ON TWO EXAMPLES BY IYAMA AND YOSHINO

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ABSTRACT. In a recent paper Iyama and Yoshino consider two interesting examples of isolated singularities over which it is possible to classify the indecomposable maximal Cohen-Macaulay modules in terms of linear algebra data. In this paper we present two new approaches to these examples. In the first approach we give a relation with cluster categories. In the second approach we use Orlov's result on the graded singularity category. We obtain some new results on the singularity category of isolated singularities which may be interesting in their own right.

CONTENTS

1. INTRODUCTION

Throughout k is a field. The explicit description of the stable category of maximal Cohen-Macaulay modules over a commutative Gorenstein ring

2000 *Mathematics Subject Classification.* Primary 13C14, 16E65.
Key words and phrases. Singularity category, maximal Cohen-Macaulay modules.
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The results of this paper were partially obtained during visits of the second author to the Université Paris Diderot-Paris 7 and the Centre de Recerca Matemàtica in Barcelona. He hereby thanks these institutions for their hospitality.

(also known as the singularity category [7, 6, 17]) is a problem that has received much attention over the years. This seems to be in general a difficult problem and perhaps the best one can hope for is a reduction to linear algebra, or in other words: the representation theory of quivers.

In [10] Iyama and Yoshino consider the following two examples.

Example 1.1. Let $S = k[[x_1, x_2, x_3]]$ and let $C_3 = \langle\sigma\rangle$ be the cyclic group of three elements. Consider the action of C_3 on S via $\sigma x_i = \omega x_i$ where $\omega^3 = 1$, $\omega \neq 1$. Put $R = S^{C_3}$.

Example 1.2. Let $S = k[[x_1, x_2, x_3, x_4]]$ and let $C_4 = \langle\sigma\rangle$ be the cyclic group of four elements. Consider the action of C_4 on S via $\sigma x_i = -x_i$. Put $R = S^{C_4}$.

In both examples Iyama and Yoshino reduce the classification of maximal Cohen-Macaulay modules over R to the representation theory of certain generalized Kronecker quivers. They use this to classify the rigid Cohen-Macaulay modules over R. As predicted by deformation theory, the latter are described by discrete data. The proofs of Iyama and Yoshino are based on the machinery of mutation in triangulated categories, a general theory developed by them. In the current paper we present two alternative approaches to the examples. Hopefully the thus obtained additional insight may be useful elsewhere.

Our first approach applies to Example 1.2 and is inspired by the treatment in [14] of Example 1.1 where the authors used the fact that in this case the stable category $\underline{\mathrm{MCM}}(R)$ of maximal Cohen-Macaulay R-modules is a 2-Calabi-Yau category which has a cluster tilting object whose endomorphism ring is the path algebra kQ_3 of the Kronecker quiver with 3 arrows. From their acyclicity result [14, §1,Thm] they obtain immediately that $\underline{\mathrm{MCM}}(R)$ is the corresponding cluster category $D^b(\mathrm{mod}(kQ_3))/(\tau[-1])$. This gives a very satisfactory description of $\underline{\mathrm{MCM}}(R)$ and implies in particular the results by Iyama and Yoshino.

In the first part of this paper we show that Example 1.2 is amenable to a similar approach. Iyama and Yoshino prove that $\underline{\mathrm{MCM}}(R)$ is a 3-Calabi-Yau category with a 3-cluster tilting object T such that $\mathrm{End}(T) = k$ [10, Theorem 9.3]. We show that under these circumstances there is an analogue of the acyclicity result of the first author and Reiten.

Theorem 1.3 (see §4.4)**.** *Assume that $\mathcal{T}$ is k-linear algebraic Krull-Schmidt 3-Calabi-Yau category with a 3-cluster tilting object T such that see* $\mathrm{End}(T) = k$. *Then there is an equivalence of $\mathcal{T}$ with the orbit category* $D^b(\mathrm{mod}(kQ_n))/(\tau^{1/2}[-1])$, $n = \dim \mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)$, *where Q_n is the generalized Kronecker quiver with n arrows and $\tau^{1/2}$ is a natural square root of the*

Auslander-Reiten translate of $D^b(\mathrm{mod}(kQ_n))$*, which on the pre-projective/ pre-injective component corresponds to "moving one place to the left".*

In the second part of this paper, which is logically independent from the first we give yet another approach to the examples 1.1, 1.2 based on the following observation which might have independent interest. It is obtained as a consequence of some results on the generation of the bounded derived category of coherent sheaves which are exhibited in §5.

Proposition 1.4 (see §6)**.** *Let* $A = k + A_1 + A_2 \cdots$ *be a finitely generated commutative graded Gorenstein* k*-algebra with an isolated singularity. Let* $\widehat{A}$ *be the completion of* A *at* $A_{\geq 1}$*. Let* $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ *be the stable category of graded maximal Cohen-Macaulay* A*-modules. Then the obvious functor* $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to \underline{\mathrm{MCM}}(\widehat{A})$ *induces an equivalence*

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1) \cong \underline{\mathrm{MCM}}(\widehat{A}) \tag{1.1}$$

where $M \mapsto M(1)$ *is the shift functor for the grading.*

In this proposition the quotient $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1)$ has to be understood as the triangulated/Karoubian hull (as explained in [13]) of the naive quotient of $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ by the shift functor ?(1). This result is similar in spirit to [3] which treats the finite representation type case. Note however that one of the main results in loc. cit. is that in case of finite representation type case *every* indecomposable maximal Cohen-Macaulay $\hat{A}$-module is gradable. This does not seem to be a formal consequence of Proposition 1.4. It would be interesting to investigate this further.

Hence in order to understand $\mathrm{MCM}(\widehat{A})$ it is sufficient to understand $\mathrm{MCM}_{\mathrm{gr}}(A)$. The latter is the graded singularity category [16] of A and by [16, Thm 2.5] it is related to $D^b(\mathrm{coh}(X))$ where $X = \operatorname{Proj} A$.

In Examples 1.1, 1.2 R is the completion of a graded ring A which is the Veronese of a polynomial ring. Hence $\operatorname{Proj} A$ is simply a projective space. Using Orlov's results and the existence of exceptional collections on projective space we get very quickly in Example 1.1

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \cong D^b(\mathrm{mod}(kQ_3))$$

and in Example 1.2

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \cong D^b(\mathrm{mod}(kQ_6))$$

(where here and below $\cong$ actually stands for a quasi-equivalence between the underlying DG-categories). Finally it suffices to observe that in Example 1.1 we have $?(-1) = \tau[-1]$ and in Example 1.2 we have $?(-1) = \tau^{1/2}[-1]$ (see §8 below).

Our proof of Proposition 1.4 uses the deep general Neron desingularization theorem of Popescu (which implies the Artin approximation theorem). However in Remark 6.2 we point out that in the situation of Examples 1.1, 1.2 a substantial simplification is possible.

Furthermore in §9 we show that rigid Cohen-Macaulay modules are in fact gradable so they are automatically in the image of $\mathrm{MCM}_{\mathrm{gr}}(A)$. We expect this to be well known in some form but we have been unable to locate a reference.

Finally we mention the following side result which we think may also be of independent interest.

Proposition 1.5 (see §6)**.** *Let (R, m) be a local Gorenstein ring with residue field k which is essentially finite over k and has an isolated singularity. Then the natural functor*

$$\widehat{R} \otimes_R ?: \underline{\mathrm{MCM}}(R) \to \underline{\mathrm{MCM}}(\widehat{R}) \tag{1.2}$$

is an equivalence up to direct summands. In partular every maximal Cohen-Macaulay module over $\widehat{R}$ is a direct summand of the completion of a maximal Cohen-Macaulay module over R.

2. Acknowledgement

We thank Osamy Iyama and Idun Reiten for commenting on a preliminary version of this manuscript.

3. Notations and conventions

We hope most notations are self explanatory but nevertheless we list them here. If R is a ring then $\mathrm{Mod}(R)$ and $\mathrm{mod}(R)$ denote respectively the category of all left R-modules and the full subcategory of finitely generated R-modules. The derived category of all R-modules is denoted by $D(R)$. If R is graded then we use $\mathrm{Gr}(R)$ and $\mathrm{gr}(R)$ for the category of graded left modules and its subcategory of finitely generated modules. The shift functor on $\mathrm{Gr}(R)$ is denoted by $?(1)$. Explicitly $M(1)_i = M_{i+1}$. If we want to refer to right modules then we use R° instead of R. If X is a scheme then $\mathrm{Qch}(X)$ is the category of quasi-coherent $\mathcal{O}_X$-modules. If X is noetherian then $\mathrm{coh}(X)$ is the category of coherent $\mathcal{O}_X$-modules. We are generally very explicit about which categories we use. E.g. we write $D^b(\mathrm{mod}(R))$ rather than something like $D^b_f(R)$. If R is graded and M, N are graded R-modules then $\mathrm{Ext}^i_R(M, N)$ is the ungraded Ext between M and N. If we need Ext in the category of graded R-modules then we write $\mathrm{Ext}^i_{\mathrm{Gr}(R)}(M, N)$.

4. First approach to the second example

4.1. **Some preliminaries on tilting complexes.** Let C, E be rings. We denote the unbounded derived category of right C-modules by $D(C^\circ)$. We let $\mathrm{Eq}(D(C^\circ), D(E^\circ))$ be the set of triangle equivalences of $D(C^\circ) \to D(E^\circ)$ modulo natural isomorphisms. Define $\mathrm{Tilt}(C, E)$ as the set of pairs (ϕ, T) where T is a perfect complex generating $D(E^\circ)$ and ϕ is an isomorphism $C \to \mathrm{RHom}_E(T)$. Associated to $(\phi, T) \in \mathrm{Tilt}(C, E)$ there is a canonical equivalence $\theta\colon D(C^\circ) \to D(E^\circ)$ such that $\theta(C) = T$. It may be constructed either directly [19] or using DG-algebras [12]. The induced map

$$\mathrm{Tilt}(C, E) \to \mathrm{Eq}(D(C^\circ), D(E^\circ))$$

is obviously injective (as it is canonically split), but unknown to be surjective. Below we will informally refer to the elements of $\mathrm{Tilt}(C, E)$ as tilting complexes.

4.2. **A square root of τ for a generalized Kronecker quiver.** Let W be a finite dimensional k-vector space and let C be the path algebra of the quiver[1]

$$\underset{1}{\bullet} \xleftarrow{\;W\;} \underset{2}{\bullet} \tag{4.1}$$

Let E be the path algebra of the quiver

$$\underset{1}{\bullet} \xleftarrow{\;W^*\;} \underset{2}{\bullet}$$

which we think of as being obtained from (4.1) by "inverting the arrows" and renumbering the vertices $(1, 2) \mapsto (2, 1)$.

Let P_i,I_i, S_i be respectively the projective, injective and simple right C-module corresponding to vertex i. For E we use P'_i, I'_i, S'_i. Let $r_i\colon \mathrm{mod}(C^\circ) \to \mathrm{mod}(E^\circ)$ be the reflection functor at vertex i. Recall that if (U, V) is a representation of C then $r_1(U, V)$ is given by (V, U') where $U' = \ker(V \otimes W \to U)$ (taking into account the renumbered vertices).

The right derived functor Rr_1 of r_1 defines an equivalence $D(C^\circ) \to D(E^\circ)$. It is obtained from the tilting complex $S'_2[-1] \oplus P'_1$ [2]. One has (see [8])

$$Rr_1 \circ Rr_1 = \tau_C \tag{4.2}$$

where τ_C is the Auslander Reiten translate on $D(C^\circ)$. Assume now that W is equipped with an isomorphism $\pi\colon W \to W^*$. Then π yields an equivalence

[1] We use the convention that multiplication in the path algebra is concatenation. So representations correspond to right modules.

$D(E^\circ) \cong D(C^\circ)$, which we denote by the same symbol. We use the same convention for the transpose isomorphism $\pi^*\colon W \to W^*$.

Lemma 4.2.1. *We have $r_1 \circ \pi^{-1} = \pi^* \circ r_1$ as functors $D(C^\circ) \to D(C^\circ)$.*

Proof. Let (U, V) be a representation of C determined by a linear map $c\colon V \otimes W \to U$ and put $(V, U'') = (r_1 \circ \pi^{-1})(U, V)$. Then one checks that U'' is given by the exact sequence

$$0 \to U'' \to V \otimes W^* \xrightarrow{c\circ(\pi^{-1}\otimes \mathrm{id})} U \to 0$$

where the first non-trivial map induces the action $U'' \otimes W \to V$. Similarly if we put $(V, U') = (\pi^* \circ r_1)(U, V)$ then one gets the same sequence

$$0 \to U' \to V \otimes W^* \xrightarrow{c\circ(\pi^{-1}\otimes \mathrm{id})} U$$

where the first non-trivial map again yields the action $U' \otimes W \to V$. Thus we have $(V, U') = (V, U'')$. □

Below we put $a = \pi \circ Rr_1$.

Lemma 4.2.2. *One has $(\pi^* \circ \pi^{-1}) \circ a^2 = \tau$. In particular $\tau \cong a^2$ if and only if π is self-adjoint or anti self-adjoint.*

Proof. This is a straightforward verification using Lemma 4.2.1 and (4.2). □

For use below we record

$$\begin{aligned} aP_2 &= P_1 \\ aP_1 &= I_2[-1] \\ aI_2 &= I_1. \end{aligned}$$

4.3. **A 3-Calabi-Yau category with a 3-cluster tilting object.** We let the notations be as in the previous section,

Put $\mathcal{H} = D^b(\mathrm{mod}(C^\circ))$, $\mathcal{D} = \mathcal{H}/a[-1]$. As $\mathcal{H}$ is hereditary we have

$$\mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\mathcal{H})/a[-1].$$

Inspection reveals that

$$\mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\}. \tag{4.3}$$

Lemma 4.3.1. *$\mathcal{D}$ is 3-Calabi-Yau if and only is π is self-adjoint or anti self-adjoint.*

Proof. Let S be the Serre-functor for $\mathcal{H}$. Being canonical S commutes with the auto-equivalence $a[-1]$. Hence S induces an autoequivalence on $\mathcal{D}$ which is easily seen to be the Serre functor of $\mathcal{D}$.

In $\mathcal{D}$ we have $S = \tau[1] = (\pi^* \circ \pi^{-1}) \circ a^2[1] = (\pi^* \circ \pi^{-1})[3]$. Thus $\mathcal{D}$ is 3-Calabi-Yau if and only if $\pi^* \circ \pi^{-1}$ is isomorphic to the identity functor. It is easy to see that this is the case if and only is $\pi^* \circ \pi^{-1} = \pm 1$ in $\mathrm{End}_k(W)$. □

Lemma 4.3.2. *The object P_1 in $\mathcal{D}$ satisfies*

$\mathrm{Ext}^i_{\mathcal{D}}(P_1, P_1) = 0$ *for* $i = 1, 2$, $\mathrm{Hom}_{\mathcal{D}}(P_1, P_1) = k$ *and* $\mathrm{Ext}^{-1}_{\mathcal{D}}(P_1, P_1) = W$.

Proof. For $N \in \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\}$ one computes

$$\mathrm{Hom}_{\mathcal{D}}(P_1, N) = \mathrm{Hom}_{\mathcal{H}}(P_1, N) \tag{4.4}$$

Thus we find

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, P_1[-1]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, a^{-1}P_1)\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, P_2)\\ &= W\end{aligned}$$

$$\mathrm{Hom}_{\mathcal{D}}(P_1, P_1) = k$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, P_1[1]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, aP_1)\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_2[-1])\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, P_1[2]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, aP_1[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_2)\\ &= 0\end{aligned}$$

□

The following lemma is not used explicitly.

Lemma 4.3.3. *The object P_1 in $\mathcal{D}$ has the properties of a 3-cluster tilting object, i.e. if $\mathrm{Ext}^i_{\mathcal{D}}(P_1, N) = 0$ for $i = 1, 2$ then N is a sum of copies of P_1.*

Proof. Assume that $N \in \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\}$ is such that $\mathrm{Hom}_{\mathcal{D}}(P_1, N[1]) = \mathrm{Hom}_{\mathcal{D}}(P_1, N[2]) = 0$. We have to prove $N = P_1$.

We may rewrite

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, N[2]) &= \mathrm{Hom}_{\mathcal{D}}(P_1[-1], N[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(a^{-1}P_1, N[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(P_2, N[1]).\end{aligned}$$

Thus we find $\mathrm{Hom}_{\mathcal{D}}(P_1, aN) = \mathrm{Hom}_{\mathcal{D}}(P_2, aN) = 0$. Hence $aN \notin \mathrm{Ind}(\mathcal{H})$. We deduce $N \in \{P_1, I_2[-1]\}$.

But if $N = I_2[-1]$ then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, N[2]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_2[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, aI_2)\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_1)\\ &\neq 0.\end{aligned}$$

So we are left with the possibility $N = P_1$ which finishes the proof. □

4.4. **Proof of Theorem 1.3.** Let $\mathcal{T}$ be an algebraic Ext-finite Krull-Schmidt 3-Calabi-Yau category containing a 3-cluster tilting object T such that $\mathrm{End}_{\mathcal{T}}(T) = k$.

Lemma 4.4.1. *Let $N \in \mathcal{T}$. Then there exists a distinguished triangle in $\mathcal{T}$*

$$T^a \to T^b \oplus T[-1]^c \to N[1] \to \ . \tag{4.5}$$

Proof. Let Y be defined (up to isomorphism) by the following distinguished triangle[2]

$$Y \to T^{\mathrm{Ext}^1_{\mathcal{T}}(T,N)} \oplus T[-1]^{\mathrm{Ext}^2_{\mathcal{T}}(T,N)} \to N[1] \to \ .$$

A quick check reveals that $\mathrm{Ext}^1_{\mathcal{T}}(T, Y) = \mathrm{Ext}^2_{\mathcal{T}}(T, Y) = 0$. Hence $Y = T^a$ for some a. □

We need to consider the special case $N = T[1]$. Then the distinguished triangle (4.5) (constructed as in the proof) has the form

$$T^{\mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)} \xrightarrow{\phi} T[-1] \xrightarrow{\alpha} T[2] \xrightarrow{\beta} \tag{4.6}$$

where ϕ is the universal map (this follows from applying $\mathrm{Hom}_{\mathcal{T}}(T, -)$). Since $\mathrm{End}_{\mathcal{C}}(T[2]) = k$ it follows that α,β are determined up to (the same) scalar.

This has a surprizing consequence. Applying $\mathrm{Hom}_{\mathcal{T}}(-, T)$ to the triangle (4.6) we find that $\mathrm{Hom}_{\mathcal{T}}(\beta[-1], T)^{-1}$ defines an isomorphism

$$\pi\colon \mathrm{Ext}^{-1}_{\mathcal{T}}(T, T) \to \mathrm{Ext}^{-1}_{\mathcal{T}}(T, T)^*.$$

Thus $W \overset{\mathrm{def}}{=} \mathrm{Ext}^{-1}_{\mathcal{C}}(T, T)$ comes equipped with an isomorphism $\pi\colon W \to W^*$ which is canonical up to a scalar. In other words we are in the setting of §4.2 and we now use the notations introduced in sections 4.2 and 4.3.

[2]It would be more logical to write e.g. $\mathrm{Ext}^1_{\mathcal{T}}(T, N) \otimes_k T$ for $T^{\mathrm{Ext}^1_{\mathcal{T}}(T,N)}$ but this would take a lot more space.

As a is obtained from the reflection in vertex 1, one verifies (see §4.2) that a is associated to the element of $\operatorname{Tilt}(C,C)$ given by $(\theta, I_2[-1]\oplus P_1)$ where $\theta\colon C\to \operatorname{End}_C(I_2[-1]\oplus P_1)$ is the composition

$$C=\begin{pmatrix} k & 0\\ W & k\end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} k & 0\\ W^* & k\end{pmatrix} = \operatorname{End}_C(I_2[-1]\oplus P_1). \tag{4.7}$$

Since the autoequivalence a is a derived functor that commutes with coproducts it is isomorphic to a derived tensor functor $-\overset{L}{\otimes}_C X$ for some $X\in D(C^e)$, by [11, 6.4]. As a right C-module we have $X\cong I_2[-1]\oplus P_1$.

Now we use the assumption that $\mathcal{H}$ is algebraic and we proceed more or less as in the appendix to [14]. By [11, Thm. 4.3] we may assume that $\mathcal{T}$ is a strict (= closed under isomorphism) triangulated subcategory of a derived category $D(\mathcal{A})$ for some DG-category $\mathcal{A}$. We denote by ${}_C\mathcal{T}$ the full subcategory of $D(C\otimes\mathcal{A})$ whose objects are differential graded $C\otimes\mathcal{A}$-modules which are in $\mathcal{T}$ when considered as $\mathcal{A}$-modules. Clearly ${}_C\mathcal{T}$ is triangulated. By [14, Lemma A.2.1(a)] T may be lifted to an object in ${}_C\mathcal{T}$, which we also denote by T. Put $S=T\oplus T[-1]$.

Lemma 4.4.2. *One has an isomorphism in* ${}_C\mathcal{T}$

$$X\overset{L}{\otimes}_B S\cong S[1].$$

Proof. As objects in $\mathcal{T}$ we have

$$\begin{aligned} X\overset{L}{\otimes}_C S &= (I_2[-1]\oplus P_1)\overset{L}{\otimes}_C S\\ &= I_2\overset{L}{\otimes}_C S[-1]\oplus P_1\overset{L}{\otimes}_C S. \end{aligned}$$

Clearly $P_1\overset{L}{\otimes}_C S\cong T$. To compute $I_2\overset{L}{\otimes}_C S$ we use the resolution

$$0\to P_1^{\operatorname{Ext}^{-1}_{\mathcal{T}}(T,T)}\to P_2\to I_2\to 0.$$

Tensoring with S we get a distinguished triangle

$$T^{\operatorname{Ext}^{-1}_{\mathcal{T}}(T,T)}\to T[-1]\to I_2\overset{L}{\otimes}_C S\to .$$

Comparing with (4.6) we find $I_2\overset{L}{\otimes}_C S\cong T[2]$. Thus, we have indeed an isomorphism

$$\varphi\colon X\overset{L}{\otimes}_B S\to S[1]$$

in $\mathcal{T}$.

Now we check that φ is C-equivariant in $\mathcal{T}$. The left C-module structure on $X\overset{L}{\otimes}_B S$ is obtained from the (homotopy) C-action on $I_2[-1]\oplus P_1$ as given in (4.7).

Let μ be an element of $W = \mathrm{Hom}_C(P_1, P_2) = \mathrm{Ext}^{-1}_{\mathcal{T}}(T, T)$. We need to prove that the following diagram is commutative in $\mathcal{T}$.

$$\begin{array}{ccc} I_2[-1] \overset{L}{\otimes}_B S & \xrightarrow{\cong} & T[1] \\ {\scriptstyle \pi(\mu)\overset{L}{\otimes}_B \mathrm{id}_S}\downarrow & & \downarrow{\scriptstyle \mu} \\ P_1 \overset{L}{\otimes}_B S & \xrightarrow[\cong]{} & T\,. \end{array}$$

We write this out in triangles

$$\begin{array}{ccccccc} T^{\mathrm{Ext}^{-1}(T,T)} & \xrightarrow{\phi} & T[-1] & \xrightarrow{\alpha} & T[2] & \xrightarrow{\beta} & \\ {\scriptstyle \pi(\mu)}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\ T & \longrightarrow & 0 & \longrightarrow & T[1] & \xrightarrow[\mathrm{id}]{} & . \end{array}$$

Rotating the triangles we need to prove that the following square is commutative

$$\begin{array}{ccc} T[1] & \xrightarrow{\beta[-1]} & T^{\mathrm{Ext}^{-1}(T,T)} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \pi(\mu)} \\ T & = & T\,. \end{array}$$

This commutivity holds precisely because of the definition of π. So ϕ is indeed C-equivariant.

But according to [14, Lemma A.2.2], any C-equivariant morphism in $\mathcal{T}$ between objects in ${}_C\mathcal{T}$ may be lifted to a morphism in ${}_C\mathcal{T}$. This finishes the proof. □

We now have a functor

$$? \overset{L}{\otimes}_C T : \mathcal{C} \to \mathcal{T}$$

and by Lemma 4.4.2 one finds that $a[-1](?) \overset{L}{\otimes}_C T$ is isomorphic to $? \overset{L}{\otimes}_C T$. By the universal property of orbit categories [13] we obtain a triangulated functor

$$Q : \mathcal{D} \to \mathcal{T}$$

which sends P_1 to T.

Lemma 4.4.3. *Q is an equivalence.*

Proof. We observe that analogues of the distinguished triangles (4.5) exist in $\mathcal{D}$ (with P_1 replacing T). Indeed, let $N \in \mathrm{Ind}(\mathcal{D})$. By (4.3) we have $N \in \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\}$. If $N \in \mathrm{Ind}(\mathcal{H})$ then $N[1] \cong aN$ and the analog of (4.5) is simply the image in $\mathcal{D}$ of the projective resolution of aN in $\mathcal{H}$ (taking into account that $P_2 = a^{-1}P_1 = P_1[-1]$).

If $N = I_2[-1]$ then $N[1] = I_2$ and the analog of (4.5) is the image in $\mathcal{D}$ of the projective resolution of I_2 in $\mathcal{H}$.

To prove that Q is fully faithful we have to prove that Q induces an isomorphism $\operatorname{Hom}_{\mathcal{D}}(M,N) \to \operatorname{Hom}_{\mathcal{T}}(QM,QN)$. Using the analogues of (4.5) we reduce to $M = P_1[i]$. But since $\operatorname{Hom}_{\mathcal{D}}(P_1[i],N) = \operatorname{Hom}_{\mathcal{D}}(P_1[-1], N[-i-1])$ we reduce in fact to $M = P_1[-1]$. It now suffices to apply $\operatorname{Hom}_{\mathcal{D}}(P_1[-1],-)$ to

$$P_1^a \to P_1^b \oplus P_1[-1]^c \to N[1] \to$$

taking into account that $\operatorname{Hom}_{\mathcal{D}}(M,N) \to \operatorname{Hom}_{\mathcal{T}}(QM,QN)$ is an isomorphism for $M = P_1$, $N = P_1$, $P_1[1]$, $P_2[2]$ by Lemma 4.3.2.

As a last step we need to prove that Q is essentially surjective. But this follows from the distinguished triangles (4.5) together with the fact that $QP_1 = T$. □

To finish the proof of Theorem 1.3 we observe that since $\mathcal{T}$ is 3-Calabi-Yau, so is $\mathcal{D}$. Hence by Lemma 4.3.1 π is either self-adjoint or anti self-adjoint. By Lemma 4.2.2 we deduce $a^2 \cong \tau$ and hence we may write $a = \tau^{1/2}$.

Remark 4.4.4. It would be interesting to deduce the fact that π is (anti) self-adjoint directly from the Calabi-Yau property of $\mathcal{T}$, without going through the construction of $\mathcal{D}$ first. This would have made our arguments above more elegant.

Remark 4.4.5. Iyama and Yoshino also consider $2n+1$-Calabi-Yau categories $\mathcal{T}$ equipped with a $2n+1$-cluster tilting object T such that $\operatorname{End}(T) = k$ and $\operatorname{Ext}^{-i}(T,T) = 0$ for $0 < i < n$. They relate such $\mathcal{T}$ to the representation theory of the generalized Kronecker quiver Q_m where $m = \dim \operatorname{Ext}^{-n}(T,T)$.

One may show that our techniques are applicable to this case as well and yield $\mathcal{T} \cong D^b(\operatorname{mod}(kQ_m))/(\tau^{1/2}[-n])$. We thank Osama Iyama for bringing this point to our attention.

5. Observations on generation

We now start with the second approach to the examples 1.1, 1.2. The results in this section are of the type where we have an exact functor $\mathcal{A} \to \mathcal{B}$ between triangulated categories and we would like to show that $\mathcal{B}$ is classically generated by objects in $\mathcal{A}$ (see below for terminology). We suspect most results are true in far greater generality then they are stated here.

5.1. **Generalities.** We use similar notations as in [5] and [20]. Thus if $\mathcal{T}$ is a triangulated category (not necessarily closed under direct sums) then $\mathcal{T}^c$ is the collection of objects X in $\mathcal{T}$ such that $\mathcal{T}(X,-)$ commutes with (existing) direct sums. If $\mathcal{E} \subset \mathcal{T}$ then $\langle \mathcal{E} \rangle_d$ is the set of objects in $\mathcal{T}$ which can be obtained by taking finite sums, shifts, summands and at most d cones. $\overline{\langle \mathcal{E} \rangle}_d$ is defined similarly except that we allow arbitrary direct sums. Finally we put[3] $\langle \mathcal{E} \rangle = \bigcup_d \langle \mathcal{E} \rangle_d$, $\overline{\langle \mathcal{E} \rangle} = \bigcup_d \overline{\langle \mathcal{E} \rangle}_d$. We say that $\mathcal{T}$ is classically generated by $\mathcal{E}$ if $\mathcal{T} = \langle \mathcal{E} \rangle$. Below we use "generated by" as a synonym for "classically generated by".

A crucial result, is the following

Proposition 5.1.1. [20, Corollary 3.13] *Let $\mathcal{E}$ be contained in $\mathcal{T}^c$. Then for any d we have*

$$\overline{\langle \mathcal{E} \rangle}_d \cap \mathcal{T}^c = \langle \mathcal{E} \rangle_d \,.$$

This result is essentially due to Neeman (see e.g. [15, Lemma 1.5]). In [5, Proposition 2.2.4] the result was stated under somewhat stronger hypotheses on $\mathcal{T}$. Note that by distributivity we get

$$\overline{\langle \mathcal{E} \rangle} \cap \mathcal{T}^c = \langle \mathcal{E} \rangle.$$

Let R be a left noetherian ring. Then by [20, Cor 6.16] we have

$$D^b(\mathrm{Mod}(R))^c = D^b(\mathrm{mod}(R)).$$

Similarly if X is a separated noetherian scheme then

$$D^b(\mathrm{Qch}(X))^c = D^b(\mathrm{coh}(X)). \tag{5.1}$$

The following is a slight strengthening of a beautiful result by Rouquier.

Proposition 5.1.2. *Let X be a separated scheme of finite type over k. Then there is $E \in D^b(\mathrm{coh}(X))$ and $d \in \mathbb{N}$ such that*

$$D^b(\mathrm{coh}(X)) = \langle E \rangle_d \tag{5.2}$$

$$D^b(\mathrm{Qch}(X)) = \overline{\langle E \rangle}_d \,. \tag{5.3}$$

Proof. By (5.1) and Proposition 5.1.1 it suffices to prove (5.3). If k is perfect then this is [20, Thm 7.39]. To do the general case we use base extension. If l/k is a field extension and $F \in D^b(\mathrm{Qch}(X)$ then we write F_l for pullback of F under $\pi\colon X_l \to X$. Similarly if $G \in D^b(\mathrm{Qch}(X))$ then we write ${}_lG$ for the push forward of G under π.

Since $\bar{k}$ is always perfect there is some $E_1 \in D^b(\mathrm{coh}(X_{\bar{k}}))$ such that $D^b(\mathrm{Qch}(X_{\bar{k}})) = \overline{\langle E_1 \rangle}_d$. Then there is a finite field extension l/k and an object $E_2 \in D^b(\mathrm{coh}(X_l))$ such that $E_1 = (E_2)_{\bar{k}}$. We put $E = {}_l(E_2) \in D^b(\mathrm{coh}(X))$. Hence ${}_k(E_1)$ is a direct sum of copies of E.

Let F be an arbitrary object in $D^b(\mathrm{Qch}(X))$. Then $F_{\bar{k}} \in \langle E_1 \rangle_d$. Hence ${}_k(F_{\bar{k}}) \in \langle {}_k(E_1) \rangle_d = \langle E \rangle_d$. Since F is a summand of ${}_k(F_{\bar{k}})$ we are done. $\square$

[3]Rouquier uses $\langle \mathcal{E} \rangle_\infty$ instead of $\langle \mathcal{E} \rangle$.

5.2. **Smooth descent.** We say that a commutative k-algebra is essentially finitely generated if it is a localization of a finitely generated k-algebra. A k-algebra morphism $R \to S$ of essentially finitely generated k-algebras is essentially smooth if S/R is flat and $S \otimes_R S$ has finite global dimension.

Proposition 5.2.1. *Assume that $R \to S$ is an essentially smooth ring morphism of essentially finitely generated k-algebras. Then there exists $E \in D^b(\mathrm{mod}(R))$ such that*

$$D^b(\mathrm{mod}(S)) = \langle S \otimes_R E \rangle_d \tag{5.4}$$

$$D^b(S) = \overline{\langle S \otimes_R E \rangle}_d \tag{5.5}$$

for certain d.

Proof. As above it is sufficient to prove (5.5). Since $S \otimes_R S$ is noetherian of finite global dimension we have $S \in \langle S \otimes_R S \rangle_e$ for some e. Assume $F \in D^b(S)$. We find immediately $F \in \langle S \otimes_R F \rangle_e$.

Since R is a localization of a finitely generated ring over k we deduce from Proposition 5.1.2 the existence of $E \in D^b(\mathrm{mod}(R))$ such that $D^b(\mathrm{Mod}(R)) = \overline{\langle E \rangle}_f$ for certain f. In particular $S \otimes_R F \in \overline{\langle S \otimes_R E \rangle}_f$. We conclude $F \in \overline{\langle S \otimes_R E \rangle}_{e+f}$. □

5.3. **Graded descent.** Here is our result.

Proposition 5.3.1. *Let R be a graded ring, essentially finitely generated over k. Then there exists $E \in D^b(\mathrm{gr}(R))$ such that*

$$D^b(\mathrm{mod}(R)) = \langle E \rangle_d \,.$$

Proof. One reduces immediately to the case where R is finitely generated over k. One then checks easily that Rouquier's proof of [20, Thm 7.39] furnishes a generator E which may be assumed to be graded (by Proposition 5.1.2 the hypothesis on perfectness of the ground field is unnecessary). □

5.4. **Descent for complete local rings.** Here is our result.

Proposition 5.4.1. *Let (R, m) be a Cohen-Macaulay local ring essentially of finite type over a field k, with residue field k, and let $\widehat{R}$ be its completion. Then there exists $E \in D^b(\mathrm{mod}(R))$ such that*

$$D^b(\mathrm{mod}(\widehat{R})) = \langle \widehat{R} \otimes_R E \rangle_d \,.$$

Proof. Since R essentially of finite type over k it is excellent [9, §7.8] and hence the morphism $R \to \widehat{R}$ is regular [9, (7.8.3)(v)]. By the general Neron desingularization theorem of Popescu (see e.g. [18, Thm 1.8]) $\widehat{R}$ is a filtered colimit of $(R_i)_{i \in I}$ where the R_i are smooth finitely generated R-algebras.

It will be convenient to replace R_i by the localization at the kernel of $R_i \to \widehat{R} \to k$. So now the R_i are local and essentially smooth, essentially of finite type over R.

Claim Assume that M is a finitely generated Cohen-Macaulay $\widehat{R}$-module. Then there exist $i \in I$ and $M_1 \in \operatorname{mod}(R_i)$ such that $M \cong \widehat{R} \overset{L}{\otimes}_{R_i} M_1$.

This claim proves the proposition. Indeed as $\widehat{R}$ is Cohen-Macaulay $D^b(\operatorname{mod}(\widehat{R}))$ is classically generated by $\widehat{R}$ together with the class of maximal Cohen-Macaulay $\widehat{R}$-modules. So by the claim we find that $D^b(\operatorname{mod}(\widehat{R}))$ is classically generated by $D^b(\operatorname{mod}(R_i))_i$. But by Proposition 5.2.1 $D^b(\operatorname{mod}(\widehat{R}))$ is then generated by $D^b(\operatorname{mod}(R))$ which in turn is generated by a single element using Proposition 5.1.2.

We now prove the claim. The idea of the proof is to show that the (infinite) projective resolution of M is determined by a finite amount of data. So it is defined over some R_i.

Let $(x_i)_{i=1,\ldots,d}$ be a system of parameters of $\widehat{R}$. Then $\widehat{R}$ is a finite $C \overset{\text{def}}{=} k[[x_1, \ldots, x_d]]$-module. Since $\widehat{R}$ is Cohen-Macaulay it follows that $\widehat{R}$ is a free C-module. Write $\widehat{R} = \oplus_{i=1}^m C y_i$. We have

$$y_i y_j = \sum_k c_{ijk} y_k \tag{5.6}$$

where the $c_{ijk} \in C$ satisfy the associativity equation

$$\sum_l c_{ijl} c_{lkm} = \sum_l c_{jkl} c_{ilm}\,. \tag{5.7}$$

Since we have assumed that M is maximal Cohen-Macaulay, M is free over C as well. Thus $M = \oplus_{i=1}^n C m_i$. We again have

$$y_i m_j = \sum_k c'_{ijk} m_k$$

where the $c'_{ijk} \in C$ satisfy the associativity equation

$$\sum_l c_{ijl} c'_{lkm} = \sum_l c'_{jkl} c'_{ilm}\,. \tag{5.8}$$

We now consider the bar resolution of M (which is exact since it is contractible over C)

$$\cdots \to \widehat{R} \otimes_C \widehat{R} \otimes_C M \to \widehat{R} \otimes_C M \to M \to 0.$$

The differential

$$d \colon \widehat{R}^{\otimes_C p} \otimes_C M \to \widehat{R}^{\otimes_C p-1} \otimes_C M$$

is given by

$$d(r_1 \otimes \cdots \otimes r_p \otimes m) = r_1 r_2 \otimes r_3 \otimes \cdots \otimes r_p \otimes m - r_1 \otimes r_2 r_3 \otimes \cdots \otimes r_p \otimes m + \cdots \pm r_1 \otimes \cdots \otimes r_p m. \tag{5.9}$$

We now equip $\widehat{R}^{\otimes_C p} \otimes_C M$ with the $\widehat{R}$-basis $m_{i_1\cdots i_p} \overset{\text{def}}{=} 1 \otimes y_{i_1} \otimes \cdots \otimes y_{i_{p-1}} \otimes m_{i_p}$. We have

$$d(m_{i_1\cdots i_p}) = y_{i_1} m_{i_2\cdots i_p} - \sum_k c_{i_1 i_2 k} m_{k i_3 \cdots i_p} + \sum_k c_{i_2 i_3 k} m_{i_1 k i_4 \cdots i_p} - \cdots \pm \sum_k c'_{i_{p-1} i_p k} m_{i_1 \cdots i_{p-2} k}$$

and (5.6)(5.7)(5.8) insure that $d^2 = 0$.

We may now find $i \in I$ as well as $\bar{y}_i$, $\bar{c}_{ijk}$, $\bar{c}'_{ijk} \in R_i$ such that (5.6) (5.7) (5.8) hold in R_i when replacing the symbols by their overlined versions.

Let P_p, $p \geq 0$ be the free R_i-module with basis $\bar{m}_{i_0\cdots i_p}$ where the latter are now just formal symbols. We make $(P_p)_p$ into a complex by defining $d\bar{m}_{i_0\cdots i_p}$ using the formula (5.9), replacing again all symbols by their overlined versions. Then by construction $\widehat{R} \otimes_{R_i} P_\bullet$ is the bar resolution of M. We claim that $P_\bullet$ is exact in homological degree ≥ 1. Since $\widehat{R}_i/R_i$ is faithfully flat, it suffices to consider $\widehat{P}_\bullet = \widehat{R}_i \otimes_{R_i} P_\bullet$. Now since R_i is essentially smooth over R, the same is true for $(\widehat{R} \otimes_R R_i)/\widehat{R}$ and the map $\widehat{R} \to \widehat{R} \otimes_R R_i$ is split using the map $R_i \to \widehat{R}$. Hence $\widehat{R}_i = (\widehat{R} \otimes_R R_i)\widehat{\ } = \widehat{R}[[z_1, \ldots, z_t]]$ in such a way that the map $\widehat{R}_i \to \widehat{R}$ (coming from the map $R_i \to \widehat{R}$) is given by killing $(z_i)_i$. Invoking lemma 5.4.3 below we deduce that $P_\bullet$ is indeed acyclic in degrees ≥ 1.

To finish put $M_1 = \operatorname{coker}(P_1 \to P_0)$. Then $P_\bullet$ is a projective resolution of M_1 and we have $\widehat{R} \overset{L}{\otimes}_{R_i} M_1 = \widehat{R} \otimes_{R_i} P_\bullet \cong M$. □

Remark 5.4.2. It is easy to see that the claim contained in the above proof is in fact true for M an arbitrary element of $D^b(\operatorname{mod}(R))$. Indeed M is quasi-isomorphic to a complex of the form

$$0 \to M' \to Q_l \to \cdots \to Q_0 \to 0$$

with Q_i finitely generated projective and M' maximal Cohen-Macaulay. It now suffices to choose R_i in such a way that the maps in this complex are also defined over R_i.

We have used the following lemma.

Lemma 5.4.3. *Let R be a commutative local noetherian ring and let I be an ideal in the Jacobson radical of R such that for all n one has that I^n/I^{n+1}*

is projective over R/I. Let

$$P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0$$

be a complex of finitely generated projective R-modules such that $R/I \otimes_R P_\bullet$ is exact (in the middle). Then $P_\bullet$ was already exact.

Proof. By induction we show that the cohomology of $P.$ lies in the image of the cohomology of $I^n P.$ for arbitrary n. The lemma now follows by an application of Artin-Rees + Nakayama. □

Remark 5.4.4. Some condition on I is necessary for the result to be true. Consider the following example: $R = k[[x, y]]/(y(x, y))$ and $I = (y)$. Consider the map $R \to R$ given by multiplication by x. This map is not injective (it kills y) but it becomes injective after tensoring by R/I.

6. Some consequences for the singularity category

We can now prove two results mentioned in the introduction.

Proof of Proposition 1.5. We first claim that the functor (1.2) is fully faithful. Let M, N be two maximal Cohen-Macaulay R-modules. Then we have

$$\operatorname{Ext}^1_R(M, N) = \widehat{R} \otimes_R \operatorname{Ext}^1_R(M, N) = \operatorname{Ext}^1_{\widehat{R}}(\widehat{R} \otimes_R M, \widehat{R} \otimes_R N).$$

The first equality follows from the fact that $\operatorname{Ext}^1_R(M, N)$ is finite dimensional. Since stable Ext^1 and ordinary Ext^1 coincide, fully faithfulness follows.

It remains to show that (1.2) is essentially surjective. Now $\underline{\mathrm{MCM}}(\widehat{R})$ is a quotient of $D^b(\operatorname{mod}(\widehat{R}))$ and by Proposition 5.4.1, $D^b(\operatorname{mod}(\widehat{R}))$ is generated by objects in $D^b(\operatorname{mod}(R))$. This finishes the proof. □

Remark 6.1. In the setting of the Proposition 1.5 it is of course not true that every maximal Cohen-Macaulay $\widehat{R}$-module is obtained from one over R. Consider the following example. Let R be the localization of $k[x, y, z, t]/(xy - z^2 - t^2(1+t))$ at (x, y, z, t). This is a three-dimensional terminal singularity of type A_1. It is easily seen to be factorial and hence there are no non projective maximal Cohen-Macaulay modules of rank one. On the other hand $\widehat{R}$ does have non projective maximal Cohen-Macaulay modules of rank one. For example $I = (x, z - t\sqrt{1+t})$. In this case one may check directly that I is a direct summand of the completion of a Cohen-Macaulay R-module. Put $S = R[\sqrt{1+t}]$. Then S/R is Galois. Let σ be the non-trivial automorphism. Put $J = (x, z - t\sqrt{1+t}) \subset S$. We view J as a Cohen-Macaulay R

module of rank two. Then we have

$$\begin{aligned}\widehat{R} \otimes_R J &= \widehat{R} \otimes_R S \otimes_S J \\ &= (\widehat{R} \oplus \widehat{R}) \otimes_S J.\end{aligned}$$

The morphism $S \to \widehat{R} \oplus \widehat{R}$ is given by $\sqrt{1+t} \mapsto (\sqrt{1+t}, -\sqrt{1+t})$. Hence $\widehat{R} \otimes_R J = \widehat{J} \oplus {}_\sigma\widehat{J} = I \oplus \cdots$.

Proof of Proposition 1.4. We claim first that (1.1) is fully faithful. It suffices to check this on generators. Let M, N be two graded maximal Cohen-Macaulay A-modules. Then we have

$$\begin{aligned}\operatorname{Ext}^1_{\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1)}(M,N) &= \bigoplus_n \operatorname{Ext}^1_{\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)}(M, N(n)) \\ &= \bigoplus_n \operatorname{Ext}^1_{\mathrm{Gr}(A)}(M, N(n)) \\ &= \operatorname{Ext}^1_A(M,N) \\ &= \widehat{A} \otimes_A \operatorname{Ext}^1_A(M,N) \\ &= \operatorname{Ext}^1_{\widehat{A}}(\widehat{A} \otimes_A M, \widehat{A} \otimes_A N).\end{aligned}$$

Essential surjectivity follows as in the proof of Proposition 1.5. Let $\widetilde{A}$ be the localization of A at $A_{>0}$. By Proposition 1.5 $D^b(\mathrm{mod}(\widehat{A}))$ is generated by objects in $D^b(\mathrm{mod}(\widetilde{A}))$. Obviously $D^b(\mathrm{mod}(\widetilde{A}))$ is generated by objects in $D^b(\mathrm{mod}(A))$ and finally by Proposition 5.3.1, $D^b(\mathrm{mod}(A))$ is generated by $D^b(\mathrm{gr}(A))$. □

Remark 6.2. In the Iyama-Yoshino examples A is a Veronese of a polynomial ring. In that case there is a substantial shortcut for the essential surjectivity of (1.1) (which is the only subtle point). Put $B = k[x_1, \ldots, x_m]$, $\deg x_i = 1$, and $A = B^{(m)} = \oplus_i B_{mi}$. For $\bar{\jmath} \in \mathbb{Z}/m\mathbb{Z}$ put $A\langle \bar{\jmath} \rangle = \oplus_i B_{mi+j}$.

To prove essential surjectivity we have to prove that $D^b(\mathrm{mod}(\widehat{A}))$ is classically generated by gradable objects. Let $M \in \mathrm{mod}(\widehat{A})$. We consider $\widehat{B}$ as a $\mathbb{Z}/m\mathbb{Z}$-graded$\widehat{A}$-algebra. Put $M' = \widehat{B} \otimes_{\widehat{A}} M \in \mathrm{gr}_{\mathbb{Z}/m\mathbb{Z}} B$. Then M' has a finite resolution by finitely generated projective $\mathbb{Z}/m\mathbb{Z}$-graded $\widehat{B}$-modules. It is easy to see that all such modules are free as $\mathbb{Z}/m\mathbb{Z}$-graded $\widehat{B}$-modules. Restricting to degree $\bar{0} \in \mathbb{Z}/m\mathbb{Z}$ we find that M has a resolution by completions of the $A\langle \bar{\jmath} \rangle$. Since the $A\langle \bar{\jmath} \rangle$ are obviously graded we are done.

7. The singularity category of graded Gorenstein rings

7.1. **Orlov's results.** Let $A = k + A_1 + A_2 + \cdots$ be a commutative finitely generated graded k-algebra. As in [1] we write $\mathrm{qgr}(A)$ for the quotient of $\mathrm{gr}(A)$ by the Serre subcategory of graded finite length modules. We write $\pi\colon \mathrm{gr}(A) \to \mathrm{qgr}(A)$ for the quotient functor. If A is generated in degree one and $X = \operatorname{Proj} A$ then by Serre's theorem [21] we have $\mathrm{coh}(X) = \mathrm{qgr}(A)$.

Now assume that A is Gorenstein. Then we have $\mathrm{RHom}_A(k, A) \cong k(a)[-d]$ where d is the Krull dimension of R and $a \in \mathbb{Z}$. The number a is called the Gorenstein parameter of A (see [16, Definition 2.1]).

Example 7.1.1. If A is a polynomial ring in n variables (of degree one) then $d = n$, $a = n$.

For use below we record another incarnation of the Gorenstein parameter. Let A' be the graded k-dual of A. Then

$$R\Gamma_{A_{>0}}(A) \cong A'(a)[-d] \tag{7.1}$$

where $R\Gamma_{A_{>0}}$ denotes cohomology with support in the ideal $A_{>0}$.

The following is a particular case of [16, Thm 2.5].

Theorem 7.1.2. *If $a \geq 0$ then there are fully faithful functors*

$$\Phi_i\colon \underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to D^b(\mathrm{qgr}(A))$$

such that for $\mathcal{T}_i = \Phi_i \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ there is a semi-orthogonal decomposition

$$D^b(\mathrm{qgr}(A)) = \langle \pi A(-i+a+1), \ldots, \pi A(-i), \mathcal{T}_i \rangle\,.$$

Hence under the hypotheses of the theorem we obtain in particular that

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \cong {}^{\perp}\langle \pi A(-i+a+1), \ldots, \pi A(-i)\rangle \subset D^b(\mathrm{qgr}(A))$$

for arbitrary i.

7.2. **The action of the shift functor on the singularity category.** Unfortunately the functors Φ_i introduced in the previous section are not compatible with ?(1). Our aim in this section is to understand how ?(1) acts on the image of Φ_i. This requires us to dig deeper into Orlov's construction which has the unusual feature of depending on the category $D^b(\mathrm{gr}_{\geq i} A)$ where $\mathrm{gr}_{\geq i} A$ are the finitely generated graded A-modules with non zero components concentrated in degrees $\geq i$. The quotient functor

$$D^b(\mathrm{gr}_{\geq i} A) \hookrightarrow D^b(\mathrm{gr}\, A) \xrightarrow{\pi} D^b(\mathrm{qgr}\, A)$$

has a right adjoint $R\omega_i A$. Its image is denoted by $\mathcal{D}_i$.

We let P_i be the graded projective A-module of rank one generated in degree i (i.e. $P_i = A(-i)$). Likewise S_i is the simple A-module concentrated in degree i. As in Orlov [16] we put $\mathcal{P}_{\geq i} = \langle (P_j)_{j \geq i} \rangle$, $\mathcal{S}_{\geq i} = \langle (S_j)_{j \geq i} \rangle$ and obvious variants with other types of inequality signs. In [16] it is proved that the image $\mathcal{T}_i$ of Φ_i is the left orthogonal to $\mathcal{P}_{\geq i}$ inside $D^b(\operatorname{gr}_{\geq i} A)$. The identification of $\mathcal{T}_i$ with the graded singularity category is through the composition

$$\mathcal{T}_i \cong D^b(\operatorname{gr}_{\geq i} A)/\mathcal{P}_{\geq i} \cong D^b(\operatorname{gr} A)/\operatorname{perf}(A) \cong \underline{\mathrm{MCM}}_{\mathrm{gr}}(A). \tag{7.2}$$

Assume $a \geq 0$. Then the relation between $\mathcal{T}_i$, $\mathcal{D}_i$ is given by the following semi-orthogonal decompositions

$$D^b(\operatorname{gr} A) = \langle \mathcal{S}_{<i}, \overbrace{\mathcal{P}_{\geq i+a}, \underbrace{P_{i+a-1}, \ldots, P_i, \mathcal{T}_i}_{\mathcal{D}_i \cong D^b(\operatorname{qgr}(A))}}^{D^b(\operatorname{gr}_{\geq i} A)} \rangle \, .$$

This is a refinement of Theorem 7.1.2.

The category $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ comes equipped with the shift functor ?(1). We denote the induced endofunctor on $\mathcal{T}_i$ by σ_i. We will now compute it.

Lemma 7.2.1. *For $M \in \mathcal{T}_i \subset D^b(\operatorname{qgr}(A))$ we have*

$$\sigma_i M = \operatorname{cone}(\operatorname{RHom}_{\operatorname{qgr}(A)}(\pi A(-i), M) \otimes_k \pi A(-i+1) \to M(1)) \tag{7.3}$$

where the symbol "cone" is to be understood in a functorial sense, for example by computing it on the level of complexes after first replacing M by an injective resolution.

Proof. Let $N \in \mathcal{T}_i \subset D^b(\operatorname{gr}(A))$. To compute $\sigma_i N$ we see by (7.2) that we have to find $\sigma_i N \in \mathcal{T}_i$ such that $\sigma_i N \cong N(1)$ up to projectives. It is clear we should take

$$\begin{aligned} \sigma_i N &= \operatorname{cone}(\operatorname{RHom}_{\operatorname{gr}(A)}(P_{i-1}, N(1)) \otimes_k P_{i-1} \to N(1)) \\ &= \operatorname{cone}(\operatorname{RHom}_{\operatorname{gr}(A)}(P_i, N) \otimes_k P_{i-1} \to N(1)). \end{aligned}$$

Now we note that the RHom can be computed in $\mathcal{D}_i \cong D^b(\operatorname{qgr}(A)$. Furthermore since the result lies in $\mathcal{T}_i \subset \mathcal{D}_i$ we can characterize it uniquely by applying π to it. Since π commutes with ?(1) we obtain (7.3) with $M = \pi N$. □

7.3. **The Serre functor for a graded Gorenstein ring.** Let A, a, d be as above but now assume that A has an isolated singularity and let $M, N \in \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$. Then by a variant of [10, Thm 8.3] we have a canonical graded isomorphism

$$\operatorname{Ext}^d_A(\underline{\operatorname{Hom}}_A(M,N), A) \cong \underline{\operatorname{Hom}}_A(N, M[d-1])$$

and furthermore an appropriate version of local duality yields

$$\operatorname{Ext}^d_A(\underline{\operatorname{Hom}}_A(M,N), A) = \underline{\operatorname{Hom}}_A(M,N)^*(a).$$

In other words we find

$$\underline{\mathrm{Hom}}_A(M,N)^* = \underline{\mathrm{Hom}}_A(N, M[d-1](-a))$$

and hence the Serre functor S on $\underline{\mathrm{MCM}}(A)$ is given by $?[d-1](-a)$.

It is customary to write $S = \tau[1]$ so that we have the usual formula

$$\underline{\mathrm{Hom}}_A(M,N)^* = \mathrm{Ext}^1(N, \tau M).$$

In this setting we find

$$\tau = ?[d-2](-a). \tag{7.4}$$

7.4. **The Gorenstein parameter of Veronese subring.** We remind the reader of the following well-known result.

Proposition 7.4.1. *Let B be a polynomial ring in n variables of degree one. Assume $m \mid n$ and let $B^{(m)}$ be the corresponding Veronese subring of B. I.e. $B^{(m)}_i = B_{mi}$. Then $B^{(m)}$ is Gorenstein with Gorenstein parameter n/m.*

Proof. The Gorenstein property is standard. To compute the Gorenstein invariant we first let A be the "blown up" Veronese. I.e.

$$A_i = \begin{cases} B_i & \text{if } m \mid i \\ 0 & \text{otherwise.} \end{cases}$$

Let $a, b = n$ be respectively the Gorenstein parameters of A and B. If M is a B-module write M^+ for $\oplus_i M_{mi}$, considered as graded A-module. We have

$$\begin{aligned} A'(a)[-n] &= R\Gamma_{A_{>0}}(A) \qquad \text{(see (7.1))} \\ &= R\Gamma_{A_{>0}}(B)^+ \\ &= R\Gamma_{B_{>0}}(B)^+ \\ &= (B'(b)[-n])^+ \\ &= A'(b)[-n]\,. \end{aligned}$$

In the 3rd equality we have used that local homology is insensitive to finite extensions. We deduce $a = b = n$. Since $B^{(m)}$ is obtained from A by dividing the grading by m obtain n/m as Gorenstein parameter for $B^{(m)}$. ☐

Remark 7.4.2. In characteristic zero we could have formulated the result for invariant rings of finite subgroups of $\mathrm{Sl}_n(k)$ (with the same proof). However in finite characteristic Veronese subrings are not always invariant rings (consider the case where the characteristic divides m).

8. The Iyama-Yoshino examples (again)

8.1. **Example 1.1.** Let $B = k[x_1, x_2, x_3]$ and $A = B^{(3)}$. We have $X \overset{\text{def}}{=} \operatorname{Proj} A = \operatorname{Proj} B = \mathbb{P}^2$. By Proposition 7.4.1 A has Gorenstein invariant 1.

Unfortunately we have to deal with the unpleasant notational problem that the shift functors on $\operatorname{coh}(\mathbb{P}^2)$ coming from A and B do not coincide. To be consistent with the sections 7.1, 7.2 we will denote them respectively by $?(1)$ and $?\{1\}$. Thus $?(1) = ?\{3\}$. Note that this choice is rather unconventional.

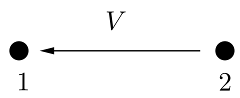
According to Theorem 7.1.2 we have a semi-orthogonal decomposition

$$D^b(\operatorname{coh}(X)) = \langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{T}_0 \rangle .$$

From the fact that $D^b(\operatorname{coh}(X))$ has a strong exceptional collection $\mathcal{O}_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}\{1\}$. $\mathcal{O}_{\mathbb{P}^2}\{2\}$ we deduce that there is a semi-orthogonal decomposition

$$\mathcal{T}_0 = \langle \mathcal{O}_{\mathbb{P}^2}\{1\}, \mathcal{O}_{\mathbb{P}^2}\{2\} \rangle .$$

In particular $\operatorname{RHom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}\{1\} \oplus \mathcal{O}_{\mathbb{P}^2}\{2\}, -)$ defines an equivalence between $\mathcal{T}_0$ and the representations of the quiver Q_3



where $V = kx_1 + kx_2 + kx_3$ and where $\mathcal{O}_{\mathbb{P}^2}\{i\}$ corresponds to the vertex labeled by i. By (7.4) the Auslander-Reiten translate on $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ is given by $?[1](-1)$. In other words: the shift functor on $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ is given by $(\tau[-1])^{-1}$. By Proposition 5.3.1 we find (using $R = \widehat{A}$)

$$\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1) \cong D^b(\operatorname{mod}(kQ_3))/(\tau[-1])$$

which is what we wanted to show.

Remark 8.1.1. Note that this in this example we had no need for the somewhat subtle formula (7.3).

8.2. **Example 1.2.** We use similar conventions as in the previous section, Let $B = k[x_1, x_2, x_3, x_4]$ and $A = B^{(2)}$. We have $X = \operatorname{Proj} A \cong \operatorname{Proj} B = \mathbb{P}^3$ and we denote the corresponding shift functors by $?(1)$, $?\{1\}$ so that $?(1) = ?\{2\}$. By Proposition 7.4.1 A has Gorenstein invariant 2. By Theorem 7.1.2 we have a semi-orthogonal decomposition

$$D^b(\operatorname{coh}(X)) = \langle \mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}\{2\}, \mathcal{T}_{-1} \rangle .$$

Now $D^b(\operatorname{coh}(X))$ has a strong exceptional collection $\mathcal{O}_{\mathbb{P}^3}$, $\mathcal{O}_{\mathbb{P}^3}\{1\}$. $\mathcal{O}_{\mathbb{P}^3}\{2\}$, $\mathcal{O}_{\mathbb{P}^3}\{3\}$. This sequence is geometric [4, Prop. 3.3] and hence by every mutation is strongly exceptional [4, Thm. 2.3]. We get in particular the following strongly exceptional collection

$\mathcal{O}_{\mathbb{P}^3}$, $\mathcal{O}_{\mathbb{P}^3}\{2\}$. $\Omega^*_{\mathbb{P}^3}\{1\}$, $\mathcal{O}_{\mathbb{P}^3}\{3\}$ where $\Omega_{\mathbb{P}^3}$ is defined by the exact sequence

$$0 \to \Omega_{\mathbb{P}^3} \to V \otimes \mathcal{O}_{\mathbb{P}^3}\{-1\} \to \mathcal{O}_{\mathbb{P}^3} \to 0 \tag{8.1}$$

where $V = kx_1 + kx_2 + kx_3 + kx_4$. Thus there is a semi-orthogonal decomposition

$$\mathcal{T}_{-1} = \langle \Omega^*_{\mathbb{P}^3}\{1\}, \mathcal{O}_{\mathbb{P}^3}\{3\} \rangle .$$

An easy computation yields

$$\operatorname{RHom}_{\mathbb{P}^3}(\Omega^*_{\mathbb{P}^3}\{1\}, \mathcal{O}_{\mathbb{P}^3}\{3\}) = \wedge^2 V.$$

$\operatorname{RHom}_{\mathbb{P}^3}(\Omega^*_{\mathbb{P}^3}\{1\} \oplus \mathcal{O}_{\mathbb{P}^3}\{3\}, -)$ defines an equivalence between $\mathcal{T}_{-1}$ and the representations of the quiver Q_6

$$\underset{1}{\bullet} \xleftarrow{\;\wedge^2 V\;} \underset{2}{\bullet}$$

Put $W = \wedge^2 V$ and choose an arbitrary trivialization $\wedge^4 V \cong k$. Let $\pi \colon W \to W^*$ be the resulting (self-adjoint) isomorphism. We are in the setting of §4.2 and hence can define $\tau^{1/2}$ as acting on the derived category of Q_6.

We will now compute $\sigma_{-1}(\Omega^*_{\mathbb{P}^3}\{1\})$, $\sigma_{-1}(\mathcal{O}_{\mathbb{P}^3}\{3\})$. An easy computation yields

$$\operatorname{RHom}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}\{2\}, \Omega^*_{\mathbb{P}^3}\{1\}) = V^*$$
$$\operatorname{RHom}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}\{2\}, \mathcal{O}_{\mathbb{P}^3}\{3\}) = V.$$

Using the formula (7.3) we find

$$\sigma_{-1}(\mathcal{O}_{\mathbb{P}^3}\{3\}) = \operatorname{cone}(V \otimes \mathcal{O}_{\mathbb{P}^3}\{4\} \to \mathcal{O}_{\mathbb{P}^3}\{5\}) = \Omega_{\mathbb{P}^3}\{5\}[1] \tag{8.2}$$

$$\sigma_{-1}(\Omega^*_{\mathbb{P}^3}\{1\}) = \operatorname{cone}(V^* \otimes \mathcal{O}_{\mathbb{P}^3}\{4\} \to \Omega^*_{\mathbb{P}^3}\{3\}) = \mathcal{O}_{\mathbb{P}^3}\{3\}[1] \tag{8.3}$$

where in the second line we have used the dual version of (8.1).

Let P_i be the projective representation of Q_6 generated in vertex i. The endofunctor on $D^b(\operatorname{mod}(kQ_6))$ induced by σ_{-1} will be denoted by the same letter. We will now compute it. From (8.3) we deduce immediately $\sigma_{-1}(P_1) = P_2[1]$. To analyze (8.2) we note that a suitably shifted slice of the Koszul sequence has the form

$$0 \to \wedge^4 V \otimes \Omega^*_{\mathbb{P}^3}\{1\} \to \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^3}\{3\} \to \Omega_{\mathbb{P}^3}\{5\} \to 0.$$

Thus $\Omega_{\mathbb{P}^3}\{5\}$ corresponds to the cone of

$$\wedge^4 V \otimes P_1 \to \wedge^2 V \otimes P_2$$

which is easily seen to be equal to $\wedge^4 V \otimes \tau^{-1} P_1$.

If we use our chosen trivialization $\wedge^4 V \cong k$ then we see that at least on objects σ_{-1} coincides with $\tau^{-1/2}[1]$. It is routine to extend this to an isomorphism of functors by starting with a bounded complex of projectives in $\mathrm{mod}(kQ_6)$.

By Proposition 5.3.1 we find (using $R = \widehat{A}$)

$$\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1) \cong D^b(\mathrm{mod}(kQ_6))/(\tau^{1/2}[-1])$$

which is what we wanted to show.

9. A remark on gradability of rigid modules

We keep notations as in the previous section. Since in the Iyama-Yoshino examples $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ is the derived category of a hereditary category the functor

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1)$$

is essentially surjective [13] and hence

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to \underline{\mathrm{MCM}}_{\mathrm{gr}}(\widehat{A})$$

is also essentially surjective. In more complicated examples there is no reason however why this should be the case. Nevertheless we have the following result which is probably well-known.

Proposition 9.1. *Assume that k has characteristic zero. Let $A = k + A_1 + A_2 + \cdots$ be a left noetherian graded k-algebra. Put $R = \widehat{A}$. Let $M \in \mathrm{mod}(R)$ be such that $\mathrm{Ext}^1_R(M, M) = 0$. Then M is the completion of a finitely generated graded A-module N.*

In the rest of this section we let the notations and hypotheses be as in the statement of the proposition (in particular k has characteristic zero). We denote the maximal ideal of R by m.

Let E be the Euler derivation on A and R. I.e. on A we have $E(a) = (\deg a)a$ and we extend E to R in the obvious way. If $M \in \mathrm{mod}(R)$ then we will define an Euler connection as a k-linear map $\nabla\colon M \to M$ such that $\nabla(am) = E(a)m + a\nabla(m)$. If $M = \widehat{N}$ for N a graded A-module then M has an associated Euler connection by extending $\nabla(n) = (\deg n)n$ for n a homogeneous element of N.

Lemma 9.2. *Let M be a finitely generated R module. Then M has an Euler connection if and only if M is the completion of a finitely generated graded A-module.*

Proof. We have already explained the easy direction. Conversely assume that M has an Euler connection. For each n we have that M/m^nM is finite dimensional and hence it decomposes into generalized eigenspaces for ∇.

$$M/m^nM = \prod_{\alpha \in k} (M/m^nM)_\alpha \qquad \text{(finite product).}$$

Considering right exact sequences

$$(m/m^2)^{\otimes n} \otimes M/mM \to M/m^{n+1}M \to M/m^nM \to 0$$

we easily deduce that the multiplicity of a fixed generalized eigenvalue in M/m^nM stabilizes a $n \to \infty$. Thus $M = \prod_{\alpha\in k} M_\alpha$ where M_α is a generalized eigenspace with eigenvalue α. We put $N' = \oplus_\alpha M_\alpha$. Then N' is noetherian since obviously any ascending chain of graded submodules of N' can be transformed into an ascending chain of submodules in M. If particular N' is finitely generated and we have $M = \widehat{N}'$.

Now N' is k-graded and not $\mathbb{Z}$-graded. But we can decompose N' along $\mathbb{Z}$-orbits and then by taking suitable shifts we obtain a $\mathbb{Z}$-graded module with the same completion as N'. □

Proof or Proposition 9.1. Let $\epsilon^2 = 0$ and consider $M[\epsilon]$ where A acts via $a \cdot m = (a + E(a)\epsilon)m$. We have a short exact sequence of A-modules

$$0 \to M\epsilon \to M[\epsilon] \to M \to 0$$

which is split by hypotheses. Denote the splitting by $m+\nabla(m)\epsilon$. For $a \in A$ we have

$$am + \nabla(am)\epsilon = (a + E(a)\epsilon)(m + \nabla(m)\epsilon)$$

and hence

$$\nabla(am) = E(a)m + a\nabla(m).$$

Hence ∇ is an Euler connection and so we may invoke Lemma 9.2 to show that $M = \widehat{N}$. □

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