

THE ENTROPY CONJECTURE FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH 1-D CENTER

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ABSTRACT. We prove that if f is a partially hyperbolic diffeomorphism on the compact manifold M with one dimensional center bundle, then the logarithm of the spectral radius of the map induced by f on the real homology groups of M is smaller or equal to the topological entropy of f . This is a particular case of the Shub's entropy conjecture, which claims that the same conclusion should be true for any C^1 map on any compact manifold.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a m -dimensional compact Riemannian manifold without boundary and let $f: M \rightarrow M$ be a differentiable map.

The map f will induce a linear action on the real homology groups of M , denoted $f_{*,k}: H_k(M, \mathbb{R}) \rightarrow H_k(M, \mathbb{R})$. The *spectral radius* of these maps are denoted $sp(f_{*,k})$ and they are equal to the largest eigenvalue in absolute value of the linear map $f_{*,k}$. The *spectral radius of f_** is

$$sp(f_*) = \max_k sp(f_{*,k}).$$

We will also use the common notation $h(f)$ for the *topological entropy* of f , for a definition we send the reader to [HK] for example.

The diffeomorphism f is called *partially hyperbolic* if there exist an invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, with at least two subbundles nontrivial, and there exist $\alpha, \beta > 1$, $C, D > 0$ such that:

- (1) E^u is uniformly expanding:

$$\|Df^k(v_u)\| \geq C\alpha^k\|v_u\|, \quad \forall v_u \in E^u, k \in \mathbb{N},$$

- (2) E^s is uniformly contracting:

$$\|Df^k(v_s)\| \leq D\beta^{-k}\|v_s\|, \quad \forall v_s \in E^s, k \in \mathbb{N},$$

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(3) E^u dominates E^c , and E^c dominates E^s :

$$\|Df|_{E_x^s}\| < \|Df|_{E_x^c}^{-1}\|^{-1} \leq \|Df|_{E_x^c}\| < \|Df|_{E_x^u}^{-1}\|^{-1}, \quad \forall x \in M.$$

Condition (3) could be replaced with some weaker condition, of eventual domination for a power of f , but this doesn't make any difference in the following considerations, because by taking that power of f or by changing the Riemannian metric on M we can always assume this strong domination condition.

We will prove the following result:

Theorem 1. *Suppose that M is a compact Riemannian manifold without boundary and $f: M \rightarrow M$ is a partially hyperbolic diffeomorphism with one dimensional center bundle. Then*

$$h(f) \geq \log sp(f_*).$$

We will prove the theorem in the next section. We remark that this is a special case of the entropy conjecture formulated by Shub in [Sh]:

Conjecture 1. *In f is a C^1 map on the compact manifold without boundary M then*

$$h(f) \geq \log sp(f_*).$$

This conjecture was proven for C^∞ maps by Yomdin ([Yo]), and it is not true for Lipschitz maps ([Pu]). It is also true if M is an infra-nilmanifold for C^0 maps (Marzantowicz and Przytycki, [MP2]), or a manifold of dimension at most three for C^1 maps (combine [MP] with [Ma] and use duality). There are other weaker versions known to be true, when one replaces the spectral radius of f_* by some smaller invariants: the degree for C^1 maps (Misiurewicz and Przytycki, [MP]), the spectral radius on the first homology group for C^0 maps (Manning, [Ma]), the growth on the fundamental group for C^0 maps (Bowen, [Bo]), the asymptotic Nielsen number for C^0 maps (Ivanov, [Iv]).

The conjecture is also true for diffeomorphisms satisfying the Axiom A and no-cycle conditions, so in particular it is true for Anosov diffeomorphisms (Shub and Williams, [SW]; Ruelle and Sullivan, [RS]). The partially hyperbolic diffeomorphisms are natural generalizations of hyperbolic diffeomorphisms, and it is expected that they have similar properties, at least in the generic setting and/or for small dimensions of the center distribution. Our result is another fact that support this claim.

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2. PROOFS

In this section we will prove the Theorem 1. We will use two propositions interesting on their own right which we will state after we introduce some notions.

Suppose $TM = E \oplus F$ is a *dominated splitting* for f , in the sense that

$$m(Df|_{F_x}) := \|Df|_{F_x}^{-1}\|^{-1} < \|Df|_{E_x}\|, \quad \forall x \in M.$$

Denote by $\mathcal{T}(E)$ the family of C^1 disks in M uniformly transverse to E (the angle between the tangent plane to the disk and E is bounded away from zero) and with the same dimension as F :

$$\mathcal{T}(E) = \{D \subset M, C^1 \text{ disk} : \dim D = \dim F, D \pitchfork E, \inf_{x \in D} \angle(T_x D, E_x) > 0\}.$$

Define the *volume growth of a disk D under f* to be the exponential rate of growth of the volume of the iterates of the disk:

$$\chi(D, f) = \limsup_{n \rightarrow \infty} \frac{\log(\text{vol}(f^n(D)))}{n},$$

and the *volume growth of $\mathcal{T}(E)$ under f* :

$$\chi(\mathcal{T}(E), f) = \sup_{D \in \mathcal{T}(E)} \chi(D, f).$$

The first proposition relates the volume growth of $\mathcal{T}(E)$ under f with the topological entropy of f :

Proposition 2. *Suppose $TM = E \oplus F$ is a dominated splitting for f . Then the topological entropy of f is greater or equal to the volume growth of $\mathcal{T}(E)$:*

$$h(f) \geq \chi(\mathcal{T}(E), f).$$

Proof. We have to prove that for every disk $D \in \mathcal{T}(E)$ we have $h(f) \geq \chi(D, f)$. Because $\chi(A \cup B, f) = \max\{\chi(A, f), \chi(B, f)\}$, we may assume that the disk D is arbitrarily small in diameter. Because $\chi(D, f) = \chi(f^n(D), f)$ and

$$\lim_{n \rightarrow \infty} \angle(T_{f^n(x)} f^n(D), F_{f^n(x)}) = 0$$

uniformly with respect to $x \in D$ (this is because the splitting is dominated and the starting disk D is transversal to E), we may also assume that $\angle(T_y f^n(D), F_y) < \frac{\epsilon}{2}$ for all $n \geq 0$ and $y \in f^n(D)$, and some fixed $\epsilon > 0$ small. A dominated splitting is also continuous, so we can assume that there is $\delta > 0$ such that if $x, y \in f^n(D)$ with $d(x, y) < \delta$ then $\angle(T_y f^n(D), F_x) < \epsilon$. Here d is the Riemannian metric on the manifold M . This implies that at the scale δ the Riemannian metric d on M is equivalent to the Riemannian

metric \tilde{d} induced on the submanifolds $f^n(D)$, meaning that there exists $C > 0$ such that if $x, y \in f^n(D)$ for some n and $\tilde{d}(x, y) < \delta$ then

$$d(x, y) \leq \tilde{d}(x, y) \leq Cd(x, y).$$

This can be proved using some small charts and eventually making δ slightly smaller. In the same way one can prove that for any $\delta' < \delta$ there is an upper bound $B_{\delta'} > 0$ for the volumes of the balls in $f^n(D)$ of \tilde{d} -radius δ' , independent of n :

$$\text{vol}(B_{\tilde{d}}(x, \delta')) \leq B_{\delta'}, \quad \forall x \in f^n(D), n \geq 0.$$

Now let $K = \sup_{x \in M} \|Df_x\|$ and choose $\delta' > 0$ such that $C\delta' < \frac{\delta}{K}$, and assume that $\text{diam}_{\tilde{d}}(D) < C\delta'$. Let S_n be a maximal $C\delta'$ -separated set in $f^n(D)$ w.r.t. \tilde{d} . Then

$$f^n(D) \subset \bigcup_{x \in S_n} B_{\tilde{d}}(x, C\delta'),$$

so

$$\text{vol}(f^n(D)) \leq \sum_{x \in S_n} \text{vol}(B_{\tilde{d}}(x, C\delta')) \leq B_{C\delta'} |S_n|,$$

where $|S_n|$ is the cardinality of S_n . Now suppose that $x, y \in f^{-n}S_n$, so $\tilde{d}(x, y) < C\delta'$ and $\tilde{d}(f^n(x), f^n(y)) > C\delta'$. Then there exist $k \in \{0, 1, 2, \dots, n-1\}$ such that:

$$\tilde{d}(f^k(x), f^k(y)) \leq C\delta' \quad \tilde{d}(f^{k+1}(x), f^{k+1}(y)) > C\delta'.$$

Then

$$\tilde{d}(f^{k+1}(x), f^{k+1}(y)) \leq K\tilde{d}(f^k(x), f^k(y)) < \delta$$

so

$$d(f^{k+1}(x), f^{k+1}(y)) \geq \frac{1}{C}\tilde{d}(f^{k+1}(x), f^{k+1}(y)) > \delta',$$

which means that the set $f^{-n}(S_n)$ is (n, δ') -separated w.r.t. d . So if we denote by $N(n, \delta', f)$ the maximal cardinality of a (n, δ') -separated set for f , we get that

$$N(n, \delta', f) \geq |S_n| \geq \frac{1}{B_{C\delta'}} \text{vol}(f^n(D)).$$

But this implies that $h(f) \geq \chi(D, f)$ and consequently

$$h(f) \geq \chi(\mathcal{T}(E), f). \quad \square$$

The second proposition relates the volume growth of $\mathcal{T}(E)$ under f with the spectral radii of $f_{*,l}$ for $l \leq \dim F$ in the case when F is uniformly expanding:

Proposition 3. *Suppose that $TM = E \oplus F$ is a dominated splitting for f and F is uniformly expanding under Df . Then for any $l < \dim F$ we have:*

$$\log(sp(f_{*,l})) < \chi(\mathcal{T}(E), f),$$

and for $\dim F$ we have:

$$\log(sp(f_{*,\dim F})) \leq \chi(\mathcal{T}(E), f).$$

Proof. Let $\dim F = u$. First we will prove that $\log(sp(f_{*,u})) \leq \chi(\mathcal{T}(E), f)$.

Let $\sigma = \sum_{i=1}^p a_i \sigma_i$, $a_i \in \mathbb{R}$, be a u -dimensional cycle corresponding to an eigenvalue of $f_{*,u}$ with maximal absolute value. Let ω be a dual differential form, so

$$\limsup_{n \rightarrow \infty} |f_*^n \sigma(\omega)|^{\frac{1}{n}} = sp(f_{*,u}).$$

This is true if the eigenvalue is both real or complex. We can also assume that σ is transverse to E , meaning that each disk (simplex) σ_i is transverse to E . Now

$$\begin{aligned} \log(sp(f_{*,u})) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log |f_*^n \sigma(\omega)| \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left| \sum_{i=1}^p a_i \int_{f^n(\sigma_i)} \omega \right| \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i=1}^p \text{vol}(f^n(\sigma_i)) \right) \\ &= \max_{1 \leq i \leq p} \chi(\sigma_i, f) \\ &\leq \chi(\mathcal{T}(E), f). \end{aligned}$$

Here we used the fact that $|\int_D \omega| \leq C \text{vol}(D)$ and the constants disappear in the limit after taking the log and dividing by n . We should remark here that for this inequality which we obtained in the case $l = \dim F$ we didn't use neither the dominated splitting nor the uniform expansion of F .

Now assume that $l < u$ and we will prove that $\log(sp(f_{*,l})) < \chi(\mathcal{T}(E), f)$.

Let $\sigma = \sum_{i=1}^p a_i \sigma_i$, $a_i \in \mathbb{R}$, be again a l -dimensional cycle corresponding to an eigenvalue of $f_{*,l}$ with maximal absolute value, and η be a dual differential form, so

$$\limsup_{n \rightarrow \infty} |f_*^n \sigma(\eta)|^{\frac{1}{n}} = sp(f_{*,l}).$$

Again we can assume that $\sigma_i \pitchfork E$.

Let $K = \cup_{i=1}^p \sigma_i$ be the geometric complex corresponding to σ , with the Riemannian metric as submanifolds of M on each σ_i and the corresponding measure m_i . Let $D = [0, 1]^{u-l}$ be the unit cube in \mathbb{R}^{u-l} with the

Lebesgue measure m_D . Following [SW], one can construct a continuous map $H: K \times D \rightarrow M$ such that:

- (1) $H(\cdot, 0) = id_K$;
- (2) $H|_{\sigma_i \times D}$ is a diffeomorphism from $\sigma_i \times D$ to $D_i := H(\sigma_i \times D) \subset M$;
- (3) D_i is transverse to E , or $D_i \in \mathcal{T}(E)$.

For each $y \in D$ consider the cycle in M

$$\sigma_y = \sum_{i=1}^p a_i H(\sigma_i \times \{y\}).$$

Because for every $y \in D$ the cycles σ_y and σ are homotopic, they will have the same homology, so we have:

$$\sigma_y(f^{*n}\eta) = \sigma(f^{*n}\eta).$$

Then

$$\begin{aligned} \log(sp(f_{*,l})) &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log |f_*^n \sigma(\eta)| = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |\sigma(f^{*n}\eta)| \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \int_D |\sigma_y(f^{*n}\eta)| dm_D \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \int_D \left| \sum_{i=1}^p a_i \int_{H(\sigma_i \times \{y\})} f^{*n}\eta \right| dm_D \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \int_D \left| \sum_{i=1}^p a_i \int_{\sigma_i \times \{y\}} H^* f^{*n}\eta \right| dm_D \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p |a_i| \int_D \left| \int_{\sigma_i \times \{y\}} H^* f^{*n}\eta \right| dm_D \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p |a_i| \int_D \int_{\sigma_i \times \{y\}} \|H^* f^{*n}\eta|_{T(\sigma_i \times D)}\| dm_i dm_D \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p |a_i| \int_{\sigma_i \times D} \|H^* f^{*n}\eta|_{T(\sigma_i \times D)}\| d(m_i \times m_D). \end{aligned}$$

But now we know that H is a diffeomorphism from $\sigma_i \times D$ to D_i , so the Jacobian is uniformly bounded away from zero and infinity, and H^* also affects the norm of differential forms in an uniformly bounded way. Denote by m_{D_i} the Riemannian measure on D_i . Because again the constants will

disappear in the limit we get:

$$\log(sp(f_{*,l})) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p \int_{D_i} \|f^{*n} \eta|_{TD_i}\| dm_{D_i}.$$

Because F is uniformly expanding there exist $\lambda > 1$ and $C > 0$ such that:

$$\|Df^n(v)\| \geq C\lambda^n \|v\|, \quad \forall v \in F.$$

Because $TM = E \oplus F$ is a dominated splitting then the same is true for all the vectors inside some small invariant cone field around F . By taking iterates if necessary, we may also assume that the disks D_i are tangent to this cone field, so the same relation holds for vectors in TD_i . But this in turn implies that the ratio between the u -dimensional volume expansion on TD_i , or the Jacobian of f restricted to $D_i - |Df|_{TD_i}|$, and the maximal l -dimensional volume expansion on TD_i under n iterates of f is greater than $C^{u-l} \lambda^{n(u-l)}$, and consequently

$$\|f^{*n} \eta|_{TD_i}\| \leq \frac{C'}{\lambda^{n(u-l)}} |Df|_{TD_i}|.$$

So going back to the logarithm of the spectral radius, we get:

$$\begin{aligned} \log(sp(f_{*,l})) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p \int_{D_i} \frac{C'}{\lambda^{n(u-l)}} |Df|_{TD_i}| dm_{D_i} \\ &= -(u-l) \log \lambda + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p \int_{D_i} |Df|_{TD_i}| dm_{D_i} \\ &= -(u-l) \log \lambda + \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^p \text{vol}(f^n(D_i)) \\ &= -(u-l) \log \lambda + \max_{1 \leq i \leq p} \chi(D_i, f) < \chi(\mathcal{T}(E), f). \quad \square \end{aligned}$$

Now we can give the proof of the theorem.

Proof of Theorem 1. First we make the observation that it is enough to prove the result for finite covers of M , so by taking a double cover if necessary, we can assume that M is orientable (see [SW]).

Denote $m := \dim(M)$, $u := \dim(E^u)$ and $s := \dim(E^s)$. Because the center bundle is one-dimensional we have

$$m = u + s + 1.$$

Then $TM = E^{cs} \oplus E^u$, where $E^{cs} = E^s \oplus E^c$, is a dominated splitting for f , so by Proposition 2 we have

$$\chi(\mathcal{T}(E^{cs}), f) \leq h(f).$$

E^u is also uniformly expanding, so by Proposition 3 we have

$$\log(sp(f_{*,l})) \leq \chi(\mathcal{T}(E^{cs}), f), \quad \forall 0 \leq l \leq u.$$

Putting these two inequalities together we get

$$(1) \quad \log(sp(f_{*,l})) \leq h(f), \quad \forall 0 \leq l \leq u.$$

But $TM = E^{cu} \oplus E^s$, where $E^{cu} := E^c \oplus E^u$, is also a dominated splitting for f^{-1} , so applying again Proposition 2 we have

$$\chi(\mathcal{T}(E^{cu}), f^{-1}) \leq h(f^{-1}) = h(f).$$

Again E^s is uniformly expanding for f^{-1} , so by Proposition 3 we have

$$\log(sp(f_{*,s}^{-1})) \leq \chi(\mathcal{T}(E^{cu}), f^{-1}), \quad \forall 0 \leq k \leq s.$$

Again, combining the two previous inequalities we get

$$(2) \quad \log(sp(f_{*,k}^{-1})) \leq h(f), \quad \forall 0 \leq k \leq s.$$

But now we assumed that M is orientable, and by duality we get

$$sp(f_{*,m-k}) = sp(f_{*,k}^{-1}),$$

which together with relation (2) implies that

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall u+1 \leq l \leq m.$$

Combining this with relation (1) we get

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall 0 \leq l \leq m,$$

or

$$\log(sp(f_*)) \leq h(f). \quad \square$$

We remark that we didn't use any conditions about the integrability of the center, center-stable or center unstable distributions. Also we obtained actually strict inequalities for dimensions different from u and $u+1$, i. e.

$$\log(sp(f_{*,l})) < h(f), \quad \forall 0 \leq l \leq m, \quad l \neq u, u+1.$$

This proofs can be applied to any partially hyperbolic diffeomorphism to give that

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall l \in \{0, 1, \dots, u-1, u, m-s, m-s+1, \dots, m-1, m\}.$$

If the dimension of the center distribution is c then we get the desired inequalities for all the dimensions with the exception of $c-1$ of them: the dimensions $u+1, u+2, \dots, u+c-1 = m-s-1$.

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