HOROSPHERES IN HYPERBOLIC GEOMETRY

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Abstract. In this paper we investigate the role of horospheres in Integral Geometry and Differential Geometry. In particular we study envelopes of families of horocycles by means of "support maps". We define invariant "linear combinations" of support maps or curves. Finally we obtain Gauss-Bonnet type formulas and Chern-Lashof type inequalities.

1. Introduction

Some parts of Integral Geometry and Differential Geometry in euclidean spaces rely on the space of hyperplanes. For instance kinematic formulas, support functions, height functions in relation with the Gauss map and the total (absolute) curvature. Here the space of oriented hyperplanes is a cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ equipped with an isotropic metric invariant with respect to euclidean motions. In fact this isotropic metric is just the pullback of the metric on $\mathbb{S}^{n-1}$ under the canonical projection.

In Hyperbolic Geometry this situation looks quite different. The space of geodesic hyperplanes is topologically a cylinder but with a non-degenerated Lorentz-metric invariant with respect to hyperbolic motions (de Sitter sphere). In some sense horospheres are closer to euclidean hyperplanes. The space of horospheres is a half-cone $\mathbb{S}^{n-1} \times \mathbb{R}^+$ equipped with an invariant isotropic metric, which is a warped product of the metric on $\mathbb{S}^{n-1}$ with $\mathbb{R}^+$.

In this paper we investigate the role of horospheres in Integral Geometry and Differential Geometry. After some preliminaries we study in section 3 envelopes of families of horocycles by means of "support maps". In section 4 we define invariant "linear combinations" of support maps or curves. Finally in section 5 we obtain Gauss-Bonnet type formulas and Chern-Lashof type inequalities.

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2. Preliminaries

We use the Lorentz space model for Hyperbolic Geometry. In detail, the model lives in Lorentz space $\mathbb{R}^{n+1}_1$ with its Lorentz product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}\,.$$

The n-dimensional hyperbolic space $\mathbb{H}^n$ is realized as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = -1 \ \wedge \ x_{n+1} > 0\}$$

i.e. the upper half of an two-sheeted hyperboloid with the light cone $\mathcal{C}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0\}$ as asymptotic cone. The group G of hyperbolic motions of $\mathbb{H}^n$ is given by the subgroup of the Lorentz group leaving invariant $\mathbb{H}^n$.

The space $\mathcal{H}$ of horospheres of $\mathbb{H}^n$ is realized as the upper half of the light cone, i.e.

$$\mathcal{H} = \mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0 \ \wedge \ x_{n+1} > 0\}\,.$$

Indeed, horospheres in $\mathbb{H}^n$ are exactly the non-void sections of $\mathbb{H}^n$ with hyperplanes which are parallel to hyperplanes tangent to the light cone $\mathcal{C}^n$. Given $\theta \in \mathcal{C}^n_+$, then the affine hyperplane $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$ is parallel to the tangent hyperplane $T_\theta\mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = 0\}$ of $\mathcal{C}^n_+$ at θ and intersects $\mathbb{H}^n$ in a horosphere which we also denote by Θ. Given a horosphere Θ in $\mathbb{H}^n$, it is the intersection of $\mathbb{H}^n$ with an affine hyperplane Θ parallel to a tangent hyperplane of $\mathcal{C}^n_+$ along along a half light-ray. Then there exists exactly one θ in this half light-ray such that $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$. (In the following we shall always denote horospheres in $\mathbb{H}^n$, or the underlying affine hyperplane, by capital greek letters and the vectors representing them in $\mathcal{C}^n_+$ by the corresponding small greek letters.) The correspondence between θ and the hyperplane Θ comes exactly from the polarity relation with respect to the quadric $\pm\mathbb{H}^n \subset \mathbb{R}^{n+1}_1$. The Lorentz product induces on $\mathcal{C}^n$ a degenerated product (isotropic metric).

The light-rays in the cone $\mathcal{C}^n_+$ are exactely the pencils of "parallel" horospheres. Two parallel horospheres Θ_1 and Θ_2 touch one another at a point at infinity, and they lie in constant hyperbolic distance to each other. A little computation in the model shows that this distance is given by $|\ln\lambda|$, where $\lambda \in \mathbb{R}^+$ is given by $\theta_2 = \lambda\theta_1$. Here we use the signed distance from Θ_1 to Θ_2 by

$$\mathrm{d}(\Theta_1, \Theta_2) = -\ln\lambda \tag{1}$$

For fixed Θ_1, as $\lambda \to +\infty$ the horosphere Θ_2 shrinks to the common point at infinity whereas the signed distance $\mathrm{d}(\Theta_1, \Theta_2) \to -\infty$. On the other side, if $\lambda \to 0$, then Θ_2 expands over the whole $\mathbb{H}^n$ and $\mathrm{d}(\Theta_1, \Theta_2) \to +\infty$.

The space of horospheres $\mathcal{H} = \mathcal{C}^n_+ \subset \mathbb{R}^{n+1}_1$ is endowed with a n-form ω which is invariant under the Lorentz group. In terms of the coordinates $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_1$ this form is given by

$$\omega = \frac{1}{x_{n+1}} dx_1 \wedge \ldots \wedge dx_{n+1} = x_{n+1}^{n-2} dx_{n+1} dv \tag{2}$$

where dv is the spherical volume element at $x_{n+1}^{-1}(x_1, \dots, x_n) \in \mathbb{S}^{n-1}$, cf. [San67], [San68].

Our bridge between the point space $\mathbb{H}^n$ and the space of horospheres $\mathcal{C}^n_+$ is the following.

Definition 2.1. Let M be a smooth regular hypersurface in $\mathbb{H}^n$ and $\nu(x)$, $x \in M$, a unit normal vector field along M. Then $\theta(x) = x + \nu(x) \in \mathcal{C}^n_+$ represents the horosphere $\Theta(x)$ which is tangent to M at x such that $\nu(x)$ points into its convex side. We call

$$\theta \colon M \longrightarrow \mathcal{C}^n_+ \quad , \quad x \mapsto x + \nu(x) \tag{3}$$

the *"support map of M with respect to ν"*.

The support map θ of M is smooth, and in general tranverse to the generators of $\mathcal{C}^n_+$.

3. Envelopes of horocycles

3.1. **Support maps: from c to θ.** Let us start with a regular parametrized curve $c(s)$ in $\mathbb{H}^2$, $s \in I$, s an arc length parameter. In order to describe the differential geometry of the curve, we use the Frenet theory. That means we have the positive oriented Frenet frame along c, build by the unit tangent vector $e_1(s) = c'(s)$ and the normal unit vector $e_2(s)$. The Frenet equations $\bigtriangledown_{e_1(s)} e_1(s) = \kappa_g(s) e_2(s)$, $\bigtriangledown_{e_1(s)} e_2(s) = -\kappa_g(s) e_1(s)$ then define the geodesic curvature κ_g of c ($\bigtriangledown$ denotes the co-variant derivative in $\mathbb{H}^2$).

In order to describe the support map, let $\nu(s)$ be a unit normal vector field along c. We consider the support map θ of c with respect to ν, i.e.

$$\theta \colon I \to \mathcal{C}^2_+ \quad \text{with} \quad \theta(s) = c(s) + \nu(s).$$

The horocycle $\Theta(s)$ is tangent to c at $c(s)$ and $\nu(s)$ points into its convex side.

Then (the primes denote derivations with respect to s)

$$\theta' = c' + \nu' = c' + \epsilon e_2' = (1 - \epsilon \kappa_g) e_1$$

with $\epsilon := \langle \nu, e_2 \rangle = \pm 1$. (Note: $\langle e_2, e_2 \rangle = 1 \Rightarrow \langle e_2, e_2' \rangle = 0$, $\langle c, e_2 \rangle = 0 \Rightarrow 0 = \langle c', e_2 \rangle + \langle c, e_2' \rangle = \langle c, e_2' \rangle$, hence $e_2' \in T_c \mathbb{H}^2$.) This shows that the curve

$\theta(s)$ is regular parametrized iff $\kappa_g \neq \epsilon 1$. The curve $\theta(s)$ is then space-like, and its arc length parameter σ is given by

$$d\sigma = |1 - \epsilon\kappa_g| ds. \tag{4}$$

3.2. **Envelopes: from θ to c.** Let us start with a regular parametrized curve $\theta(\sigma)$ in $\mathcal{C}^2_+$ which is locally a graph with respect to the generators of $\mathcal{C}^2_+$, $\sigma \in I$, σ an arc length parameter, i.e $\langle \dot\theta, \dot\theta\rangle = 1$. We look for the envelope curve $c(\sigma)$ of the family $\Theta(\sigma)$ of horocycles in $\mathbb{H}^2$, i.e.

$$\begin{aligned} &\langle c, c\rangle = -1,\\ &\langle c, \theta\rangle = -1,\\ &\langle \dot c, \theta\rangle = 0 \quad \text{(envelope condition)}. \end{aligned} \tag{5}$$

For the curve θ we have

$\langle \theta, \theta\rangle = 0 \Rightarrow \langle \dot\theta, \theta\rangle = 0 \ \Rightarrow 0 = \langle \ddot\theta, \theta\rangle + \langle \dot\theta, \dot\theta\rangle = \langle \ddot\theta, \theta\rangle + 1$

(the points denote derivations with respect to σ),

$\langle \dot\theta, \dot\theta\rangle = 1 \ \Rightarrow \langle \dot\theta, \ddot\theta\rangle = 0$, and

$\langle \dot\theta, \ddot\theta\rangle = 0 \Rightarrow \langle \ddot\theta, \ddot\theta\rangle + \langle \dot\theta, \dddot\theta\rangle = 0.$

From (5) we get

$\langle c, c\rangle = -1 \Rightarrow \langle \dot c, c\rangle = 0$, and

$\langle c, \theta\rangle = -1 \Rightarrow 0 = \langle \dot c, \theta\rangle + \langle c, \dot\theta\rangle = \langle c, \dot\theta\rangle.$

Now, we assume that $\theta, \dot\theta, \ddot\theta$ are linear independent, and we try $c = \alpha\theta + \beta\dot\theta + \gamma\ddot\theta$ with unknown functions α, β, γ . We take into account the above relations, i.e.

$0 = \langle c, \dot\theta\rangle = \alpha\langle \theta, \dot\theta\rangle + \beta\langle \dot\theta, \dot\theta\rangle + \gamma\langle \ddot\theta, \dot\theta\rangle = \beta$, and

$-1 = \langle c, \theta\rangle = \alpha\langle \theta, \theta\rangle + \beta\langle \dot\theta, \theta\rangle + \gamma\langle \ddot\theta, \theta\rangle = -\gamma$, and

$-1 = \langle c, c\rangle = \alpha^2\langle \theta, \theta\rangle + \beta^2\langle \dot\theta, \dot\theta\rangle + \gamma^2\langle \ddot\theta, \ddot\theta\rangle + 2\alpha\beta\langle \theta, \dot\theta\rangle + 2\alpha\gamma\langle \theta, \ddot\theta\rangle + 2\beta\gamma\langle \dot\theta, \ddot\theta\rangle = \gamma^2\langle \ddot\theta, \ddot\theta\rangle - 2\alpha\gamma.$

And we get

$$c = \frac{1}{2}\Big(1 + \langle \ddot\theta, \ddot\theta\rangle\Big)\,\theta + \ddot\theta\,. \tag{6}$$

Now, with the expression (6) for c we directly check

$\langle c, c\rangle = -1$, i.e. $c \subset \mathbb{H}^n$, $\langle c, \theta\rangle = -1$, i.e. $c \in \Theta$, and $\langle \dot c, \theta\rangle = 0$.

And therefore, c is the envelope of Θ we looked for.

From (6) we get by differentiation

$$\dot{c} = \frac{1 - \langle \ddot{\theta}, \ddot{\theta} \rangle}{2} \, \dot{\theta} \,. \tag{7}$$

(To this, we try $\dddot{\theta}$ as a linear combination of the vectors $\theta, \dot{\theta}, \ddot{\theta}$. We take into account $\langle \ddot{\theta}, \theta \rangle = -1 \Rightarrow \langle \ddot{\theta}, \dot{\theta} \rangle + \langle \dddot{\theta}, \theta \rangle = 0$ and $\langle \ddot{\theta}, \dot{\theta} \rangle = 0 \Rightarrow \langle \ddot{\theta}, \ddot{\theta} \rangle + \langle \dddot{\theta}, \dot{\theta} \rangle = 0$, to get $\dddot{\theta} = -\langle \dddot{\theta}, \ddot{\theta} \rangle \theta - \langle \ddot{\theta}, \ddot{\theta} \rangle \dot{\theta}$.)

Formula (7) shows that the envelope c is regular iff $\langle \ddot{\theta}, \ddot{\theta} \rangle \neq 1$.

Remark 3.1. The condition $\langle \ddot{\theta}, \ddot{\theta} \rangle \neq 1$ means, that the osculating plane of the curve θ in $\mathbb{R}^3_1$ is not tangent to the model $\mathbb{H}^2$. This property characterizes curves θ in $\mathcal{C}^2_+$ which envelope regular curves in $\mathbb{H}^2$.

Remark 3.2. The osculating plane of θ at a fixed parameter defines a family of horocycles with the following geometric meaning: If the osculating plane is space-like, then the envelope curve c of θ has an osculating circle at the point under consideration. We have $|\kappa_g| > 1$ at this point. And the family of horocycles envelopes this osculating circle on their concave sides or convex sides respectively, if the plane of the family intersects $\mathbb{H}^2$ or avoids $\mathbb{H}^2$ respectively.

If the osculating plane is of mixed type, then the envelope curve c of θ has an osculating equidistant at the point under consideration. We have $|\kappa_g| < 1$ at this point. And the family of horocycles envelopes this equidistant.

Finally we compute the geodesic curvature κ_g of the curve c in terms of θ:

From (6) we have $c = \alpha\theta + \gamma\ddot{\theta}$, and further

$$c' = \frac{dc}{ds} = \frac{d\sigma}{ds}\dot{c} = \frac{d\sigma}{ds}\left(\dot{\alpha}\theta + \alpha\dot{\theta} + \dot{\gamma}\ddot{\theta} + \gamma\dddot{\theta}\right).$$

Since $c', \dot{\theta}$ are linearly dependent, and $|\dot{\theta}| = 1$ we can compute

$$\begin{aligned} 1 = |c'| = |\langle c', \dot{\theta} \rangle| &= |1 - \epsilon\kappa_g| \left| \left\langle \dot{\alpha}\theta + \alpha\dot{\theta} + \dot{\gamma}\ddot{\theta} + \gamma\dddot{\theta}, \dot{\theta} \right\rangle \right| = \\ &= |1 - \epsilon\kappa_g| \left| \left(\dot{\alpha}\langle \theta, \dot{\theta} \rangle + \alpha\langle \dot{\theta}, \dot{\theta} \rangle + \dot{\gamma}\langle \ddot{\theta}, \dot{\theta} \rangle + \gamma\langle \dddot{\theta}, \dot{\theta} \rangle \right) \right| = \\ &= |1 - \epsilon\kappa_g| \left| \left(\alpha - \gamma\langle \ddot{\theta}, \ddot{\theta} \rangle \right) \right|. \end{aligned}$$

Inserting the coefficients α, β from (6) we get

$$1 = \frac{|1 - \epsilon\kappa_g|}{2} \left| \left(1 - \langle \ddot{\theta}, \ddot{\theta} \rangle \right) \right|. \tag{8}$$

3.3. **Further relations between c and θ.** We want to write the length $L(c)$ and the total curvature $TC(c)$ of the point curve c in terms of the support curve θ.

Proposition 3.1. *Let $c(s), s \in I$, be a regular curve in $\mathbb{H}^2$ parametrized by arc length s and $\nu(s)$ a unit normal vector field along c. Let $\theta : I \to \mathcal{C}^2_+$, $\theta(s) = c(s) + \nu(s)$ denote the support map of c with respect to ν, and set $\epsilon = \langle \nu, e_2 \rangle$.*
If $\epsilon = +1$ and $\kappa_g > 1$, or $\epsilon = -1$ and $\kappa_g > -1$ respectively, then

$$L(c) = \frac{1}{2} \int_\theta \epsilon \left(\kappa_\theta^2 - 1 \right) d\sigma, \tag{9}$$

where κ_θ is the curvature of the curve θ as a curve in $\mathbb{R}^3_1$, and

$$TC(c) = \int_c \kappa_g \, ds = \frac{1}{2} \int_\theta \left(\kappa_\theta^2 + 1 \right) d\sigma. \tag{10}$$

Proof. The case $\epsilon = +1$ and $\kappa_g > 1$: From (4) and (8) we get

$$ds = \frac{1}{\kappa_g - 1} \, d\sigma = \frac{|1 - \langle \ddot{\theta}, \ddot{\theta} \rangle|}{2} \, d\sigma.$$

Locally c lies in the convex side of Θ, hence we have $\langle c, \ddot{\theta} \rangle > 0$. (This can be seen in the model: Take the intersection of $\mathcal{C}^2_+$ and the plane through θ in direction $\mathrm{span}(\dot{\theta}, \ddot{\theta})$ which represents the horocycles tangent to the osculating circle of c, and take into account that c locally lies in the convex side of Θ.) Hence through (6) we have $1 - \langle \ddot{\theta}, \ddot{\theta} \rangle < 0$. Because σ is an arc length parameter on θ we have $\langle \ddot{\theta}, \ddot{\theta} \rangle = \kappa_\theta^2$. Altogether we get (9).
From (4) and (8), taking into account $1 - \langle \ddot{\theta}, \ddot{\theta} \rangle < 0$, we get

$$\kappa_g \, ds = \frac{\langle \ddot{\theta}, \ddot{\theta} \rangle + 1}{2} \, d\sigma,$$

hence we get (10).
In case $\epsilon = -1$ and $\kappa_g > -1$ the proof runs analogously. □

4. Linear combinations of support maps

In euclidean spaces the Minkowski addition, and also linear combinations, of two convex bodies are well known, and plays a fundamental role in Convexity, for instance leading to mixed volumes. This construction is in some sense invariant with respect to translations. Previous attempts to define a Minkowski addition in hyperbolic geometry were based on the choice of an origin (cf. [Lei03]). Here we use the linear structure of the rays of $\mathcal{C}^n_+$ to define an analogue of Minkowski addition in $\mathbb{H}^n$ which only depends on the convex bodies, and not of further choices. In the following we concentrate on the 2-dimensional situation.

4.1. **The "λ-multiple".**

Definition 4.1. Let $c(s), s \in I$, be a regular curve in $\mathbb{H}^2$ parametrized by arc length s and $\nu(s)$ a unit normal vector field along c. Let $\theta : I \to \mathcal{C}^2_+$, $\theta(s) = c(s) + \nu(s)$ denote the support map of c with respect to ν.
For $\lambda \in \mathbb{R}_+$, we call the envelope of $\lambda\theta$ in $\mathbb{H}^n$, the *"λ-multiple λc of c"*.

We consider the case $\epsilon = +1$ and $\kappa_g > 1$. Then locally c lies in the convex side of each of its tangent horocycles which are supporting c, and we have $\langle \ddot{\theta}, \ddot{\theta} \rangle > 1$.
We consider $\theta^* = \lambda\theta$ with $\lambda > 0$. Using (1) we set $t = d(\Theta, \Theta^*) = -\ln \lambda$.

a) The case $\lambda > 1$: The envelope c^* of θ^* is the inner parallel curve to c at distance t.
We compute

$$\left\langle \frac{d^2\theta^*}{(d\sigma^*)^2}, \frac{d^2\theta^*}{(d\sigma^*)^2} \right\rangle = \frac{1}{\lambda^2} \langle \ddot{\theta}, \ddot{\theta} \rangle \ .$$

Taking into account (7) we get: If $\lambda^2 < \langle \ddot{\theta}, \ddot{\theta} \rangle = \kappa_\theta^2$, then the envelope c^* is regular. Singular points occur for $\lambda^2 = \langle \ddot{\theta}, \ddot{\theta} \rangle$.
In our case we have $\langle \ddot{\theta}, \ddot{\theta} \rangle > 1$. Therefore (8) implies

$$\kappa_g = \frac{\langle \ddot{\theta}, \ddot{\theta} \rangle + 1}{\langle \ddot{\theta}, \ddot{\theta} \rangle - 1} \,. \tag{11}$$

The geodesic curvature κ_g and the curvature radius ρ are related by

$$\kappa_g = \coth \rho = \frac{e^{2\rho} + 1}{e^{2\rho} - 1} \ , \tag{12}$$

hence

$$e^{2\rho} = \langle \ddot{\theta}, \ddot{\theta} \rangle \,. \tag{13}$$

This shows that singular points occur for $t = -\rho$, i.e. singular points occur when the inner parallel curve of c runs through focal points of c.

b) The case $\lambda < 1$: The envelope $c^* = c_t$ of θ^* is the outer parallel curve to c at distance t.
Because of $\lambda < 1$, formula (7) shows that c_t is regular for all $t > 0$.
We now compute the length of c_t: Using (9), $d\sigma^* = \lambda\, d\sigma$, we get

$$L(c_t) = L(c^*) = \frac{1}{2} \int_{\theta^*} \left\langle \frac{d^2\theta^*}{(d\sigma^*)^2}, \frac{d^2\theta^*}{(d\sigma^*)^2} \right\rangle d\sigma^* - \frac{1}{2} \int_{\theta^*} d\sigma^* =$$

$$= \frac{1}{2\lambda}\int_\theta \langle \ddot\theta, \ddot\theta\rangle \, d\sigma - \frac{1}{2}\,\lambda \int_\theta d\sigma =$$
$$= \frac{1}{2}\left(\frac{1}{\lambda} - \lambda\right)\int_c \kappa_g \, ds + \frac{1}{2}\left(\frac{1}{\lambda} + \lambda\right) L(c) \; .$$

And replacing $\lambda = e^{-t}$, we arrive at

$$L(c_t) = \sinh(t)\int_c \kappa_g \, ds + \cosh(t) L(c) \; . \tag{14}$$

This is a well-known Steiner formula in hyperbolic plane, cf. e.g. [San76].

4.2. **The "sum".**

Definition 4.2. Let c_1, c_2 be two regular curves in $\mathbb{H}^2$ and θ_1, θ_2 their support maps with respect to unit normal fields ν_1, ν_2 along c_1, c_2. Then we call the envelope of $\theta_1 + \theta_2$ in $\mathbb{H}^n$, in case it is well defined, the *"sum $c_1 + c_2$ of c_1 and c_2"*.

Suppose $\theta_2(\sigma_1) = \lambda(\sigma_1)\theta_1(\sigma_1)$, parametrized by the arc length parameter σ_1 on θ_1. Then we have

$$\frac{d\theta_2}{d\sigma_1} = \frac{d\lambda}{d\sigma_1}\theta_1 + \lambda\frac{d\theta_1}{d\sigma_1} \, ,$$
$$\left\langle \frac{d\theta_2}{d\sigma_1}, \frac{d\theta_2}{d\sigma_1}\right\rangle = \lambda^2 \left\langle \frac{d\theta_1}{d\sigma_1}, \frac{d\theta_1}{d\sigma_1}\right\rangle = \lambda^2,$$

hence

$$d\sigma_2 = \lambda \, d\sigma_1 \; . \tag{15}$$

We consider, in case it is well defined, $\theta^* = \theta_1 + \theta_2 = (1+\lambda)\,\theta_1$.

Then we have

$$\frac{d\theta^*}{d\sigma_1} = \frac{d\lambda}{d\sigma_1}\theta_1 + (1+\lambda)\frac{d\theta_1}{d\sigma_1}$$

and

$$\left\langle \frac{d\theta^*}{d\sigma_1}, \frac{d\theta^*}{d\sigma_1}\right\rangle = (1+\lambda)^2 \langle \frac{d\theta_1}{d\sigma_1}, \frac{d\theta_1}{d\sigma_1}\rangle = (1+\lambda)^2.$$

Therefore we get

$$d\sigma^* = (1+\lambda)\, d\sigma_1 = \frac{1+\lambda}{\lambda}\, d\sigma_2 = d\sigma_1 + d\sigma_2 \, , \tag{16}$$

and we arrive at

Proposition 4.1. *For the lengths of the support images involved in the "sum" the following relation holds*

$$L(\theta^*) = L(\theta_1 + \theta_2) = L(\theta_1) + L(\theta_2). \tag{17}$$

In order to compute the length L^* and the total curvature TC^* of the sum in terms of the summands, we need the following

Definition 4.3. Let c_1, c_2 be two regular curves in $\mathbb{H}^2$ and θ_1, θ_2 their support maps with respect to unit normal fields ν_1, ν_2 along c_1, c_2. Then $\theta_2 = \lambda\theta_1$, and the signed distance $\mathrm{d}(\Theta_1, \Theta_2)$ from Θ_1 to Θ_2 is given by $\mathrm{d}(\Theta_1, \Theta_2) = -\ln\lambda$, cf. (1). We call

$$w_{12} : \theta_1 \to \mathbb{R} \quad , \quad \sigma_1 \mapsto -\ln\lambda(\sigma_1) \tag{18}$$

the *"mixed width function of c_1 and c_2 with respect to c_1"*.

The mixed width function describes the relative position of c_1 and c_2 to one another in terms of the distance between parallel tangent horocycles.

Remark 4.1. If θ_1 is the support map of a point $O \in \mathbb{H}^2$, then w_{12} coincides with the horocycle support function of c_2 based at the point O, cf. [Fil70], [San67], [San68].

Remark 4.2. If $c_1 = c_2 = c$ with opposite normal fields and c is h-convex, then the mixed width function w_{12} coincides with the width function with respect to horocycles considered in [GRST08].

4.2.1. *The sum of "concave-sided" support maps.* We consider the following situation: Let c_1, c_2 be two regular curves in $\mathbb{H}^2$ with geodesic curvatures $(\kappa_g)_1, (\kappa_g)_2 > -1$. We take the support maps θ_1, θ_2 according to $\epsilon_1 = \epsilon_2 = -1$, that means locally the curves lie on the concave sides of their respective support horocycles.

Proposition 4.2. *Suppose the situation described above. Then, whenever well defined, the sum $\theta^* = \theta_1 + \theta_2$ envelopes a regular curve $c^* = c_1 + c_2$ in $\mathbb{H}^2$ with*

(i) $\kappa_g^* > -1$, *and*
(ii) c^* *lies locally on the concave sides of its respective support horocycles.*

Proof. The curves c_1, c_2 lie locally on the concave sides of their respective support horocycles, therefore the osculating planes of θ_1, θ_2 intersect $\mathbb{H}^2$ without being tangent (cf. Remark 3.1).

Now, we keep fixed an arbitrary parameter σ_1.

The osculating plane of θ_1 at σ_1 is given by

$$\theta_1(\sigma_1) + \mathrm{span}(\dot\theta_1(\sigma_1), \ddot\theta_1(\sigma_1)).$$

Let P_1 denote the parallel plane through $\theta^*(\sigma_1)$, i.e.

$$P_1 = \theta^*(\sigma_1) + \mathrm{span}(\dot\theta_1(\sigma_1), \ddot\theta_1(\sigma_1)).$$

The osculating plane of θ_1 at σ_1 intersects $\mathbb{H}^2$ without being tangent, and $\theta^* = \theta_1 + \theta_2$, therefore P_1 also intersects $\mathbb{H}^2$ without being tangent.

Now $\theta_2 = \lambda\theta_1$, hence

$$\dot{\theta}_2 = \dot{\lambda}\theta_1 + \lambda\dot{\theta}_1 \quad \text{and} \quad \ddot{\theta}_2 = \ddot{\lambda}\theta_1 + 2\dot{\lambda}\dot{\theta}_1 + \lambda\ddot{\theta}_1 \tag{19}$$

(where the dots denote derivatives with respect to σ_1). And the osculating plane of θ_2 at σ_1 is given by

$$\theta_2(\sigma_1) + \mathrm{span}(\dot{\theta}_2(\sigma_1), \ddot{\theta}_2(\sigma_1)).$$

Let P_2 denote the parallel plane through $\theta^*(\sigma_1)$, i.e.

$$P_2 = \theta^*(\sigma_1) + \mathrm{span}(\dot{\theta}_2(\sigma_1), \ddot{\theta}_2(\sigma_1)).$$

The osculating plane of θ_2 at σ_1 intersects $\mathbb{H}^2$ without being tangent, we have $\theta^* = \theta_1 + \theta_2$, therefore P_2 also intersects $\mathbb{H}^2$ without being tangent.

The osculating plane of θ^* at σ_1 is given by

$$P^* = \theta^*(\sigma_1) + \mathrm{span}(\dot{\theta}^*(\sigma_1), \ddot{\theta}^*(\sigma_1))$$

with

$$\dot{\theta}^* = \dot{\theta}_2 + \dot{\theta}_1 \quad \text{and} \quad \ddot{\theta}^* = \ddot{\theta}_2 + \ddot{\theta}_1. \tag{20}$$

Let T be the tangent plane of $\mathcal{C}^2_+$ along the generator $\mathbb{R}_+ \cdot \theta_1(\sigma_1)$, i.e.

$T = \theta^*(\sigma_1) + \mathrm{span}(\theta_1(\sigma_1), \dot{\theta}_1(\sigma_1))$. For $a \geq 0$ let T_a denote the plane parallel to T given by $T_a = T + a\,\ddot{\theta}_1(\sigma_1)$.

Then T_a intersects P_1 in the line

$$\ell_{1a} = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \mathbb{R} \cdot \dot{\theta}_1(\sigma_1).$$

And by (19), T_a intersects P_2 in the line

$$\ell_{2a} = \theta^*(\sigma_1) + \frac{a}{\lambda(\sigma_1)}\,\ddot{\theta}_2(\sigma_1) + \mathbb{R} \cdot \dot{\theta}_2(\sigma_1).$$

And by (20), T_a intersects P^* in the line

$$\ell^*_a = \theta^*(\sigma_1) + \frac{a}{1+\lambda(\sigma_1)}\left(\ddot{\theta}_2(\sigma_1) + \ddot{\theta}_1(\sigma_1)\right) + \mathbb{R} \cdot \left(\dot{\theta}_2(\sigma_1) + \dot{\theta}_1(\sigma_1)\right) .$$

Let g_a denote the line in T_a given by

$$g_a = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \mathbb{R} \cdot \theta_1(\sigma_1)\,.$$

Then g_a intersects ℓ_{1a} in the point

$$Q_{1a} = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1)\,.$$

And g_a intersects ℓ_{2a} in the point

$$Q_{2a} = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \frac{a(\lambda\ddot{\lambda} - 2\dot{\lambda}^2)}{\lambda^2}\Big|_{\sigma_1}\,\theta_1(\sigma_1)\,.$$

And g_a intersects ℓ_a^* in the point

$$Q_a^* = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \frac{a((1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2)}{(1+\lambda)^2}|_{\sigma_1}\,\theta_1(\sigma_1)\,.$$

The proof now splits into two cases.

- The first case, $((1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2)|_{\sigma_1} \geq 0$:

P_1 intersects $\mathbb{H}^2$ without being tangent. Therefore there exists an $a > 0$ such that ℓ_{1a} intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. The axis of the parabola is $\theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \mathbb{R}\cdot\theta_1(\sigma_1)$. Hence the half-ray $Q_{1a} + \mathbb{R}_+\cdot\theta_1(\sigma_1) \subset T_a$ lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. In the first case Q_a^* lies on this half-ray. Hence Q_a^* lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. Hence ℓ_a^* intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the osculating plane P^* of θ^* at σ_1 intersects $\mathbb{H}^2$ without being tangent.

- The second case, $((1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2)|_{\sigma_1} < 0$:

P_2 intersects $\mathbb{H}^2$ without being tangent. Therefore there exists an $a > 0$ such that ℓ_{2a} intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the half-ray $Q_{2a} + \mathbb{R}_+\cdot\theta_1(\sigma_1) \subset T_a$ lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. Through the assumption in the second case we have

$$\frac{\lambda\ddot{\lambda} - 2\dot{\lambda}^2}{\lambda^2}|_{\sigma_1} \leq \frac{(1+\lambda)\ddot{\lambda} - 2\dot{\lambda}^2}{(1+\lambda)^2}|_{\sigma_1}\,.$$

Hence Q_a^* lies on this half-ray. Hence ℓ_a^* intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the osculating plane P^* of θ^* at σ_1 intersects $\mathbb{H}^2$ without being tangent.

Altogether, this shows that the osculating planes of θ^* intersect $\mathbb{H}^2$ without being tangent. Therefore c^* is regular at σ_1, θ^* supports c^* concave-sided, and moreover $\kappa_g^* > -1$. □

Proposition 4.3. *Suppose the situation described above. Then the length L^* and the total curvature TC^* of $c^* = c_1 + c_2$ write in terms of c_1, c_2 and their relative position to each other in $\mathbb{H}^2$ as follows:*

$$L^* = -\frac{1}{2}\left(W(c_1, c_1 + c_2) - L_1 - TC_1 - L_2 - TC_2\right) \tag{21}$$

$$TC^* = \frac{1}{2}\left(W(c_1, c_1 + c_2) + L_1 + TC_1 + L_2 + TC_2\right), \tag{22}$$

with

$$W(c_1, c_1 + c_2) = TC^* - L^* = \int_{\theta_1} e^{w_{1*}}\left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + \kappa_{\theta_1}^2\right)\,d\sigma_1$$

and the mixed width function

$$w_{1*}(\sigma_1) = -\ln(1+\lambda(\sigma_1)).$$

Proof. By the assumptions on c_1, c_2 and by Proposition 4.2 we have for all three curves c_1, c_2, c^* that $\epsilon_1 = \epsilon_2 = \epsilon^* = -1$ and $(\kappa_g)_1, (\kappa_g)_2, \kappa_g^* > -1$. Therefore (6) writes $d\sigma = (\kappa_g + 1)\, ds$, hence

$$L(\theta) = \int_\theta d\sigma = \int_c (\kappa_g + 1)\, ds = \int_c \kappa_g\, ds + L(c)\,. \tag{23}$$

This and (17) gives

$$TC^* + L^* = TC_1 + L_1 + TC_2 + L_2. \tag{24}$$

From (4) and (8) we get

$$ds = \frac{1 - \langle \ddot\theta, \ddot\theta \rangle}{2}\, d\sigma,$$

$$L(c) = -\frac{1}{2}\int_\theta \kappa_\theta^2\, d\sigma + \frac{1}{2} L(\theta).$$

This applied to c^* yields

$$L(c^*) = -\frac{1}{2}\int_{\theta^*} \kappa_{\theta^*}^2\, d\sigma^* + \frac{1}{2} L(\theta^*). \tag{25}$$

Now a straightforward but lengthy computation, not acted out here, starts at $\theta^* = (1+\lambda)\,\theta_1$ and reaches

$$\begin{aligned} \langle \frac{d^2\theta^*}{d\sigma^{*2}}, \frac{d^2\theta^*}{d\sigma^{*2}} \rangle = \frac{1}{(1+\lambda)^2} \Bigg[\left(\frac{d}{d\sigma_1}\,(\ln(1+\lambda)) \right)^2 - \\ - 2\frac{d^2}{d\sigma_1^2}\,(\ln(1+\lambda)) + \langle \ddot\theta_1, \ddot\theta_1 \rangle \Bigg]. \end{aligned} \tag{26}$$

Using the mixed width function of c_1 and c_* with respect to c_1, i.e. $w_{1*} = -\ln(1+\lambda)$, formula (26) gives

$$\begin{aligned} \int_{\theta^*} \kappa_{\theta^*}^2\, d\sigma^* &= \int_{\theta^*} \langle \frac{d^2\theta^*}{d\sigma^{*2}}, \frac{d^2\theta^*}{d\sigma^{*2}} \rangle\, d\sigma^* = \\ &= \int_{\theta_1} e^{w_{1*}} \left((\dot w_{1*})^2 + 2\ddot w_{1*} + \kappa_{\theta_1}^2 \right)\, d\sigma_1\,. \end{aligned} \tag{27}$$

(Note: $d\sigma^* = (1+\lambda)\, d\sigma_1$ cf. (16).)
Hence (25), (27) and (23) yield

$$L^* = -\frac{1}{2}\int_{\theta_1} e^{w_{1*}} \left((\dot w_{1*})^2 + 2\ddot w_{1*} + \kappa_{\theta_1}^2 \right)\, d\sigma_1 + \frac{1}{2} TC^* + \frac{1}{2} L^*. \tag{28}$$

Finally (24) and (28) give the result. □

Remark 4.3. Corresponding formulas in the euclidean case are simpler because there one benefits from the invariance with respect to translations.

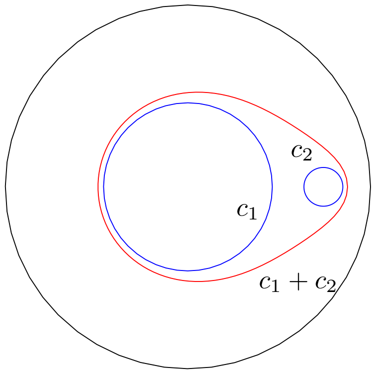

FIGURE 1. The "sum" $c_1 + c_2$ of two circles c_1, c_2 in the Poincaré disk, with radii $r_1 = 1, r_2 = 0.5$ and distance 2 between their centers

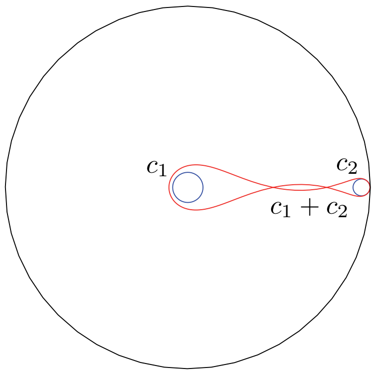

FIGURE 2. The "sum" $c_1 + c_2$ of two circles c_1, c_2 in the Poincaré disk, with radii $r_1 = 0.16, r_2 = 2$ and distance 5 between their centers

4.3. **The "rum".**

Definition 4.4. Let c_1, c_2 be two regular curves and θ_1, θ_2 their support maps with respect to unit normal fields ν_1, ν_2 along c_1, c_2. Then $\theta_2 = \lambda\theta_1$. Whenever well-defined, we call

$$c_1\#c_2 = c^* \quad \text{, given by} \quad \theta^* = \frac{\lambda}{1+\lambda}\,\theta_1 \tag{29}$$

the *"rum $c_1\#c_2$ of c_1 and c_2"*.

Geometrically, this definition is induced by the sum of the two parallel planes Θ_1 and Θ_2 in the vector space $\mathbb{R}^3_1$.

Lemma 4.1. *Let θ_1, θ_2 be support maps. Then $\theta^* = \theta_1\#\theta_2$ lies below θ_1 and θ_2 with respect to each of the generators of $\mathcal{C}^n_+$.*

Proof. This follows immediately from the geometric meaning of the definition of the rum. Alternatively:
We have $\theta_2 = \lambda\,\theta_1$ with $\lambda > 0$. Hence

$$\theta^* = \frac{\lambda}{1+\lambda}\theta_1 < \theta_1 \text{ , and}$$

$$\theta^* = \frac{\lambda}{1+\lambda}\theta_1 = \frac{1}{1+\lambda}\theta_2 < \theta_2\,.$$

□

Proposition 4.4. *Let c_1, c_2 be circles or points in $\mathbb{H}^2$. Then the rum $c_1\#c_2$ of c_1 and c_2 is a circle or a point.*

Proof. The support maps of circles or points are given by the intersection of $\mathcal{C}^2_+$ with space-like planes. Therefore θ_1, θ_2 are uniquely given by planes $\langle n_1, x\rangle = -1$ and $\langle n_2, x\rangle = -1$ with time-like vectors $n_1, n_2 \neq 0$ lying inside $\mathcal{C}^2_+ \subset \mathbb{R}^3_1$. Then $\theta_2 = \lambda\,\theta_1$ with $\lambda = -1/\langle n_2, \theta_1\rangle$. Putting $n^* = n_1 + n_2$, we compute

$$\langle n^*, \theta^*\rangle = \langle n_1 + n_2, \frac{\lambda}{1+\lambda}\theta_1\rangle =$$

$$= \frac{\lambda}{1+\lambda}\langle n_1, \theta_1\rangle + \frac{\lambda}{1+\lambda}\langle n_2, \theta_1\rangle = -1\,.$$

Therefore θ^* lies in the plane $\langle n^*, x\rangle = -1$, and hence it envelopes a circle or a point. □

Now we bring orientations into game. We assume a given orientation on hyperbolic plane $\mathbb{H}^2$. For an oriented curve c in $\mathbb{H}^2$ we now fix $\nu = e_2$, i.e. $\epsilon = +1$, and we have the support map $\theta = c + e_2$. Horocycles Θ are oriented such that the convex region is on its left-hand side (i.e. we choose the positive orientation, i.e the counter-clockwise direction).

A view on oriented circles in $\mathbb{H}^2$:
An oriented circle c is given by its center $m \in \mathbb{H}^2$ and its radius $r \in \mathbb{R}$, thereby that the hyperbolic radius is $|r|$ and the orientation is counter-clockwise for $r > 0$ and clockwise for $r < 0$. Especially for $r = 0$ we get points.
If c is oriented counter-clockwise, then its θ supports convex-sided. If c is oriented clockwise, then its θ supports concave-sided.
If the circle is given by its support map θ, then θ is the intersection of $\mathcal{C}^2_+$ with a space-like plane $\langle n, x\rangle = -1$, n time-like and inside the half-cone $\mathcal{C}^2_+ \subset \mathbb{R}^3_1$. Its center is $m = n/|n| \in \mathbb{H}^2$ and its radius is $r = \ln|n|$. Moreover: $|n| > 1$ iff θ supports c convex-sided. $|n| < 1$ iff θ supports c concave-sided. $|n| = 1$ iff c is a point.

Proposition 4.5. *Let c_1, c_2 be circles or points in $\mathbb{H}^2$ with centers m_1, m_2 and signed radii r_1, r_2. The rum $c^* = c_1\#c_2$ is a circle or a point which center m^* and signed radius r^* are given as follows:*

$$r^* = \frac{1}{2}\ln\left(e^{2r_1} + e^{2r_2} + 2e^{r_1+r_2}\cosh(\mathrm{d}(m_1, m_2))\right) \tag{30}$$

where $\mathrm{d}(m_1, m_2)$ is the hyperbolic distance between m_1 and m_2; and

$$m^* = \frac{1}{|n_1 + n_2|}(n_1 + n_2) \tag{31}$$

with $n_1 = e^{r_1} m_1$ and $n_2 = e^{r_2} m_2$. Moreover

$$\frac{\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}^*))}{\cosh(\mathrm{d}(\mathrm{m}_2, \mathrm{m}^*))} = \frac{e^{r_1} + e^{r_2}\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}_2))}{e^{r_1}\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}_2)) + \mathrm{e}^{\mathrm{r}_2}}. \tag{32}$$

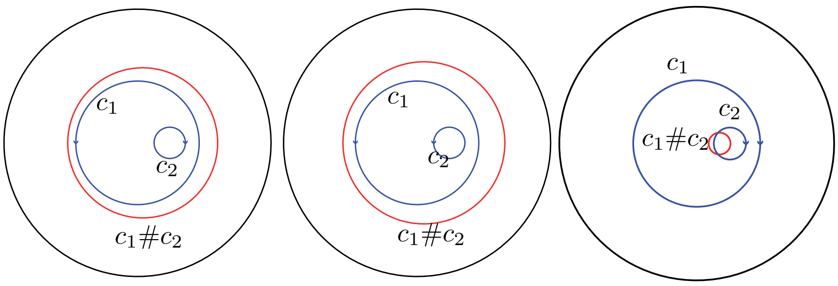

FIGURE 3. The "rum" $c_1\#c_2$ of two circles c_1, c_2 in the Poincaré disk, with signed radii $r_1 = 1, +1, -1, r_2 = -0.25, +0.25, -0.25$ and distance 0.5 between their centers

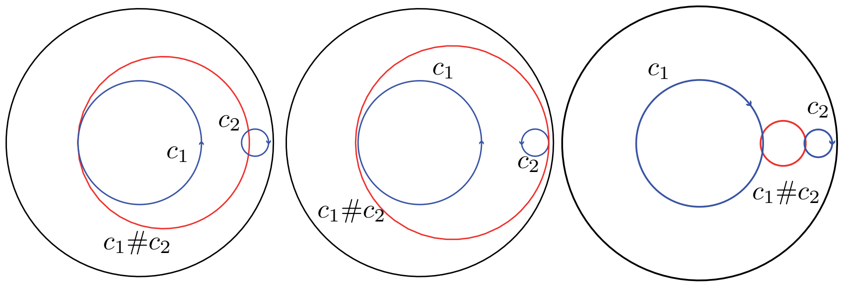

FIGURE 4. The "rum" $c_1\#c_2$ of two circles c_1, c_2 in the Poincaré disk, with signed radii $r_1 = +1, +1, -1, r_2 = -1, +1, -1$ and distance 3 between their centers

Proof. The support maps θ_1, θ_2 of c_1, c_2 are uniquely determined by their planes $\langle n_1, x\rangle = -1$, $\langle n_2, x\rangle = -1$ as described above. Then their centers and signed radii are given by $m_1 = n_1/|n_1|$, $m_2 = n_2/|n_1|$ and $r_1 = \ln|n_1|$, $r_2 = \ln|n_2|$ (cf. (13)). The support map θ^* is given by the plane $\langle n_1 + n_2, x\rangle = -1$. Then straightforward computations give the results. □

Proposition 4.6. *Let c_1, c_2 be counter-clockwise oriented circles or points in $\mathbb{H}^2$. Then the rum $c_1\#c_2$ of c_1 and c_2 is a circle containing both c_1 and c_2.*

Proof. c_1, c_2 are counter-clockwise oriented. Hence their support maps θ_1, θ_2 support convex-sided, and their planes do not intersect $\mathbb{H}^2$. Now θ^* lies below θ_1 and θ_2 with respect to each generator of $\mathcal{C}^2_+$ (cf. Lemma 4.1). Therefore the plane of θ^* does not intersect $\mathbb{H}^2$. Hence each θ^* supports $c_1\#c_2$ convex-sided and contains c_1 and c_2. □

Proposition 4.7. *Let c_1, c_2 be counter-clockwise oriented smooth regular boundaries of h-convex bodies K_1, K_2 in $\mathbb{H}^2$. Then the rum $c^* = c_1\#c_2$ of c_1 and c_2 is the counter-clockwise oriented smooth regular boundary of an h-convex body K^*, also called rum $K^* = K_1\#K_2$ of K_1 and K_2. Moreover $K_1, K_2 \subset K^*$.*

Proof. The curves c_1, c_2 are oriented counter-clockwise and h-convex, hence their θ_1, θ_2 support convex-sided.

The second order situation of c_1 and c_2 at related points determines the second order situation of c^* at the envelope point. For more details at this place, one should especially take into account:

• The support map of the osculating circle of a c in $\mathbb{H}^2$ is given by the intersection of the osculating plane of θ in $\mathbb{R}^3_1$ with $\mathcal{C}^2_+$.
• The intersection of the osculating plane of θ with $\mathcal{C}^2_+$ is the osculating circle of the curve θ in $\mathbb{R}^3_1$ (use the Meusnier formula).
• The rum in $\mathcal{C}^2_+$ of the osculating circles of θ_1 and θ_2 in $\mathcal{C}^2_+$ is equal to the osculating circle of $\theta_1 \# \theta_2$ (to this use 2) and (26)).

Now the second order situation of c_1, c_2 is given by their osculating circles osc_1, osc_2 in $\mathbb{H}^2$. Therefore the circle (cf. Proposition 4.6) $\mathrm{osc}_1 \# \mathrm{osc}_2$ describes the second order situation of $c_1 \# c_2$. Hence $\theta_1 \# \theta_2$ supports $c_1 \# c_2$ convex-sided, $c_1 \# c_2$ is regular and oriented counter-clockwise. And $\mathrm{osc}_1 \# \mathrm{osc}_2$ is the osculating circle of $c_1 \# c_2$. Therefore $c_1 \# c_2$ is h-convex. Finally by Proposition 4.6, $K_1, K_2 \subset K^*$. □

Proposition 4.8. *Let c_1, c_2 be two oriented regular curves in H^2 with $(\kappa_g)_1, (\kappa_g)_2 > 1$ and support maps θ_1, θ_2 with respect to $\nu_1 = (e_2)_1, \nu_2 = (e_2)_2$. Then the length L^* and the total curvature TC^* of the rum $c^* = c_1 \# c_2$ of c_1 and c_2, if well defined, are given by*

$$L^* = \frac{1}{2}(W - V) \tag{33}$$

and

$$TC^* = \frac{1}{2}(W + V) \tag{34}$$

with

$$W(c_1, c_1 \# c_2) = L^* + TC^* =$$
$$= \int_{\theta^*} \kappa^2_{\theta^*}\, d\sigma^* = \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + \kappa^2_{\theta_1} \right) d\sigma_1,$$
$$w_{1*}(\sigma_1) = -\ln \left(\frac{\lambda(\sigma_1)}{1 + \lambda(\sigma_1)} \right) \quad \textit{and}$$
$$V(c_1, c_1 \# c_2) = \int_{\theta^*} d\sigma^* = \int_{\theta_1} e^{-w_{1*}}(\sigma_1) d\sigma_1 = \int_{\theta_1} \frac{\lambda(\sigma_1)}{1 + \lambda(\sigma_1)} d\sigma_1 \,.$$

Proof. We can write $\theta_2 = \lambda \theta_1$, hence by the definition of the rum $\theta^* = \frac{\lambda}{1+\lambda}\,\theta_1$, $d\sigma^* = \frac{\lambda}{1+\lambda}\, d\sigma_1$. Note: we have no additivity of the length of the associated support curves as in Proposition 4.1.

By the proof of Proposition 4.7 the rum c^* fulfils $\kappa^*_g > 1$ and $\nu^* = e^*_2$. Therefore we calculate

$$L(\theta^*) = \int_{\theta^*} d\sigma^* = \int_{c^*} (\kappa^*_g - 1)\, ds^* = TC^* - L^*$$

and (cf. (9))

$$L^* = L(c^*) = \frac{1}{2} \int_{\theta^*} \left(\kappa_\theta^{*2} - 1\right) \, d\sigma^*.$$

This yields (33) and (34). □

5. Gauss-Bonnet theorems and Chern-Lashof type inequalities

For compact immersed submanifolds M in euclidean spaces, the well-known extrinsic version of the Gauss-Bonnet theorem states that the total Lipschitz-Killing curvature of M is equal to the Euler characteristic $\chi(M)$ of M.

Moreover, the well-known Chern-Lashof inequality states that the total absolute Lipschitz-Killing curvature of M is bounded from below by the sum $\beta(M)$ of the Betti numbers of M, cf. [CL57], [CL58], [Fer68]. Especially for curves, this is the classical Fenchel inequality, cf. [Fen29], [Bor47].

For compact immersed submanifolds in hyperbolic spaces, the picture completely changes:

Neither the total Lipschitz-Killing curvature of M is equal to the Euler characteristic of M, nor the total absolute Lipschitz-Killing curvature of M is in general bounded by the sum of the Betti numbers of M. The latter recently was discovered by R. Langevin and G. Solanes, cf. [LS03], [Sol07].

The following facts are known:

• For curves in hyperbolic spaces there are generalizations of the Fenchel inequality, cf. [Sze68], [BH74] [Tsu74].

• For compact immersed submanifolds M lying inside a ball of radius R, there are lower bounds for the total absolute Lipschitz-Killing curvature in terms of the Betti numbers of M and the radius R, cf. [Teu82], [Teu88], [Oka98].

• The classical Gauss-Bonnet theorem in hyperbolic spaces, especially for hypersurfaces, contains not only the Lipschitz-Killing curvature but also the other mean curvatures of M, cf. [San76], [Sol06].

On the other hand, in recent years there are investigations on differential geometric quantities on M other than the Lipschitz-Killing curvature, in order to obtain Gauss -Bonnet like theorems and Chern-Lashof type inequalities respectively, cf. [Koi03], [Koi05]. (Not only in hyperbolic spaces, but also e.g. in spheres etc. cf. [LR96], [Koi05], [DK05].)

In the following we research along this line, using height functions based on pencils of parallel horospheres. In detail: For $u \in \mathbb{H}^n_\infty$, let $h_u\colon \mathbb{H}^n \to \mathbb{R}$ be the height function which level hypersurfaces are just the parallel horospheres of the pencil through u. As a measuring rod one may use any geodesic through u. We consider in the following the height function

$h_u|_M: M \to \mathbb{R}$, which is generical a Morse function. And we shall apply Morse theory to $h_u|_M$ (cf. [Hir94]).

In order to see the relation between critical points, their index and the curvature, let us first consider a general height function h on $\mathbb{H}^n$, i.e. a submersion $h: \mathbb{H}^n \to \mathbb{R}$ (defined at least locally). Let $p \in M$ be a critical point of the induced height function $h|_M$ along M, then some level hypersurface S of h is tangent to M at p, and hence $\operatorname{grad} h(p) = \lambda(p)\,\nu(p)$. We have

Lemma 5.1.

$$\operatorname{hess} h|_M(p) = \lambda(p)\, II_M(p) - |\lambda(p)|\, II_S(p) \tag{35}$$

where $II_M(p)$ is the second fundamental form of M at p with respect to its unit normal $\nu(p)$, and $II_S(p)$ is the second fundamental form of S with respect to its unit normal $\operatorname{grad} h(p)/|\operatorname{grad} h(p)|$ *at p.*

Proof. Along M we write $\operatorname{grad} h|_M = \operatorname{grad} h - \lambda\,\nu$ with an appropriate function λ. Then

$$\begin{aligned}\operatorname{hess} h|_M(p)(X,Y) &= g(\nabla_X \operatorname{grad} h|_M, Y)|_p = \\ &= g(\nabla_X \operatorname{grad} h, Y)|_p - d\lambda(X)\, g(\nu, Y)|_p - \lambda(p)\, g(\nabla_X \nu, Y)|_p = \\ &= -|\lambda(p)|\, II_S(X,Y) + \lambda(p)\, II_M(X,Y)\end{aligned}$$

with $X, Y \in T_pM$ and g the first fundamental form of M. □

We consider a compact immersed hypersurface M^{n-1} in $\mathbb{H}^n$ (without boundary). Let $\theta : N^1M \to \mathcal{C}^n_+$ with $\theta(x,\nu) = x + \nu$ be the support map of its unit normal bundle N^1M, such that ν points into the convex side of $\Theta(x,\nu)$.

If $v_1, \dots, v_{n-1}$ is a principal basis with respect to ν at x, we have $d\theta(v_i) = (1-k_i)v_i$ where $k_i = k_i(x,\nu)$ is the corresponding principal curvature. Then the area element of $\theta(N^1M)$ is

$$dA_\theta = |1-k_1| \cdots |1-k_{n-1}| dA_{(x,\nu)} \tag{36}$$

($dA_{(x,\nu)}$ = area element of N^1M at (x,ν)). Hence

$$\int_{N^1M} |1-k_1| \cdots |1-k_{n-1}| dA_{(x,\nu)} = \int_{\theta(N^1M)} dA_\theta \tag{37}$$

(dA_θ = area element of $\theta(N^1M)$ at θ).

Proposition 5.1. *Let M be a compact hypersurface immersed in $\mathbb{H}^n$ (without boundary). Assume that M is contained in some ball of radius r. Then*

$$\int_{N^1M} |1-k_1| \cdots |1-k_{n-1}| dA_{(x,\nu)} > e^{-(n-1)r} O_{n-1} \beta(M). \tag{38}$$

Proof. Assume that M is contained in the ball $B_p(r)$ with radius $r > 0$ and center $p \in \mathbb{H}^n$. Each horosphere Θ tangent to M is interior to some parallel horosphere tangent to $B_p(r)$ leaving it to the convex side. Therefore we have $\langle \theta, -p \rangle \geq e^{-r}$; i.e. θ lies above the plane $\{\langle \theta, -p \rangle = e^{-r}\}$, which intersects $\mathcal{C}^n_+$ in a sphere $S(r)$ of radius e^{-r}. Hence the support image $\theta(N^1M)$ of N^1M lies above $S(r)$.

We take into account the following fact: Let S_1, S_2 be two hypersurfaces in the cone $\mathcal{C}^n_+$ with $\theta_2 = \lambda\theta_1$, $\theta_i \in S_i$. Then the projection $\pi : S_2 \to S_1$ (along the generators of $\mathcal{C}^n_+$) has jacobian λ^{1-n}.

In particular, $\pi\colon \theta(N^1M) \to S(r)$ locally reduces area.

Therefore, applying the co-area formula, cf. [How93], to π gives

$$\int_{\theta(N^1M)} dA_\theta \geq \int_{S(r)} \#(\pi^{-1}(\theta))\, dS(r)_\theta \tag{39}$$

($\#(\pi^{-1}(\theta))$ = number of intersection points of $\theta(N^1M)$ and $\mathbb{R}^+\theta$).

Now, we use Differential Topology in particular Morse theory, cf. [Hir94]: For $\theta \in S(r)$, the associated pencil of parallel horospheres defines a height function h_θ. Because of the construction of the support map, the number of critical points of $h_\theta|_M$ is just the number $\#(\pi^{-1}(\theta))$ of intersection points of $\mathbb{R}^+\theta$ with $\theta(N^1M)$. Generically $h_\theta|_M$ is a Morse function. Therefore, by the Morse inequalities we have

$$\#(\pi^{-1}(\theta)) \geq \beta(M) \tag{40}$$

where $\beta(M)$ is the sum of the Betti numbers of M.

This shows, that π covers $S(r)$ at least $\beta(M)$ times.

Finally, bringing together (37), (39) and (40), we arrive at (38). □

Remark 5.1. Equality in (38) can never occur: In the proof we used two estimations, first the area-decreasing property of π and secondly the Morse inequalities. Altough we may have equality in the second estimation (e.g. for h-tight immersions), we never have equality in the first estimation. Note, that for every $x \in M$ not both of the two tangent horospheres can lie in $S(r)$.

Proposition 5.2. *Let M be a compact hypersurface immersed in $\mathbb{H}^n$ (without boundary). Fix a point $p \in \mathbb{H}^n$ and let $\rho(x,\nu), (x,\nu) \in N^1M$, denote the signed distance from p to the tangent horosphere $\Theta(x,\nu)$ represented by $\theta(x,\nu) = x + \nu$. Then*

$$\int_{N^1M} e^{(n-1)\rho}|1-k_1|\cdots|1-k_{n-1}|dA_{x,\nu} \geq O_{n-1}\beta(M). \tag{41}$$

Proof. Horospheres through p are represented by the section $S = T_p\mathbb{H}^n \cap \mathcal{C}^n_+$. The projection $\pi\colon \theta(N^1M) \to S$ has jacobian $e^{(n-1)\rho}$, where $\rho = \rho(x,\nu)$ is the signed distance from p to the horosphere $\Theta(x,\nu)$ (positive when p is interior).

Therefore, the co-area formula applied to $\pi \circ \theta : N^1M \to S$ gives

$$\int_{N^1M} e^{(n-1)\rho}|1-k_1|\cdots|1-k_{n-1}|dA_{x,\nu} = \int_S \#(\pi^{-1}(\theta))\, dS_\theta.$$

Again, by Morse inequalities we have

$$\#(\pi^{-1}(\theta)) \geq \beta(M).$$

Altogether we get (41). □

Remark 5.2. For h-tight hypersurfaces we have equality in (41).

Remark 5.3. When M is oriented by a unit normal field $\nu(x), x \in M$, then its unit normal bundle N^1M splits into two copies of M, say M_+ with normals ν and M_- with normals $\hat{\nu} = -\nu$. Also its support map splits into two maps θ with $\theta(x) = x + \nu(x)$ and $\hat{\theta}$ with $\hat{\theta}(x) = x + \hat{\nu}(x) = x - \nu(x)$ respectively. Then (41) writes

$$\int_M \Big(e^{(n-1)\rho}|1-k_1|\cdots|1-k_{n-1}| + e^{(n-1)\hat{\rho}}|1+k_1|\cdots|1+k_{n-1}|\Big)\, dA_x \geq O_{n-1}\beta(M) \tag{42}$$

where $k_1, \ldots, k_n$ are the principal curvatures of M with respect to ν, and $\rho, \hat{\rho}$ are the two support maps with base point p associated to $\theta, \hat{\theta}$.

Next we bring signs into game. First, we orient $T_x\mathbb{H}^n$ through x, and similarly we orient $\mathcal{C}^n_+$ through any vector $\mathbb{H}^n$. Given a subspace $V \subset T_\theta C^n_+$ transverse to $\mathbb{R}\theta$ we orient it through θ. For $(x,\nu) \in N^1M$ we choose principal directions $v_1, \ldots, v_{n-1}$ on T_xM with respect to ν such that $\{v_1, \ldots, v_{n-1}, \nu\}$ is a positive basis of $T_x\mathbb{H}^n$. Then $d\theta(v_i) = (1-k_i)v_i$, and $\{v_1, \ldots, v_{n-1}, \theta = x + \nu\}$ is a positive basis of $T_\theta\mathcal{C}^n_+$. Thus, θ preserves orientations if and only if $(1-k_1)\ldots(1-k_{n-1}) > 0$. Hence, the signed area of $\theta(N^1M)$ is

$$A^+(\theta(N^1M)) = \int_{N^1M}(1-k_1)\cdots(1-k_{n-1})dA_{(x,\nu)}. \tag{43}$$

Proposition 5.3. *Let M be a compact hypersurface immersed in $\mathbb{H}^n$ and oriented through a unit normal vector field $\nu(x), x \in M$. Fix an origin $p \in \mathbb{H}^n$, and let $\rho(x)$ be the signed distance from p to the horosphere $\Theta(x)$*

given by $\theta(x) = x + \nu(x)$ (with $\rho(x) > 0$ if and only if p is interior to $\Theta(x)$). Then, if n is odd

$$\int_M e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1})dA_x = \frac{O_{n-1}}{2}\chi(M), \tag{44}$$

and for general n, if $M = \partial Q$ and $\nu(x), x \in M$, points into Q

$$(-1)^{n-1}\int_M e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1})dA_x = O_{n-1}\chi(Q). \tag{45}$$

Proof. The projection $\pi\colon \mathcal{C}^n_+ \to S$, $S = T_p\mathbb{H}^n \cap \mathcal{C}^n_+$, preserves orientations when restricted to hypersurfaces transverse to the light rays. In particular $\pi\colon \theta(M) \to S$ preserves orientation, and it has jacobian $e^{(n-1)\rho}$. Therefore, application of the co-area formula to $\pi \circ \theta\colon M \to S$ gives

$$\int_M e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1})dA_x = \int_S \mu_M(\theta)dS_\theta, \tag{46}$$

where $\mu_M(\theta)$ is the algebraic intersection number of $\theta(M)$ with $\mathbb{R}^+\theta$.

For $\theta \in S$, let $h_\theta\colon \mathbb{H}^n \to \mathbb{R}$ be the height function which level hypersurfaces are build by the family of horospheres parallel to Θ (i.e. represented by the light-ray $\mathbb{R}^+\theta$ in $\mathcal{C}^n_+$), and which heights are given by the signed distance of these horospheres to the point p (positive when p lies in the convex side). Then, because of the construction of the support map θ,

$$\mu_M(\theta) = \sum_{\nabla h_\theta(x) = -\nu(x)} (-1)^i \tag{47}$$

where $i = i(x,\theta)$ is the index of x as a critical point of $h_\theta|_M$. Indeed, Lemma 5.1 gives

$$\begin{aligned}(-1)^i &= \operatorname{sign}(\det \operatorname{hess}_x(h_{\theta(x)}|_M)) = \\ &= \operatorname{sign}(\det(Id - II_M(x))) = \\ &= \operatorname{sign}(1-k_1)\cdots(1-k_{n-1}).\end{aligned} \tag{48}$$

Alternatively, we compute μ_M by using a diffeomorphism $\Psi\colon \mathbb{H}^n \to \mathbb{R}^n$ such that $h_{\theta(x)} = -x_n \circ \Psi$ (for instance, we can take the half-space model with $\Theta(x)$ horizontal). Here, Lemma 5.1 gives

$$\begin{aligned}\operatorname{sign}(\det \operatorname{hess}_x(h_\theta|_M)) &= \operatorname{sign}(\det \operatorname{hess}_{\Psi(x)}(-x_n|_{\Psi(M)})) = \\ &= \operatorname{sign}(\det(-II^e_{\Psi(M)})) = (-1)^{n-1}\operatorname{sign}K_e,\end{aligned} \tag{49}$$

being $II^e_{\Psi(M)}$ the euclidean second fundamental form (in the model), and K_e the euclidean Gauss curvature of $\Psi(M)$ in $\mathbb{R}^n$ with respect to the normal $\Psi_*\nu(x)$. From (47), (48) and (49) we conclude that $(-1)^{n-1}\mu_M(\theta)$ is the degree of the euclidean Gauss map of $\Psi(M)$.

In case n is odd we get

$$\mu_M(\theta) = \chi(M)/2.$$

This follows from

$$O_{n-1}\,\mu_M = \int_{\Psi(M)} K_e\, d(\Psi(M))_x = \frac{O_{n-1}}{2}\,\chi(M).$$

Here the first equality comes by application of the co-area formula to the euclidean Gauss map, and the second one is just the formula of Gauss-Bonnet (cf. [CL57], [CL58]).

For general n, if $M = \partial Q$ we have

$$\mu_M(\theta) = (-1)^{n-1}\chi(Q),$$

see e.g. [Mor29].

Altogether this proves the result. □

Remark 5.4. In case M not orientable, one can apply the previous proposition to the unit normal bundle, which is oriented. If M is orientable, this is equivalent to taking two copies of M, each with a different orientation. In this case, we get for n odd

$$\int_M \Big[e^{(n-1)\rho}(1-k_1)\cdots(1-k_{n-1}) \\ + e^{(n-1)\widehat{\rho}}(1+k_1)\cdots(1+k_{n-1})\Big]\, dA_x = O_{n-1}\chi(M), \tag{50}$$

where $\widehat{\rho}$ is the signed distance from p to the horosphere $\widehat{\Theta}(x)$ given by $\widehat{\theta}(x) = x - \nu(x)$.

Remark 5.5. Formulas (38), (41),(42) and (44) are also stated in [Koi03], there proven by using distance functions with respect to points.

Let now M be an immersed compact hypersurface (without boundary) in $\mathbb{H}^n$, oriented by a unit normal vector field $\nu(x), x \in M$.

Given any $\theta \in \mathcal{C}^n_+ \setminus \theta(M)$ we define $\mu^+_M(\theta)$ as the algebraic intersection number of the ray $(1,\infty)\theta = \{\lambda\theta | \lambda > 1\}$ with $\theta(M)$. Let Θ be the horosphere represented by θ, Ω the convex region bounded by Θ, and $\rho : \mathbb{H}^n \to \mathbb{R}$ the signed distance function to Θ (negative in the convex side of Θ, cf. (1)) . We consider the (signed) number of critical points of $\rho|_M$ that occur inside Ω, and such that $\nabla\rho = \nu$; i.e.

$$\mu^+_M(\theta) = \sum_{\substack{(\nabla\rho)(x)=\nu(x)\\ \rho(x)<0}} \operatorname{sign}(\det \operatorname{hess}_x \rho|_M).$$

Integrating with respect to θ over $\mathcal{C}^n_+$ we get

$$(51)\qquad \int_{\mathcal{C}^n_+} \mu^+_M(\theta)\omega_\theta = \int_{\mathbb{S}^{n-1}} \int_0^\infty \mu^+_M(\theta) y_{n+1}^{n-2} dy_{n+1} d\mathbb{S}^{n-1}_v =$$

$$= \int_{\mathbb{S}^{n-1}} \sum_{y\in\mathbb{R}^+(v,1)\cap\theta(M)} (-1)^i \frac{(y_{n+1})^{n-1}}{n-1} d\mathbb{S}^{n-1}_v =$$

$$= \frac{(-1)^{n-1}}{n-1} \int_M (1-k_1)\cdots(1-k_{n-1}) dA_x.$$

To this: The first equality is just rewriting the density of horospheres (cf. (2), $\theta = y_{n+1}(v,1), v \in \mathbb{S}^{n-1} = T^1_{(0,\ldots,0,1)}\mathbb{H}^n$).

For the second equality we carry out the integration with respect to y_{n+1} for fixed $v \in \mathbb{S}^{n-1}$. This integration runs along the generator $\mathbb{R}^+(v,1)$ of $\mathcal{C}^n_+$. Note that along the generator the function $\mu^+_M(\theta)$ is locally constant with jumps exactly at the intersection points of $\mathbb{R}^+(v,1)$ and $\theta(M)$. The magnitude of jump at a $\theta(x)$ is equal to $(-1)^i = \mathrm{sign}(\det(\mathrm{hess}_x\rho|_M))$ by definition of μ^+_M.

The third equality follows with (36), (37), taking into account (47) and $(y_{n+1})^{n-1}\, d\mathbb{S}^{n-1}_v = dA_\theta$ $(\theta = (y_{n+1})(v,1) = \theta(x))$.

The following question appears: is the number $\mu^+_M(\theta)$ determined by the topology of $M \cap \{\rho < 0\}$? We can answer this question in positive assuming M is embedded, or alternatively replacing M by its oriented cover, cf. (53).

Proposition 5.4. *Let $M \subset \mathbb{H}^n$ be an embedded compact hypersurface bounding a domain$Q$, and oriented by its inner normals. Then,*

$$(52)\quad (-1)^{n-1} \int_M (1-k_1)\cdots(1-k_{n-1}) dA_x =$$

$$= (n-1) \int_{\mathcal{C}^n_+} (\chi(Q\cap\Omega) - \chi(\Theta\cap Q))\,\omega.$$

Proof. Consider the domain $\Omega \cap Q$, which has piecewise smooth boundary $(M\cap\Omega)\cup(\Theta\cap Q)$. We can deform $B = \Theta\cap Q$ to a new hypersurface B' so that $\partial B = \partial B'$, and $(M\cap\Omega)\cup B'$ is a regular hypersurface bounding a domain R homotopic to $\Omega\cap Q$. Moreover, B' can be constructed so that the unit normal ν' on B' (obtained by transporting the orientation of B to B') fulfills $\langle \nu', \nabla\rho\rangle \leq 0$ everywhere on B. Let us consider the situation in the Poincaré model. We can assume Θ is horizontal in the model, so that $\nabla\rho$ is vertical and points downwards. Then, the degree of the Gauss map γ of ∂R (in the model) is $\chi(R)$. On the other hand, this degree can be computed as the signed number of preimages of the vector $(0,\dots,0,1)\in\mathbb{S}^{n-1}$. Then we have

$$(53)\quad \chi(\Omega\cap Q)=\chi(R)=\deg\gamma=\chi(B')+\mu^+_M(\theta)=\chi(\Theta\cap Q)+\mu^+_M(\theta).$$

We finish by applying equation (51). □

Remark 5.6. For h-convex M, we have that $\chi(Q\cap\Omega)-\chi(\Theta\cap Q)$ is equal to 1 if $M\subset\Omega$, otherwise it is equal to 0. Hence (52) gives

$$(54)\quad (-1)^{n-1}\int_M(1-k_1)\cdots(1-k_{n-1})dA_x=(n-1)\int_{M\subset\Omega}d\theta=(n-1)\,m(M)$$

where $m(M)$ is the measure of horospheres having M entirely in their convex sides.

Proposition 5.5. *For an embedded compact hypersurface M in $\mathbb{H}^n$,*

$$(55)\qquad \int_M(1+\sigma_2+\cdots+\sigma_{2k})dA_x=\frac{n-1}{2}\int_{\mathcal{C}^n_+}\chi(M\cap\Omega)\omega_\theta$$

where $\sigma_i=\sum_{1\le j_1\le\cdots\le j_i\le n-1}k_{j_1}\cdots k_{j_i}$, and $2k\le n-1\le 2k+1$.

Proof. To prove the formula, we consider $M'=N^1(M)$. Then

$$\mu^+_{M'}(\theta)=\sum_i(-1)^i c^+_i(\rho)$$

where $c^+_i(\rho)$ is the number of critical points of index i of ρ restricted to $M\cap\{\rho<0\}$. By Morse theory we know

$$\mu^+_{M'}(\theta)=\chi(M\cap\Omega),$$

and we get formula (55) by using equation (51). □

Remark 5.7. If n is even, then $\chi(M\cap\Omega)=\chi(M\cap\Theta)/2$ and formula (55) coincides with a result of [GNS04].

If n is even and Q is h-convex, then formula (55) writes

$$\int_M(1+\sigma_2+\cdots+\sigma_{n-2})dA_x=\frac{n-1}{2}\int_{\mathcal{H}}\chi(M\cap\Omega)\,d\theta=$$
$$=\frac{n-1}{2}\,(m_2+m_1)$$

where m_2 is the measure of horospheres containing Q in the interior, and m_1 denotes the measure of the horospheres intersecting Q. This together with (54) gives

$$\int_{M\cap\Theta\neq\emptyset}d\theta=\frac{2}{n-1}\int_M(\sigma_1+\sigma_3+\cdots+\sigma_{n-2})\,dA_x,$$

which coincides with one of the results in [GNS04].

Remark 5.8. For an immersion $f : M^{n-1} \to \mathbb{H}^n$ Proposition 5.5 remains true with the integrand on the right-hand side of (55) replaced by $\chi(f^{-1}(f(M) \cap \Omega))$.

References

[BH74] F. Brickell and C. C. Hsiung, *The total absolute curvature of closed curves in Riemannian manifolds*, J. Differential Geometry **9** (1974), 177–193.

[Bor47] Karol Borsuk, *Sur la courbure totale des courbes fermées*, Ann. Soc. Polon. Math. **20** (1947), 251–265 (1948).

[CL57] Shiing-shen Chern and Richard K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318.

[CL58] ———, *On the total curvature of immersed manifolds. II*, Michigan Math. J. **5** (1958), 5–12.

[DK05] Franki Dillen and Wolfgang Kühnel, *Total curvature of complete submanifolds of Euclidean space*, Tohoku Math. J. (2) **57** (2005), no. 2, 171–200.

[Fen29] Werner Fenchel, *Über Krümmung und Windung geschlossener Raumkurven*, Math. Ann. **101** (1929), no. 1, 238–252.

[Fer68] Dirk Ferus, *Totale Absolutkrümmung in Differentialgeometrie und topologie*, Lecture Notes in Mathematics, no. 66, Springer-Verlag, Berlin, 1968.

[Fil70] Jay P. Fillmore, *Barbier's theorem in the Lobachevski plane*, Proc. Amer. Math. Soc. **24** (1970), 705–709.

[GNS04] E. Gallego, A. M. Naveira, and G. Solanes, *Horospheres and convex bodies in n-dimensional hyperbolic space*, Geom. Dedicata **103** (2004), 103–114.

[GRST08] E. Gallego, G. Reventós, G. Solanes, and E. Teufel, *Width of convex bodies in spaces of constant curvature*, Manuscripta Mathematica **126** (2008), no. 1, 115–134.

[Hir94] Morris W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York, 1994, Corrected reprint of the 1976 original.

[How93] Ralph Howard, *The kinematic formula in Riemannian homogeneous spaces*, Mem. Amer. Math. Soc. **106** (1993), no. 509, vi+69.

[Koi03] Naoyuki Koike, *Theorems of Gauss-Bonnet and Chern-Lashof types in a simply connected symmetric space of non-positive curvature*, Tokyo J. Math. **26** (2003), no. 2, 527–539.

[Koi05] ———, *The Gauss-Bonnet and Chern-Lashof theorems in a simply connected symmetric space of compact type*, Tokyo J. Math. **28** (2005), no. 2, 483–497.

[Lei03] Kurt Leichtweiß, *On the addition of convex sets in the hyperbolic plane*, J. Geom. **78** (2003), no. 1-2, 92–121.

[LR96] Remi Langevin and Harold Rosenberg, *Fenchel type theorems for submanifolds of S^n*, Comment. Math. Helv. **71** (1996), no. 4, 594–616.

[LS03] Rémi Langevin and Gil Solanes, *On bounds for total absolute curvature of surfaces in hyperbolic 3-space*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 1, 47–50.

[Mor29] Marston Morse, *Singular Points of Vector Fields Under General Boundary Conditions*, Amer. J. Math. **51** (1929), no. 2, 165–178.

[Oka98] Takashi Okayasu, *An extension of Chern-Lashof theorem to other space forms*, The Third Pacific Rim Geometry Conference (Seoul, 1996), Monogr. Geom. Topology, vol. 25, Int. Press, Cambridge, MA, 1998, pp. 281–293.

[San67] L. A. Santaló, *Horocycles and convex sets in hyperbolic plane*, Arch. Math. (Basel) **18** (1967), 529–533.

[San68] ———, *Horospheres and convex bodies in hyperbolic space*, Proc. Amer. Math. Soc. **19** (1968), 390–395.

[San76] Luis A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, Encyclopedia of Mathematics and its Applications, vol. 1.

[Sol06] Gil Solanes, *Integral geometry and the Gauss-Bonnet theorem in constant curvature spaces*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1105–1115 (electronic).

[Sol07] ———, *Total absolute curvature and tight submanifolds in hyperbolic space*, J. Lond. Math. Soc. (2) **75** (2007), no. 2, 420–430.

[Sze68] J. Szenthe, *On the total curvature of closed curves in Riemannian manifolds*, Publ. Math. Debrecen **15** (1968), 99–105.

[Teu88] Eberhard Teufel, *On the total absolute curvature of immersions into hyperbolic spaces*, Topics in differential geometry, vol. I, II (Debrecen, 1984), Colloq. Math. Soc. János Bolyai, vol. 46, North-Holland, Amsterdam, 1988, pp. 1201–1209.

[Teu82] ———, *Differential topology and the computation of total absolute curvature*, Math. Ann. **258** (1981/82), no. 4, 471–480.

[Tsu74] Yôtarô Tsukamoto, *On the total absolute curvature of closed curves in manifolds of negative curvature*, Math. Ann. **210** (1974), 313–319.

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