^2z - 3a^2yzt + a^2z^3 - 5abyt^2 + abz^2t + 4b^2zt^2 - a^3t^3),$$

$$A_2 = 6x^2 - 9axt + 3ayz + 4byt + 2bz^2 + 3a^2t^2 \text{ and } A_3 = -4x + 3at.$$

In particular, w is integral if and only if for every $1 \leq j \leq 3$, $\frac{A_j}{p^{ji}} \in \mathbb{Z}$.

The following theorem is an improvement and a specialization of the theorem of index on quartic number fields.

Theorem 1.2. *Let $P(X) = X^4 + mX^3 + nX^2 + aX + b \in \mathbb{Z}[X]$ be an irreducible polynomial such that for every prime p , $v_p(m) = 0$ or $v_p(n) \leq 1$ or $v_p(a) \leq 2$ or $v_p(b) \leq 3$. Let p be a prime integer. If $P(X)$ is a p -regular polynomial, then we have the following:*

- (1) *If $\bar{P}(X)$ is square free, then $(1, \alpha, \alpha^2, \alpha^3)$ is a p -integral basis of \mathbb{Z}_K .*

- (2) If $\bar{P}(X) = (\phi(X))^4$, where $\deg\phi = 1$, then $(1, \alpha, \frac{\alpha^2+a_3\alpha}{p^{h_2}}, \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}})$ is a p -integral basis of \mathbb{Z}_K , where $P(X) = \sum_{i=0}^4 a_i\phi^i$ is the ϕ_1 -adic development of $P(X)$.
- (3) If $\bar{P}(X) = (\phi(X))^3P_2$, where $\deg\phi = 1$ and $\phi(X)$ does not divide $P_2(X)$, then $(1, \alpha, \frac{\alpha^2+a_3\alpha}{p^{h_2}}, \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}})$ is a p -integral basis of \mathbb{Z}_K , where $P(X) = \sum_{i=0}^4 a_i\phi^i$ is the ϕ -adic development of $P(X)$.
- (4) If $\bar{P}(X) = (\phi(X))^2P_2$, where $\deg\phi_1 = 1$, $\phi(X)$ does not divide $P_2(X)$ and $P_2(X)$ is square free, then $(1, \alpha, \frac{\alpha^2+a_3\alpha}{p^{h_2}}, \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}})$ is a p -integral basis of \mathbb{Z}_K , where $P(X) = \sum_{i=0}^4 a_i\phi^i$ is the ϕ -adic development of $P(X)$.
- (5) If $\bar{P}(X) = (\phi_1(X))^2(\phi_2(X))^2$, where $\deg\phi_i = 1$ and $\phi_1(X)$ does not divide $\phi_2(X)$, then for every i , let $P(X) = \sum_{j=0}^4 a_{i,j}\phi_i^j$ be the ϕ_i -adic development of $P(X)$, $w_i \frac{\alpha^3+a_{i,3}\alpha^2+a_{i,2}\alpha}{p^{h_3^i}}$ and $h_3^i \leq h_3^j$. Then:
- If $h_3^i = 0$, then $(1, \alpha, \alpha^2, w_j)$ is a p -integral basis of \mathbb{Z}_K .
 - If $h_3^i \geq 1$, then $(1, \alpha, w_i - p^{h_3^i-h_3^j}w_j, w_j)$ is a p -integral basis of \mathbb{Z}_K .
- (6) If $\bar{P}(X) = (\phi(X))^2$, where $\phi(X)$ is irreducible of degree 2, then $(1, \alpha, \frac{\phi(\alpha)}{p^h}, \frac{\alpha\phi(\alpha)}{p^h})$ is a p -integral basis of \mathbb{Z}_K , where $P(X) = \phi^2 + A(X)\phi + B(X)$ is the $\phi(X)$ -adic development of $P(X)$ and h is the little of $(v_p(A(X)), \lfloor \frac{v_p(B(X))}{2} \rfloor)$.

Proof.

- (1) Case 1. By Dedekind criterion, since \bar{P} is square free, then $(1, \alpha, \alpha^2, \alpha^3)$ is a p -integral basis of \mathbb{Z}_K .
- (2) Cases 2,3,4. By theorem of index it suffices to show that every $w_i \in \mathbb{Z}_K$, where $w_2 = \frac{\alpha^2+a_3\alpha}{p^{h_2}}$ and $w_3 = \frac{\alpha^3+a_3\alpha^2+a_2\alpha}{p^{h_3}}$. Let $\phi(X) = X - x_0$. By replacing $P(X)$ by $P(X + x_0)$, we can assume that $x_0 = 0$, and then $w_2 = \frac{\alpha^2+m\theta}{p^{h_2}}$ and $w_3 = \frac{\alpha^3+m\alpha^2+n\alpha}{p^{h_3}}$. Let $Ch_{w_2}(X) = X^4 + \frac{2n}{p^{h_2}}X^3 + \frac{(n^2+2b+am)}{p^{2h_2}}X^2 + \frac{(amn+bm^2-a^2+2bn)}{p^{3h_2}}X + \frac{(bm^2n-abm+b^2)}{p^{4h_2}}$ and $Ch_{w_3}(X) = X^4 + \frac{3a}{p^{h_3}}X^3 + \frac{(bn+3a^2)}{p^{2h_3}}X^2 + \frac{(a^3+2abn-b^2m)}{p^{3h_3}}X + \frac{(b^3+a^2bn-ab^2m)}{p^{4h_3}}$ be the respective characteristic polynomial of l_{w_2} and l_{w_3} , where l_w is the endomorphism of K defined by the multiplication by w . By definition of h_2 , p^{h_2} divides n . Since $N_\phi(P)$ is convex, then $v_p(b) \geq 3h_2$ and $v_p(a) \geq 2h_2$. Thus, $Ch_{w_2} \in \mathbb{Z}[X]$, and then $w_2 \in \mathbb{Z}$. For w_3 , by definition of h_3 , p^{h_3} divides a . Since $N_\phi(P)$ is convex, then $v_p(b) \geq h_3 + (h_3 - h_1) \geq h_3 + (h_3 - h_2)$, and then $v_p(b) + v_p(n) \geq 2h_3$, $3v_p(b) \geq 4h_3$ and $2v_p(b) + v_p(m) \geq 3h_3$. Thus, $Ch_{w_3} \in \mathbb{Z}[X]$ and $w_3 \in \mathbb{Z}$.
- (3) Case 5. As in the previous cases, every $w_k \in \mathbb{Z}_K$. By Hensel lemma, let $(P_1, P_2) \in \mathbb{Z}_p[X]^2$ such that $P_1P_2 = P$ and $\bar{P}_k = \phi_k^2$. Since \bar{P}_1 and \bar{P}_2 are coprime, then $v_p(\text{ind}(P)) = v_p(\text{ind}(P_1)) + v_p(\text{ind}(P_2))$ (cf. MN). Since P is p -regular, then for every k , $v_p(\text{ind}(P_k)) = h_3^k$. Thus, If $h_3^i = 0$,

then $(1, \alpha, \alpha^2, w_j)$ is a p -integral basis of \mathbb{Z}_K . Else, then for every k , let $\phi_k = X - x_k$. Then $a_{k,3} = 4x_k + m$. Since $x_1 \neq x_2$ modulo p , then $a_{1,3} \neq a_{2,3}$ modulo p , and then $w_i - p^{h_3^j - h_3^i} w_j = \frac{U(\alpha)}{p^{h_3^j}}$, where $U(X) \in \mathbb{Z}[X]$ of degree 2 such that the coefficient of X^2 is coprime to p . Finally, $(1, \alpha, w_i - p^{h_3^j - h_3^i} w_j, w_j)$ is a p -integral basis of \mathbb{Z}_K .

- (4) Case 6. By Theorem of index it suffices to show that every $\phi(\alpha) \in \mathbb{Z}_K$. Let $P(X) = \phi^2 + A(X)\phi + B(X)$ be the $\phi(X)$ -adic development of $P(X)$ and $k = v_p(\phi(\alpha))$. Since $P(\alpha) = 0$, then $2v_p(\phi(\alpha)) \geq v_p(B(\alpha))$, and then $k \geq h$. Thus, $\frac{\phi(\alpha)}{p^h} \in \mathbb{Z}_K$.

The following Theorem gives us a triangular p -integral basis of K , $v_p(\Delta)$ and $v_p(d_K)$ for every prime integer p .

Theorem 1.3. *Let $p \geq 5$ be a prime integer. Under the above hypotheses, a p -integral (resp. a 2-integral, resp. a 3-integral) basis of \mathbb{Z}_K is given in table A (resp. table B, B2*, B3* and B.3.2, resp. table C)*

Table A

case	$v_p(b)$	$v_p(a)$	$v_p(\Delta)$	p -integral basis	$v_p(d_K)$
A1	3	≥ 3	9	$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$	3
A2	≥ 3	2	8	$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$	2
A3	≥ 1	0	0	$(1, \alpha, \alpha^2, \alpha^3)$	0
A4	2	≥ 2	6	$((1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p}))$	2
A5	1	≥ 1	3	$(1, \alpha, \alpha^2, \alpha^3)$	3
A6	≥ 2	1	4	$((1, \alpha, \alpha^2, \frac{\alpha^3}{p}))$	2
A7	0	≥ 1	0	$(1, \alpha, \alpha^2, \alpha^3)$	0
A8	0	0	?	$(1, \alpha, \alpha^2, \frac{\alpha^3 + t\alpha^2 + t^2\alpha - 3t^3}{p^m})$	$v_p(\Delta)[2]$

In case A8, $t \in \mathbb{Z}$ such that $3at + 4b = 0$ modulo p^{m+1} , $m = \lfloor \frac{v_p(\Delta)}{2} \rfloor$ and $v_p(\Delta)[2]$ is the remainder of the Euclidean division of $v_p(\Delta)$ by 2.

Table B

case	conditions	$v_2(\Delta)$	2-integral basis	$v_2(d_K)$
B1	$v_2(b) \geq 3, v_2(a) = 2$	8	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	2
B2	$v_2(b) = 3, v_2(a) \geq 5$	17	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	11
B3	$v_2(b) = 3, v_2(a) = 4$	16	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	10
B4	$v_2(b) = 3, v_2(a) = 3$	12	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2^2})$	6
B5	$b = 4 + 16B, a = 16A, A = B[2]$	14	$(1, \alpha, \frac{\alpha^2+2\alpha+2}{2^2}, \frac{\alpha^3+2\alpha^2+(2+4B)\alpha}{2^3})$	4
B6	$b = 4 + 16B, a = 16A, A \neq B[2]$	14	$(1, \alpha, \frac{\alpha^2+2\alpha+2}{2^2}, \frac{\alpha^3+2\alpha^2+2\alpha}{2^2})$	6
B7	$b = 12 + 16B, a = 16A$	12	$(1, \alpha, \frac{\alpha^2+2}{4}, \frac{\alpha^3+2\alpha}{2^2})$	4
B8	$v_2(b) = 2, v_2(a) = 3$	12	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3+2\alpha}{4})$	6
B9	$v_2(b) = 2, v_2(a) = 2$	8	$(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2})$	4
B10	$v_2(b) \geq 2, v_2(a) = 1$	4	$(1, \alpha, \alpha^2, \frac{\alpha^3}{2})$	2
B11	$v_2(b) = 1, v_2(a) \geq 3$	11	$(1, \alpha, \alpha^2, \alpha^3)$	11
B12	$v_2(b) = 1, v_2(a) = 2$	8	$(1, \alpha, \alpha^2, \alpha^3)$	8
B13	$v_2(b) = 1, v_2(a) = 1$	4	$(1, \alpha, \alpha^2, \alpha^3)$	4
B14	$v_2(a) = 0$	0	$(1, \alpha, \alpha^2, \alpha^3)$	0
B15	$v_2(a) \geq 3, b = 1 [4]$	8	$(1, \alpha, \alpha^2, \alpha^3)$	8
B16	$v_2(a) \geq 3, b = 3 [8]$	8	$(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3-\alpha^2+\alpha-1}{2})$	4
B17	$v_2(a) \geq 3, b = 7 [8]$	8	$(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3+4\alpha^2+6\alpha}{4})$	2
B18	$v_2(a) = 2, b = 1 [4]$	9	$(1, \alpha, \alpha^2, \alpha^3)$	9
B19	$v_2(a) = 2, b = 7 [8]$	10	$(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3+\alpha}{2})$	6
B20	$v_2(a) = 1, b = 3 [4]$	4	$(1, \alpha, \alpha^2, \alpha^3)$	4
B21	$v_2(a) = 1, b = 1 [4]$	4	$(1, \alpha, \alpha^2, \frac{\alpha^3-\alpha^2+\alpha-1}{2})$	2
B2*	$b = 3[8], v_2(a) = 2$	*	cf table B2*	*

If $b = 3[4]$ and $v_2(a) = 2$, then let $A = 4 + a$, $B = 1 + a + b$. Consider $F(X) = P(X + 1) = X^4 + 4X^3 + 6X^2 + AX + B$, $d = A_2^2 - 3B_2$ and $\theta = \alpha - 1$. Then

Table B2*

conditions	$v_2(\Delta)$	2-integral basis	$v_2(d_K)$
$v_2(B) + 1 \geq 2v_2(A)$ $v_2(A) = m$	$5 + 2v_2(A)$	$(1, \alpha, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^m}), m = v_2(A)$	3
$v_2(B) = 2m, v_2(A) = m + 1$	$2m + 7$	$(1, \alpha, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^m})$	5
$v_2(B) = 2m, v_2(A) \geq m + 2$	$2m + 8$	$(1, \alpha, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^m})$	6
$v_2(B)+1 < 2v_2(A), v_2(B)=1[2]$	*	cf Table B3*	*

B.3* : $v_2(B) + 1 < 2v_2(A), v_2(B) = 2k + 1$.

Let $t \in \mathbb{Z}$ such that $v_2(n_2t + A_2) = s, s_1 = L(\lfloor \frac{v_2(d)-1}{2} \rfloor, \lfloor \frac{k+2}{2} \rfloor)$ and $H(X) = F(X + 2^kt) = X^4 + m_1X^3 + n_1X^2 + A_1X + B_1$.

Table B3*

conditions	$v_2(\Delta)$	2-integral basis	$v_2(d_K)$
$2v_2(A) > v_2(B) + 1$ $v_2(d) = 1$	$2k+9$	$(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+4\theta^2+6\theta}{2^{k+1}})$	5
$v_2(A) \geq k + 3$	$2k+9$	$s = 1$ $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(-3.2^kt+4)\theta^2+6\theta-6.2^kt}{2^{k+1}})$	3
$k = 1, v_2(d) \geq 2$	11	$(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+6\theta}{2^2})$	5
$v_2(A) = k + 2, k \geq 2$ $v_2(d) \geq 3$	$2(k+s+1)+7$	$s = s_1$ $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^kt)\theta^2+(6-4.2^{k+1}t)\theta-6.2^kt}{2^{k+s_1+2}})$	5
$v_2(A)=k+2, v_2(d)=2$	$2k+11$	let $s = 1$, and go to B4*.	

Table B4*

conditions	2-integral basis	$v_2(d_K)$
$v_2(B_1) = 2(k + 2)$	$(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^kt)\theta^2+6\theta-6.2^kt}{2^{k+2}})$	5
$v_2(B_1) \geq 2(k + 2) + 1$	$(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3+(4-3.2^kt)\theta^2+6\theta-6.2^kt}{2^{k+3}})$	3

Tables B2*, B3* and B4* could be summarized as follows: Let $s \in \mathbb{Z}$ such that $P(X + s)$ is 2-regular: $P(X + s) = X^4 + 4sX^3 + 6s^2X^2 + AX + B$ ($v_2(A) = r$ and $v_2(B) \geq 2r - 1$) or ($v_2(A) \geq r + 1$ and $v_2(B) = 2r$). A such s could be computed

by iterations: Let $s_0 = 1$. If $P(X + s_0)$ is not 2-regular: ($v_2(B) = 2k + 1$ and $v_2(A) \geq k + 2$), then replace s by $s_1 = s_0 + 2^k$. Repeat this iteration until to obtain $P(X + s)$ is 2-regular. Then $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3 + 4s\theta^2 + 6s^2\theta}{3^r})$ is a 2-integral basis of \mathbb{Z}_K , where $r = \lfloor \frac{v_2(\Delta) - 2 - v_2(d_K)}{2} \rfloor$ ($r \geq 2$).

Table C

case	conditions	$v_3(\Delta)$	3-integral basis	$v_p(d_K)$
C1	$v_3(b) \geq 4, v_3(a) = 2$	11	$(1, \alpha, \frac{\alpha^2}{3}, \frac{\alpha^3}{3^2})$	5
C2	$v_3(b) \geq 4, v_3(a) = 1$	7	$(1, \alpha, \alpha^2, \frac{\alpha^3}{3})$	5
C3	$v_3(b) \geq 4, v_3(a) = 0, a^2 \neq 1[9]$	3	$(1, \alpha, \alpha^2, \alpha^3)$	3
C4	$v_3(b) \geq 4, v_3(a) = 0, a^2 = 1[9]$	3	$(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha}{3})$	1
C5	$v_3(b) = 3, v_3(a) \geq 2,$	9	$(1, \alpha, \frac{\alpha^2}{3}, \frac{\alpha^3}{3^2})$	3
C6	$v_3(b) = 3, v_3(a) = 1$	7	$(1, \alpha, \alpha^2, \frac{\alpha^3}{3})$	5
C7	$v_3(b) = 3, a^2 = 1[9]$	3	$(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha}{3})$	1
C8	$v_3(b) = 3, v_3(a) = 0, a^2 \neq 1[9]$	3	$(1, \alpha, \alpha^2, \alpha^3)$	3
C9	$v_3(b) = 2, v_3(a) \geq 2,$	6	$((1, \alpha, \frac{\alpha^2}{3}, \frac{\alpha^3}{3}))$	2
C10	$v_3(b) = 2, v_3(a) = 1$	6	$(1, \alpha, \alpha^2, \frac{\alpha^3}{3})$	4
C11	$v_3(b) = 2, a^2 = 1[9]$	3	$(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha}{3})$	1
C12	$v_3(b) = 2, v_3(a) = 0, a^2 \neq 1[9]$	3	$(1, \alpha, \alpha^2, \alpha^3)$	3
C13	$v_3(b) = 1, v_3(a) \geq 1,$	3	$(1, \alpha, \alpha^2, \alpha^3)$	3
C14	$b = 6[9], v_3(a) = 0, a^2 \neq 4[9],$	3	$(1, \alpha, \alpha^2, \alpha^3)$	3
C15	$b = 6[9], a^2 = 4[9],$	3	$(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha}{3})$	1
C16	$b = 3[9], v_3(a) = 0, a^2 \neq 7[9],$	4	$(1, \alpha, \alpha^2, \alpha^3)$	4
C17	$b = 3[9], a^2 = 7[9], a^4 - a^2 + b = 0[27]$	≥ 6	$(1, \theta, \frac{\theta^2}{3}, \frac{\theta^3 + 4s\theta^2 + 6s^2\theta}{3^m})$ $as = -4b_3 [3^{v_3(\Delta)+1}]$ $\theta = \alpha - s$	$v_3(\Delta) [2]$
C18	$b = 3[9], a^2 = 7[9], v_3(a^4 - a^2 + b) = 2$	5	$(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha}{3})$	3
C19	$v_3(b) = 0$	0	$(1, \alpha, \alpha^2, \alpha^3)$	0

Proof. First, $\Delta = 2^8b^3 - 3^3a^4$ and the proof is based on the Newton polygon. For every prime p , let $u_3 = v_p(a)$, $u_4 = v_p(b)$, $\bar{P}(X)$ the reduction of P modulo p , N the X -Newton polygon of P and N^+ its principal part.

- (1) Case 1 ($v_p(b) = 3$ and $v_p(a) \geq 2$): A1, B2, B3, B4, C5.

$\bar{P}(X) = X^4$ and $\text{ind}_N(P) = 3$.

1) If $u_3 \geq 3$, then $N = S$ is one side and $P_S(Y) = Y + \bar{b}_p$ is square free and $(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$ is a p -integral basis of \mathbb{Z}_K .

2) If $u_3 = 2$, then $N = S_1 + S_2$ with slopes respectively $1/3$ and 1 . $P_{S_1}(Y) = Y + \bar{a}_p$ and $P_{S_2}(Y) = \bar{a}_p Y + \bar{b}_p$ are square free. Thus, $(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2})$ is a p -integral basis of \mathbb{Z}_K .

- (2) Case 2 ($v_p(b) \geq 2$ and $v_3(a) = 1$): A6, B9, C2, C6, C10.

$N = S_1 + S_2$ with slopes respectively $1/3$ and 1 . $P_{S_1}(Y)$ and $P_{S_2}(Y)$ are of degree 1. So, Thus, $(1, \alpha, \alpha^2, \frac{\alpha^3}{p})$ is a p -integral basis of \mathbb{Z}_K .

- (3) Case 3 ($v_p(b) \geq 1$ and $v_3(a) = 0$): A3, B3, C3, C4, C7, C8, C11, C12, C14, C15, C16, C17, C18.

If $p \neq 3$, then $\bar{P}(X)$ is square free and then $v_p(\text{ind}(P)) = 0$.

For $p = 3$, let $F(X) = P(X - a) = X^4 - 4aX^3 + 6a^2X^2 + AX + B$, where $A = -a(4a^2 - 1)$ and $B = (a^4 - a^2 + b)$. Then $v_3(A) \geq 1$ and $v_3(B) \geq 1$. It follows that $v_3(\text{ind}(P)) = 0$ if and only if $v_3(B) = 1$, i.e., if $(a^2 = 1 \text{ modulo } 9 \text{ and } v_3(b) = 1)$ or $(a^2 \neq 1 \text{ modulo } 9 \text{ and } v_3(b) \geq 2)$, then $v_3(\text{ind}(P)) = 0$. Else, then $u_1 = 0$, $u_2 = 1$, $u_3 = v_3(A) \geq 1$ and $u_4 = v_3(B) \geq 2$.

(a) If $u_3 = 1$ or $u_4 = 2$, then $v_3(\text{ind}(P)) = 1$, $\frac{\theta^3 - 4a\theta^2 + 6a^2\theta}{3}$ is integral, and then $(1, \alpha, \alpha^2, \frac{\alpha^3 - a\alpha^2 + \alpha}{3})$ is a 3-integral basis of \mathbb{Z}_K : ($b = 6$ and $a^2 = 4$ modulo 9) or ($b = 3$, $a^2 = 7$ modulo 9 and $v_3(a^4 - a^2 + b) = 2$).

(b) If $u_3 \geq 2$ and $u_4 \geq 3$: ($b = 3$, $a^2 = 7$ modulo 9 and $a^4 - a^2 + b = 0$ modulo 27), then let $m = \lfloor \frac{v_2(\Delta) - 2}{2} \rfloor$, $s \in \mathbb{Z}$ such that $as = -4b_3 [3^{v_3(\Delta)+1}]$ and $\theta = \alpha - s$. Then $K = \mathbb{Q}[\theta]$ and $f(X) = X^4 + 4sX^3 + 6s^2X^2 + AX + B$ is the minimal polynomial of θ , where $A = 4s^3 + a$ and $B = s^4 + as + b$. It follows that $u_1 = 0$, $u_2 = 1$, $u_3 = v_2(\Delta) - 3 = u_4$. Finally, $(1, \theta, \frac{\theta^2}{3}, \frac{\theta^3 + 4s\theta^2 + 6s^2\theta}{3^m})$ is a 3-integral basis of \mathbb{Z}_K and $v_3(d_K) = v_2(\Delta) [2]$.

- (4) Case 4 ($v_p(b) = 2$ and $v_3(a) \geq 2$): A4, C9, B5, B6, B7, B8:

For $p \neq 2$, $N = S$ is one side and $P_S(Y) = Y^2 + \bar{b}_p$ is square free. Hence $v_p(\text{ind}(P)) = 2$

For $p = 2$, $N = S$ is one side and $P_S(Y) = (Y + 1)^2$. Since $P(X)$ is not 2-regular, we will use a higher order. Let $t_1 = (X, 1/2, Y + 1)$ and $\phi_2(X) = X^2 + 2$ as defined in[3], page 16 and let V_2 be the 2-adic valuation of 2^d -order as defined in[3], page 17. Then $V_2(\phi_2) = 2$, $V_2(X) = 1$ and for every $x \in \mathbb{Z}$, $V_2(x) = 2v_2(x)$. Let $P(X) = \phi_2^2(X) - 4\phi_2(X) + (4AX + 4b_2 + 4)$, $R_0 = V_2(\phi_2^2(X)) = 4$, $R_1 = V_2(4\phi_2(X)) = 6$ and $R_2 = V_2(4AX + 4b_2 + 4)$. From [3, Th 4.18, p:48], it follows that:

(a) If $v_2(a) = 2$, then $R_2 = 5$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{2})$ is a 2-integral basis of \mathbb{Z}_K .

- (b) If $v_2(a) = 3$, then $R_2 = 7$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3+2\alpha}{4})$ is a 2-integral basis of \mathbb{Z}_K .
- (c) If $b_2+1=0$ modulo 4 and $v_2(a) \geq 4$, then $R_2 \geq 8$ and $(1, \alpha, \frac{\alpha^2+2}{4}, \frac{\alpha^3+2\alpha}{4})$ is a 2-integral basis of \mathbb{Z}_K .
- (d) If $b_2+1 = 2$ modulo 4 and $v_2(a) \geq 4$, then let $b = 4 + 16B$, $a = 16A$, $\phi_2(X) = X^2 + 2X + 2t$ and $P(X) = \phi_2^2(X) - 4(X+t-1)\phi_2(X) + 8(t-1+2A)X + 4(1+t^2+4B-2t)$. Then $R_0 = 4$ $R_1 = 7$. Since $1+t^2+4B-2t = 2(1-t+2B)$ modulo 8, let $t \in \mathbb{Z}$ such that $1-t+2B = 0$ modulo 4. It follows that: if $B = A$ modulo 2, then $R_2 \geq 10$, $v_2(\text{ind}(P)) = 5$ and $(1, \alpha, \frac{\alpha^2+2\alpha+2}{4}, \frac{\alpha^3+2\alpha^2+2(1+2B)\alpha}{2^3})$ is a 2-integral basis of \mathbb{Z}_K .
If $B \neq A$ modulo 2, then $R_2 = 9$, $v_2(\text{ind}(P)) = 4$ and $(1, \alpha, \frac{\alpha^2+2\alpha+2}{4}, \frac{\alpha^3+2\alpha^2+2\alpha}{2^2})$ is a 2-integral basis of \mathbb{Z}_K .
- (5) Case 5 ($v_p(b) = 1$ and $v_3(a) \geq 1$): A5, C13, B10, B11, B12.
 $N = S$ is one side and $P_S(Y) = Y + \bar{b}_p$ is square free. Hence $v_p(\text{ind}(P)) = 0$.
- (6) Case 6 $v_p(b) \geq 3$ and $v_3(a) = 2$: A2, C1, B1:
 $N = S_1 + S_2$ with slopes respectively $2/3$ and $u_4 - 2$. Since every $P_{S_i}(Y)$ is square free, then $v_p(\text{ind}(P)) = 3$.
- (7) Case 7 $v_p(b) = 0$ and $v_3(a) \geq 1$: A7, B14, B15, ..., B19, B2*, B3* and C19.

If $p \neq 2$, then $\bar{P}(X)$ is square free.

For $p = 2$, according to the Dedekind criterion, let $f(X) = \frac{P(X)-(X+1)^4}{2} = -2X^3 - 3X^2 + \frac{a-4}{2}X + \frac{b-1}{2}$ and $f(-1) = 1 + \frac{a}{2} + \frac{b-1}{2}$. Thus,

If $(v_2(a) \geq 2$ and $b = 1$ modulo 4) or $(v_2(a) = 1$ and $b = 3$ modulo 4), then $(1, \alpha, \alpha^2, \alpha^3)$ is a 2-integral basis of \mathbb{Z}_K .

Else, let $P(X+1) = X^4 + 4bX^3 + 6X^2 + (4+a)X + (1+a+b)$. Then $u_1 = 2$, $u_2 = 1$ and

- (a) If $v_2(a) = 1$ and $b = 1$ modulo 4, then $u_3 = 1$, $u_4 \geq 2$ and $(1, \alpha, \alpha^2, \frac{\alpha^3-\alpha^2-\alpha-1}{2})$ is a 2-integral basis of \mathbb{Z}_K .
- (b) If $(v_2(a) \geq 3$ and $b = 3$ modulo 8) or $(v_2(a) = 2$ and $b = 7$ modulo 8), then $u_3 = u_4 = 2$ and $N = S$ is one side such that $P_S(Y) = Y^2 + Y + 1$ is irreducible. Hence, $(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3-\alpha^2-\alpha-1}{2})$ is a 2-integral basis of \mathbb{Z}_K .
- (c) If $(v_2(a) \geq 3$ and $b = 7$ modulo 8), then $u_3 = 2$, $u_2 = 1$, $u_3 = 2$ and $u_4 \geq 3$. Thus, $(1, \alpha, \frac{\alpha^2+1}{2}, \frac{\alpha^3+2\alpha+1}{4})$ is a 2-integral basis of \mathbb{Z}_K .
- (d) If $v_2(a) = 2$ and $b = 3$ modulo 4, let $F(X) = P(X+1) = X^4 + 4X^3 + 6X^2 + (4+a)X + (1+b+a) = X^4 + 4X^3 + 6X^2 + AX + B$, where $A = 4+a$, $B = 1+b+a$, $v_2(A) \geq 3$ and $v_2(B) \geq 3$. First, we

have $\Delta = \text{disc}(F) = 256B^3 - 768B^2A + 768BA^2 + 176A^3 + 2304B^2 - 4608BA - 288A^2 - 27A^4 + 6912B$. Consequently,

- (i) If $v_2(B) + 1 \geq 2v_2(A)$, then $v_2(\Delta) = 5 + 2v_2(A)$, $v_2(\text{ind}(P)) = 1 + v_2(A)$, $v_2(d_K) = 3$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3 + 4\alpha^2 + 6\alpha}{2^k})$ is a 2-integral basis of \mathbb{Z}_K , where $k = v_2(A)$.
- (ii) If $v_2(B) = 2k$ is even and $v_2(B) + 1 < 2v_2(A)$, then if $v_2(A) \geq k + 2$, then $v_2(\Delta) = 8 + v_2(B)$, $v_2(\text{ind}(P)) = k + 1$, $v_2(d_K) = 6$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3 + 4\alpha^2 + 6\alpha}{2^k})$ is a 2-integral basis of \mathbb{Z}_K . If $v_2(A) = k + 1$, then $v_2(\Delta) = 2k + 7$, $v_2(\text{ind}(P)) = k + 1$, $v_2(d_K) = 5$ and $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3 + 4\alpha^2 + 6\alpha}{2^k})$ is a 2-integral basis of \mathbb{Z}_K .
- (iii) $v_2(B)$ is odd and $v_2(B) + 1 < 2v_2(A)$. By the Theorem of the polygon $F(X) = H(X)G(X)$ in $\mathbb{Z}_2[X]$, where $H(X) = X^2 + rX + s$, $G(X) = X^2 + RX + S$,

$$\begin{cases} v_2(s) = 1, & v_2(S) = v_2(B) - 1, & v_2(r) \geq 1, & v_2(R) > \frac{v_2(B) - 1}{2} \\ B = sR + rS, & A = Sr + Rs, & 4 = r + R \end{cases}$$

Since $4 = r + R$, $v_2(r) \geq 3$ and $v_2(r) = 2$. So, $v_2(\text{disc}(H)) = v_2(r^2 - 4s) = 3$, $v_2(\text{Res}(H, G)) = 2v_2(H(\theta)) = 2$, where θ is a root of $G(X)$. As $v_2(r) \geq 1$ and $v_2(s) = 1$, we have $H(X)$ is irreducible in $\mathbb{Z}_2[X]$. On the other hand, as $\text{disc}(F) = \text{disc}(H)\text{disc}(G)(\text{Res}(H, G))^2$,

$$v_2(\text{disc}(F)) = v_2(\text{disc}(H)) + 2v_2(\text{Res}(H, G)) + v_2(\text{disc}(G)) = 7 + v_2(\text{disc}(G)). \text{ Thus,}$$

- (A) If $G(X)$ is irreducible in $\mathbb{Z}_2[X]$, then from [4], $v_2(\text{ind}(F)) = 0 + 2 + v_2(\text{ind}(G))$. Let θ be a root of $G(X)$ and $u = \frac{\theta + x}{2^k} \in \mathbb{Q}_2[\theta]$. Since the characteristic polynomial of u is $Ch_u = X^2 - \frac{2x - R}{2^k}X + \frac{(2x - R)^2 + \text{disc}(G)}{2^{2k+2}}$, where $\text{disc}(G) = 4S - R^2$, then u is integral if and only if 2^{2k+2} divides $\text{disc}(G)$. Therefore, $v_2(\text{ind}(G)) = \lfloor \frac{v_2(\text{disc}(G))}{2} \rfloor - 1$. Thus, $v_2(\text{ind}(F)) = 2 + \lfloor \frac{v_2(\text{disc}(G))}{2} \rfloor - 1$, and then if $v_2(\Delta)$ is even, then $v_2(\text{ind}(F)) = \frac{v_2(\text{disc}(F)) - 6}{2}$ and $v_2(d_K) = 6$. Else, then $v_2(\text{ind}(F)) = \frac{v_2(\text{disc}(F)) - 5}{2}$ and $v_2(d_K) = 5$.
- (B) If $G(X) = (X - \theta_1)(X - \theta_2)$ in $\mathbb{Z}_2[X]$, then $v_2(\text{disc}(G)) = 2v_2(\theta_1 - \theta_2)$ and $v_2(\text{disc}(F)) = 7 + 2v_2((\theta_1 - \theta_2))$. Let $G_1 = X - \theta_1$ and $G_2 = X - \theta_2$. Then $v_2(\text{ind}(F)) = 2 + v_2(\text{Res}(G_1, G_2)) = 2 + v_2((\theta_1 - \theta_2))$, and then $v_2(d_K) = 3$.

In these cases, $(1, \alpha, \frac{\alpha^2}{2}, \frac{\alpha^3 + z\alpha^2 + y\alpha^2 + x}{2^m})$ is a 2-integral basis of \mathbb{Z}_K , where $m = \lfloor \frac{v_2(\text{disc}) - v_2(d_K)}{2} \rfloor - 1$, x, y and z are integers such that $\frac{\alpha^3 + z\alpha^2 + y\alpha^2 + x}{2^m} \in \mathbb{Z}_K$.

Since we can not compute the coefficients of $G(X)$ in \mathbb{Q}_p neither to test if $G(X)$ is irreducible in $\mathbb{Q}_p[X]$, we must give, in C.1,

a method which allows us to compute the integers x , y and z independently of the knowledge of the irreducibility of $G(X)$.

- (8) C.1: Let $F(X) = X^4 + 4X^3 + 6X^2 + AX + B \in \mathbb{Z}[X]$, $\theta = \alpha - 1$, $d = A_2^2 - 3B_2$, where $v_2(A) \geq k + 2$, $v_2(B) = 2k + 1$ and $k \geq 1$. Let $t \in \mathbb{Z}$ such that $v_2(3t + A_2) = s$, $H(X) = F(X + 2^k t) = X^4 + m_1 X^3 + n_1 X^2 + A_1 X + B_1$. Then $m_1 = 4 + 2^{k+2}t$, $n_1 = 6 + 3 \cdot 2^{k+2}t + 3 \cdot 2^{2k+1}t^2$, $A_1 = A + 3 \cdot 2^{k+2}t + 3 \cdot 2^{2k+2}t^2 + 2^{3k+2}t^3$ and $B_1 = B + 2^k A t + 3 \cdot 2^{2k+1}t^2 + 2^{3k+2}t^3 + 2^{4k}t^4$. Therefore $3B_1 = 2^{2k+1}(3B - A_2^2) + 3 \cdot 2^{3k}t^3(2^2 + 2^k t) + 2^{2k+1+2s}L$, where L is an odd integer. It follows that: $v_2(m_1) = 2$, $v_2(n_1) = 1$ and
- (a) If $v_2(d) = 1$, then $v_2(B_1) = 2k + 2$, $v_2(\text{ind}(F)) = k + 2$, and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3 + 4\theta^2 + 6\theta}{2^{k+1}})$ is a 2-integral basis of \mathbb{Z}_K . Moreover, if $v_2(A) = k + 2$, then $v_2(\Delta) = 2k + 9$ and $v_2(d_K) = 5$. if $v_2(A) \geq k + 3$, then $v_2(\Delta) = 2k + 10$ and $v_2(d_K) = 6$.
 - (b) If $v_2(d) \geq 2$ and $v_2(A) > k + 2$, then $v_2(A_1) = k + 2$ and $v_2(B_1) \geq 2k + 3$. Hence $v_2(\text{ind}(F)) = k + 3$, $v_2(\Delta) = 2k + 9$ and $v_2(d_K) = 3$ and $(1, \theta, \frac{\theta^2}{2}, w_3)$ is a 2-integral basis of \mathbb{Z}_K , where $w_3 = \frac{\theta^3 + (4 - 3 \cdot 2^k t)\theta^2 + 6\theta - 3 \cdot 2^{k+1}t}{2^{k+2}}$.
 - (c) If $k = 1$, $v_2(d) \geq 2$ and $v_2(A) = 3$, then for $s = 1$, $v_2(B_1) = 4$, $v_2(\Delta) = 11$, $v_2(d_K) = 5$, $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3 + 4\theta^2 + 6\theta}{2^2})$ is a 2-integral basis of \mathbb{Z}_K .
 - (d) If $v_2(A) = k + 2$, $k \geq 2$ and $v_2(d) \geq 3$, then for $s = L(\lfloor \frac{v_2(d)-1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor)$, $v_2(A_1) = (k + 1 + s) + 1$, $v_2(B_1) \geq 2(k + s + 1) + 1$. Hence $v_2(\text{ind}(F)) = k + 2 + s$, $v_2(\Delta) = 2(k + s + 1) + 7$, $v_2(d_K) = 5$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3 + (4 - 3 \cdot 2^k t)\theta^2 + 6\theta - 3 \cdot 2^{k+1}t}{2^{k+1+s}})$ is a 2-integral basis of \mathbb{Z}_K .
 - (e) If $v_2(A) = k + 2$ and $v_2(d) = 2$, then $v_2(\Delta) = 2k + 11$. For $s = 1$, $v_2(A_1) = k + 3$ and $3B_1 = -2^{2k+1}d + 3 \cdot 2^{3k+2}t^3(1 + 2^{k-2}t) + 2^{2k+3}L = 2^{2k+3}(-d_2 + L) + 3 \cdot 2^{3k+2}t^3(1 + 2^{k-2}t)$, where L is an odd integer. Thus, $v_2(A_1) = k + 3$ and $v_2(B_1) \geq 2(k + 2)$. It follows that:
 - (i) If $v_2(B_1) = 2(k + 2)$, then $v_2(\text{ind}(F)) = k + 3$, $v_2(d_K) = 5$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3 + (4 - 3 \cdot 2^k t)\theta^2 + 6\theta - 3 \cdot 2^{k+1}t}{2^{k+2}})$ is a 2-integral basis of \mathbb{Z}_K .
 - (ii) If $v_2(B_1) \geq 2(k + 2) + 1$, then $v_2(\text{ind}(F)) = k + 4$, $v_2(d_K) = 3$ and $(1, \theta, \frac{\theta^2}{2}, \frac{\theta^3 + (4 - 3 \cdot 2^k t)\theta^2 + 6\theta - 3 \cdot 2^{k+1}t}{2^{k+3}})$ is a 2-integral basis of \mathbb{Z}_K .
- (9) Case 8 $v_p(ab) = 0$: A8.
If $p \in \{2, 3\}$, then $v_p(\Delta) = 0$, $(1, \alpha, \alpha^2, \alpha^3)$ is a p -integral basis of \mathbb{Z}_K and $v_p(d_K) = 0$.
Let $p \geq 5$. If $v_p(\text{disc}(2^8 b^3 - 3^3 a^4)) \leq 1$, then $(1, \alpha, \alpha^2, \alpha^3)$ is a p -integral basis of \mathbb{Z}_K and $v_p(d_K) = 0$.
Else, since $3a \neq 0$ modulo p , let $t \in \mathbb{Z}$ such that $3at + 4b = 0$ modulo p^s , where $s = m + 1$ and $m = \lfloor \frac{v_p(\Delta)}{2} \rfloor$ ($3at + 4b = p^s L$). Then $(3a)^3 P'(t) = -\Delta + 3 \cdot 4^3 b^2 p^s L$ modulo p^{2s} . Thus, $v_p(P'(t)) \geq s$. Moreover, $(3a)^4 P(t) = b\Delta - p^s \Delta$ modulo p^{2s} . Thus, $v_p(P(t)) = v_p(\Delta)$. Let $P(X + t) = X^4 +$

$4tX^3 + 6t^2X^2 + P'(t)X + P(t)$. Since $6t^2 \neq 0$ modulo p , then $N = S_0 + S_1$ with slopes respectively 0 and $\frac{v_p(\Delta)}{2}$ and $P_{S_1}(Y)$ is square free. Hence, $v_p(\text{ind}(P)) = \lfloor \frac{v_p(\Delta)}{2} \rfloor$, $\frac{\theta^3 + 4t\theta^2 + 6t^2\theta}{p^m} \in \mathbb{Z}_K$, where $\theta = \alpha - t$. Thus, $(1, \alpha, \alpha^2, \frac{\alpha^3 + t\alpha^2 + t^2\alpha - 3t^3}{p^m})$ is a p -integral basis of z_K and $v_p(d_K) = v_p(\Delta)$ modulo 2.

- (10) Case 9 $v_2(a) = 0$ or $v_3(b) = 0$ (B13, C19). Since $v_p(\Delta) = 0$, $(1, \alpha, \alpha^2, \alpha^3)$ is a p -integral basis of \mathbb{Z}_K and $v_p(d_K) = 0$.

2. AN INTEGRAL BASIS OF A QUARTIC NUMBER FIELD DEFINED BY

$$X^4 + aX + b$$

- Remarks 2.1.** (1) Let p be a prime integer such that p^2 divides Δ . For every $1 \leq i \leq 3$, let $w_{i,p} = \frac{L_i^p(\alpha)}{p^{r_{i,p}}}$, where $L_i^p(X) \in \mathbb{Z}[X]$ is a monic polynomial of degree i such that $\mathcal{F} = (1, w_{1,p}, w_{2,p}, w_{3,p})$ is a triangular p -integral basis of K . Then $r_{1,p} \leq r_{2,p} \leq r_{3,p}$, $v_p(\Delta) = r_1 + r_2 + r_3$ and $v_p(d_K) = v_p(\Delta) - 2(r_1 + r_2 + r_3)$.
- (2) Let p_1, \dots, p_r be the primes such that every p_i^2 divides Δ . For every $1 \leq i \leq 3$, denote $d_i = \prod_{j=1}^r p_j^{r_{ij}}$, where for every j , $w_{i,j} = \frac{L_i^{p_j}(\alpha)}{p_j^{r_{ij}}}$ and $(1, w_{1,j}, w_{2,j}, w_{3,j})$ is a p_j -integral basis of K . Then $1 \mid d_1 \mid d_2 \mid d_3$ are the elementary divisors of $\mathbb{Z}_K/\mathbb{Z}[\alpha]$. In particular, d_3 is the conductor of the order $\mathbb{Z}[\alpha]$ and $d_1 d_2 d_3 = \mp \text{ind}(P)$.
- (3) We can always assume that a triangular p -integral basis has the property: if $r_i = r_{i+1}$, then we can take $w_{i+1} = \alpha w_i$.

One can recover a triangular integral basis from different triangular p -integral basis for all p as follows :

Proposition 2.2. Let p_1, \dots, p_s the prime integers such that p^2 divides Δ and $1, d_1, d_2$ and d_3 the elementary divisors of the abelian group $\mathbb{Z}_K/\mathbb{Z}[\alpha]$. For every j , let $\mathcal{F}_j = (1, w_{1,j}, w_{2,j}, w_{3,j})$ be a triangular p_j -integral basis of K , i.e., $w_{i,j} = \frac{L_i^j(\alpha)}{p_j^{r_{ij}}}$ such that every $L_i^j(X)$ is a monic polynomial of $\mathbb{Z}[X]$ of degree i . Then $\mathcal{B} = (1, w_1, w_2, w_3)$ is a triangular integral basis of K , where every $w_i = \frac{L_i(\alpha)}{d_i}$, $L_i(X) = L_i^j(X)$ modulo $p_j^{r_{ij}}$.

Proof Since $\text{ind}(P) = d_1 d_2 d_3$, we need only to check that every $w_i \in \mathbb{Z}_K$. Let $1 \leq i \leq 3$. Since for every i the integers $(\frac{d_i}{p_j^{r_{ij}}})_{1 \leq j \leq s}$ are pairwise coprime, there exist integers t_1, \dots, t_s such that $\sum_{j=1}^s t_j \frac{d_i}{p_j^{r_{ij}}} = 1$. Hence, $\frac{L_i(\alpha)}{d_i} = \sum_{j=1}^s t_j \frac{L_i(\alpha)}{p_j^{r_{ij}}} \in \mathbb{Z}_K$, because all $\frac{L_i(\alpha)}{p_j^{r_{ij}}} \in \mathbb{Z}_K$.

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