

ON THE DISTRIBUTION OF SOME MEANS CONCERNING THE EULER FUNCTION

JEAN-MARC DESHOULLERS AND FLORIAN LUCA

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To Professor Wladislaw Narkiewicz for his seventieth birthday

1. INTRODUCTION

Let $\phi(n)$ be the Euler function of the positive integer n . At the Czech-Slovak Number Theory Conference in Smolenice in August 2007, the second author asked whether the sequences of general term

$$a_n = \frac{1}{n} \sum_{m \leq n} \phi(m) \quad \text{or} \quad g_n = \left(\prod_{m \leq n} \phi(m) \right)^{1/n}$$

which give the arithmetic and the geometric means of the first n values of the Euler function, respectively, are uniformly distributed modulo 1. A. Schinzel modified these questions by asking whether these sequences are dense modulo 1. This question was repeated at the meeting on Uniform Distribution in Luminy in January 2008. In this paper, we give an affirmative answer to Schinzel's question. Our results are the following:

Theorem 1. *Each one of the sequences of general terms:*

$$(i) \quad s_n = \sqrt{\sum_{m \leq n} \phi(m)};$$

$$\begin{aligned}
\text{(ii)} \quad a_n &= \frac{1}{n} \sum_{m \leq n} \phi(m); \\
\text{(iii)} \quad g_n &= \left(\prod_{m \leq n} \phi(m) \right)^{1/n}; \\
\text{(iv)} \quad h_n &= \frac{n}{\sum_{m \leq n} \frac{1}{\phi(m)}}
\end{aligned}$$

is dense modulo 1.

Regarding the sequence (ii), it was shown in [6] that for a large positive real number x the number of $n \leq x$ such that s_n is an integer is $\ll x/(\log x)^{0.0003}$. The exponent 0.0003 was later improved to 0.2 in [1]. These results were extended by I. Kátai [4] to the instance when the Euler function $\phi(n)$ is replaced by a multiplicative function satisfying some technical conditions. This includes the sum of the divisors function $\sigma(n)$, for example. Some of the above sequences when the Euler function $\phi(n)$ is replaced by the n th prime number function p_n have been treated in [5] and [2]. For example, in [5] it was shown that the set of n such that $\sum_{m \leq n} p_m$ is a square is of asymptotic density zero although no nontrivial upper bound on the counting function of the set of such positive integers $n \leq x$ was given. This was achieved in [2], where it was shown that for a large positive real number x the number of such positive integers $n \leq x$ is $\leq x \exp(-c_0(\log x)^{3/5}(\log \log x)^{-1/5})$ for some positive constant c_0 unconditionally and $\ll (x \log x)^{5/6}$ under the Riemann Hypothesis. A similar result was obtained for the number of positive integers $n \leq x$ such that the arithmetic mean of the first n primes is an integer. It was also shown in [2] that both sequences of general terms $\sqrt{\sum_{m \leq n} p_m}$ and $\frac{1}{n} \sum_{m \leq n} p_m$ are uniformly distributed modulo 1.

In what follows, we use p and q for prime numbers. We also use the Landau symbols O and o as well as the Vinogradov symbols \gg , \ll and \asymp with their usual meaning.

2. PROOF OF THEOREM 1

(i) We let $\varepsilon \in (0, 1/10)$ be arbitrarily small and let M be a positive integer depending on ε to be fixed later. For $j \in \{1, \dots, M\}$ we put $c_j = \phi(j)/j$. Let $\alpha = 3/\pi^2$ and choose finite disjoint sets of primes \mathcal{P}_j consisting of primes $p > M$ such that

$$(1) \quad \prod_{p \in \mathcal{P}_j} \left(1 - \frac{1}{p}\right) \in \left[\frac{\sqrt{\alpha\varepsilon}}{c_j}, \frac{2\sqrt{\alpha\varepsilon}}{c_j}\right].$$

We shall show that this is possible when ε is sufficiently small: this will come from the condition

$$(2) \quad 2\sqrt{\alpha\varepsilon} < c_j \quad \text{holds for all } j = 1, \dots, M,$$

and the fact that the infinite product $\prod_{p \geq 2} \left(1 - \frac{1}{p}\right)$ diverges to zero. Put $P_j = \prod_{p \in \mathcal{P}_j} p$. Let x be a positive real number whose natural logarithm exceeds the largest member in $\bigcup_{j=1}^M \mathcal{P}_j$ and let

$$\mathcal{Q} = \{M < q \leq \log x\} \setminus \bigcup_{j=1}^M \mathcal{P}_j.$$

Put $Q = \prod_{q \in \mathcal{Q}} q$. We use the Chinese Remainder Lemma to deduce that there exists a positive integer n such that

$$(3) \quad n \equiv 0 \pmod{M!Q} \quad \text{and} \quad n \equiv -j \pmod{P_j} \quad \text{for all } j = 1, \dots, M.$$

The above system is solvable and all its positive integer solutions n form an arithmetic progression $n_0 \pmod{N}$, where

$$N = M! \left(\prod_{j=1}^M P_j \right) Q.$$

Here, we take n_0 to be the least positive integer in the above progression. By the Prime Number Theorem, keeping M fixed and taking x large, we get that

$$N = M! \prod_{M < p < \log x} p = M! \exp((1 + o(1)) \log x), \quad \text{as } x \rightarrow \infty,$$

therefore we may choose x to be sufficiently large such that $N < x^2$. Let $n \equiv n_0 \pmod{N}$ be in the interval $[x^2, 2x^2)$. Write $n = n_0 + N\ell$ for some $\ell \geq 1$. Observe that for $j \in \{1, \dots, M\}$ the positive integer $n + j$ is a multiple of both j and P_j , which are coprime. Further, if $p \leq \log x$ is a prime and $p \mid n + j$, then $p \mid jP_j$. Indeed, let $p \leq \log x$ be a prime factor of $n + j$. If $p \leq M$ or $p \in \mathcal{Q}$, then since $M!Q \mid n$, we get that $p \mid n$, therefore $p \mid (n + j) - n = j$, which is what we wanted. If $p > M$ and $p \notin \mathcal{P}_j$, then $p \in \mathcal{P}_i$ for some $i \neq j$. System (3) leads to $p \mid n + i$, therefore $p \mid (n + i) - (n + j) = i - j$, which is impossible because $0 < |i - j| < M < p$. Note that since $n + j \leq 2x^2$, it follows that

$$\omega(n + j) \ll \frac{\log(n + j)}{\log \log(n + j)} \ll \frac{\log x}{\log \log x}.$$

Here, for a positive integer m we write $\omega(m)$ for the number of distinct prime factors of m . Hence,

$$\begin{aligned}
\frac{\phi(n+j)}{n+j} &= \prod_{p|n+j} \left(1 - \frac{1}{p}\right) = \prod_{p|jP_j} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n+j \\ p \nmid jP_j}} \left(1 - \frac{1}{p}\right) \\
&= \frac{\phi(j)}{j} \prod_{p \in \mathcal{P}_j} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{\log x}\right)^{O\left(\frac{\log x}{\log \log x}\right)} \\
(4) \quad &= \frac{\phi(j)}{j} \frac{\phi(P_j)}{P_j} \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \in \left[\frac{\sqrt{\alpha\varepsilon}}{2}, 3\sqrt{\alpha\varepsilon}\right],
\end{aligned}$$

where the last containment holds by (1) provided that x is sufficiently large with respect to ε and M .

We put $S(n) = \sum_{m \leq n} \phi(m)$. We use the elementary estimate [3]

$$(5) \quad S(n) = \alpha n^2 + O(n \log n),$$

as well as the fact that

$$(6) \quad \sqrt{1+t} = 1 + \frac{t}{2} + O(t^2),$$

to deduce that for each fixed $j \in \{0, \dots, M-1\}$ we have

$$\begin{aligned}
\sqrt{S_{n+j+1}} &= \sqrt{S_{n+j}} \left(1 + \frac{\phi(n+j+1)}{S_{n+j}}\right)^{1/2} \\
(7) \quad &= \sqrt{S_{n+j}} + \frac{\phi(n+j+1)}{2\sqrt{S_{n+j}}} + O\left(\frac{\phi(n+j)^2}{S_{n+j}^{3/2}}\right).
\end{aligned}$$

Clearly, by estimates (5) and (6), we have

$$\frac{n+j+1}{2\sqrt{S_{n+j}}} = \frac{1}{2\sqrt{\alpha}} \left(1 + O\left(\frac{\log n}{n}\right)\right) = \frac{1}{2\sqrt{\alpha}} \left(1 + O\left(\frac{\log x}{x^2}\right)\right),$$

while the error term in the above approximation (7) is

$$\ll \frac{x^4}{x^6} = \frac{1}{x^2}.$$

Thus, for large x , we get that

$$\begin{aligned} s_{n+j+1} - s_{n+j} &= \sqrt{S_{n+j+1}} - \sqrt{S_{n+j}} \\ &= \frac{\phi(n+j+1)}{n+j+1} \left(\frac{1}{2\sqrt{\alpha}} + O\left(\frac{\log x}{x^2}\right) \right) + O\left(\frac{1}{x^2}\right) \\ &= \frac{\phi(n+j+1)}{2\sqrt{\alpha}(n+j+1)} + O\left(\frac{\log x}{x^2}\right). \end{aligned}$$

Using containment (4), we get that

$$s_{n+j+1} - s_{n+j} \in \left[\frac{\varepsilon}{5}, 3\varepsilon \right]$$

holds for all $j = 0, \dots, M-1$ provided that x is sufficiently large. We now choose $M = \lfloor 5/\varepsilon \rfloor + 1$. The above estimates show that

$$s_{n+M} - s_n = \sum_{j=0}^{M-1} (s_{n+j+1} - s_{n+j}) > 1.$$

This easily implies that for each interval $I \subset [0, 1]$ of length $> 3\varepsilon$ there exists $j \in \{1, \dots, M\}$ such that $\{s_{n+j}\} \in I$, which finishes the proof of (i) because $\varepsilon > 0$ was arbitrary. It remains to check that (2) is fulfilled with this choice of M versus ε . However, from the minimal order of the Euler function in the interval $[1, M]$, we know that if $M \geq 3$, then

$$(8) \quad \frac{\phi(m)}{m} \geq \frac{\beta}{\log \log M} \quad \text{holds for all } m \in [1, M],$$

where β is some positive constant. Thus, we have that the inequality

$$c_j \geq \frac{\beta}{\log \log(3/\varepsilon)} \quad \text{holds for all } j = 1, \dots, M.$$

Hence, in order for inequality (2) to hold, it suffices that

$$2\sqrt{\alpha}\varepsilon < \frac{\beta}{\log \log(3/\varepsilon)}$$

and this last inequality is certainly fulfilled if ε is sufficiently small.

(ii) We follow the same method as at (i). First, we note that for each nonnegative integer a there is a positive integer b such that

$$(9) \quad \sum_{a < j \leq b} \frac{\phi(j)}{j} > \frac{4(b-a)}{\pi^2}.$$

Indeed, if this were not so, then for some positive integer a we would have

$$\frac{1}{b-a} \sum_{a < j \leq b} \frac{\phi(j)}{j} \leq \frac{4}{\pi^2}$$

for all $b > a$. However, passing to the limit when b tends to infinity, the left hand side above tends to

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} > \frac{4}{\pi^2},$$

which is a contradiction. In fact, using the elementary estimate

$$(10) \quad \sum_{n \leq y} \frac{\phi(n)}{n} = 2\alpha y + O(\log y)$$

valid for all $y \geq 2$ we get easily that for every $a \geq 2$ the minimal b satisfying (9) satisfies the estimate $b = a + O(\log a)$.

Let now $(m_i)_{i \geq 0}$ be the increasing sequence of integers given by $m_0 = 0$ and for $i \geq 0$, m_{i+1} is the smallest positive integer b such that the inequality (9) holds with $a = m_i$. The above remark shows that the estimate

$$m_{i+1} = m_i + O(\log m_i) \quad \text{holds for all } i \geq 2.$$

Put

$$T_i = \frac{\sum_{m_i < j \leq m_{i+1}} c_j}{\alpha(m_{i+1} - m_i)} > \frac{4}{3}.$$

Now let $\varepsilon \in (0, 1/10)$ be small and let L be the minimal positive integer such that $M = m_L > 7/\varepsilon$. Let $\varepsilon_1 = 7/M \in (0, 1/10)$. For $j \in \{1, \dots, M\}$ we let \mathcal{P}_j be finite disjoint sets of prime numbers exceeding M with the following property. Let $i \in \{0, \dots, L-1\}$ be such that $j \in [m_i + 1, m_{i+1}]$. We impose that

$$\prod_{p \in \mathcal{P}_j} \left(1 - \frac{1}{p}\right) - \frac{1}{T_i} \in (\varepsilon_1, 2\varepsilon_1).$$

Note that since $T_i > 4/3$ and $\varepsilon_1 < 1/10$, we have that $1/T_i + 2\varepsilon_1 < 1$. Put as at (i) $P_j = \prod_{p \in \mathcal{P}_j} p$. We now let again x be a large positive real number whose logarithm exceeds all primes in $\bigcup_{j=1}^M \mathcal{P}_j$ put \mathcal{Q} for the set of primes $M < q \leq \log x$ which do not belong to any of the \mathcal{P}_j 's for $j = 1, \dots, M$ and set $Q = \prod_{q \in \mathcal{Q}} q$. Let n be a positive integer satisfying the system of congruences (3). By the arguments at (i) it follows that if x is large, then we may find such an integer n in the interval $[x^2, 2x^2)$. Further, every prime

factor of $n + j$ either divides jP_j , or exceeds $\log x$. Let again i be such that $j \in [m_i + 1, m_{i+1}]$. Thus, for some $\theta_j \in (1, 2)$ we have

$$\begin{aligned} \frac{\phi(n+j)}{n+j} &= \prod_{p|jP_j} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n+j \\ p \nmid jP_j}} \left(1 - \frac{1}{p}\right) \\ &= \frac{\phi(j)}{j} \left(\frac{1}{T_i} + \theta_j \varepsilon_1\right) \left(1 - \frac{1}{\log x}\right)^{O\left(\frac{\log x}{\log \log x}\right)} \\ &= \frac{\phi(j)}{j} \left(\frac{1}{T_i} + \theta_j \varepsilon_1\right) \left(1 + O\left(\frac{1}{\log \log x}\right)\right) \\ &= \frac{\phi(j)}{jT_i} + \lambda_j \varepsilon_1, \end{aligned}$$

where $\lambda_j \in (c_j/2, 3c_j)$ provided that x is sufficiently large with respect to ε . Observe that

$$\begin{aligned} a_{n+m_{i+1}} - a_{n+m_i} &= \frac{S_{n+m_{i+1}}}{n+m_{i+1}} - \frac{S_{n+m_i}}{n+m_i} \\ &= \sum_{m_i < j \leq m_{i+1}} \frac{\phi(n+j)}{n+m_{i+1}} - \frac{(m_{i+1} - m_i)}{(n+m_i)(n+m_{i+1})} \sum_{\ell \leq n+m_i} \phi(\ell) \\ &= \sum_{m_i < j \leq m_{i+1}} \frac{\phi(n+j)}{n+j} \left(1 + O\left(\frac{1}{n}\right)\right) \\ &\quad - \frac{(m_{i+1} - m_i)}{(n+m_i)^2} (\alpha(n+m_i)^2 + O(n \log n)) \left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \sum_{m_i < j \leq m_{i+1}} \left(\frac{\phi(j)}{jT_i} + \lambda_j \varepsilon_1\right) - (m_{i+1} - m_i) \alpha + O\left(\frac{\log x}{x^2}\right) \\ &= \left(\sum_{m_i < j \leq m_{i+1}} \lambda_j\right) \varepsilon_1 + O\left(\frac{\log x}{x^2}\right). \end{aligned}$$

In the above calculation, we used estimate (5). Furthermore, the constants implied by the above O 's depend on ε . Put $d_i = \sum_{m_i < j \leq m_{i+1}} c_j$ for $i = 0, 1, \dots, L-1$. Since $\varepsilon_1 = 7/M$ and $\lambda_j \geq c_j/2$, we get, by the minimal order

of the Euler function (8), that

$$d_i \varepsilon_1 \geq \frac{7\beta}{2M \log \log M} \gg \frac{\varepsilon}{\log(\log(7/\varepsilon))},$$

while from the growth condition on the sequence $(m_i)_{i \geq 0}$, we also have

$$d_i \varepsilon_1 \leq \frac{7(m_{i+1} - m_i)}{M} \ll \frac{\log M}{M} \ll \varepsilon \log(7/\varepsilon).$$

Hence, if ε is sufficiently small, then the inequalities

$$d_i \varepsilon_1 \in [\varepsilon^2, \varepsilon^{1/2}]$$

hold for all $i = 0, \dots, L-1$. In particular, if x is sufficiently large with respect to ε , then the containments

$$a_{n+m_{i+1}} - a_{n+m_i} \in [d_i \varepsilon_1/2, 2d_i \varepsilon_1]$$

hold for all $i = 0, \dots, L-1$. Now observe that, by using estimate (10), we have

$$\begin{aligned} a_{n+M} - a_n &= \sum_{i=0}^{L-1} (a_{n+m_{i+1}} - a_{n+m_i}) > \frac{\varepsilon_1}{2} \sum_{i=0}^{L-1} d_i \\ &\geq \frac{\varepsilon_1}{4} \sum_{1 \leq j \leq M} c_j = \frac{7}{4M} (2\alpha M + O(\log M)) \\ &= \frac{21}{\pi^2} + O\left(\frac{\log M}{M}\right) = \frac{21}{\pi^2} + O(\varepsilon \log(7/\varepsilon)) \end{aligned}$$

and the last expression above is > 1 provided that $\varepsilon > 0$ is sufficiently small. The above calculations show that for such a small ε and for each interval $I \subset [0, 1]$ of length $\varepsilon^{1/2}$, there exists $i \in \{0, \dots, L-1\}$ such that $\{a_{n+m_i}\} \in I$ which takes care of (ii).

(iii) We first give an asymptotic evaluation of g_n when n tends to infinity. We use Stirling's relation as $\log n! = n \log n - n + O(\log n)$. We have

$$\begin{aligned}
\log g_n &= \frac{1}{n} \sum_{m \leq n} \log \phi(m) = \frac{1}{n} \sum_{m \leq n} \left(\log m + \sum_{p|m} \log \left(1 - \frac{1}{p} \right) \right) \\
&= \log n - 1 + O\left(\frac{\log n}{n}\right) + \frac{1}{n} \sum_{p \leq n} \log \left(1 - \frac{1}{p} \right) \left[\frac{n}{p} \right] \\
&= \log n - 1 + O\left(\frac{\log n}{n}\right) + \sum_{p \leq n} \frac{1}{p} \log \left(1 - \frac{1}{p} \right) + O\left(\frac{1}{n} \sum_{p \leq n} \frac{1}{p}\right) \\
&= \log n - 1 + \sum_{p=1}^{\infty} \frac{1}{p} \log \left(1 - \frac{1}{p} \right) + O\left(\frac{\log n}{n}\right) \\
&= \log(\alpha n) + O\left(\frac{\log n}{n}\right),
\end{aligned}$$

therefore

$$(11) \quad g_n = \alpha n + O(\log n), \quad \text{where} \quad \alpha = \frac{1}{e} \prod_{p \geq 2} \left(1 - \frac{1}{p} \right)^{1/p}.$$

We now turn our attention to the difference $g_{n+1} - g_n$. Using the previous relation and the fact that $g_{n+1}^{n+1} = \phi(n)g_n^n$, we have

$$g_{n+1} - g_n = \left(\prod_{m \leq n} \phi(m) \right)^{\frac{1}{n+1}} \left(\phi(n+1)^{\frac{1}{n+1}} - \left(\prod_{m \leq n} \phi(m) \right)^{\frac{1}{n(n+1)}} \right).$$

Observe that

$$\left(\prod_{m \leq n} \phi(m) \right)^{\frac{1}{n+1}} = g_n^{\frac{n}{n+1}} = g_n \exp \left(O\left(\frac{\log g_n}{n}\right) \right) = \alpha n + O(\log n),$$

that

$$\begin{aligned}
\left(\prod_{m \leq n} \phi(m) \right)^{\frac{1}{n(n+1)}} &= g_n^{\frac{1}{n+1}} = \exp \left(\frac{1}{n+1} \log \left(\alpha n \left(1 + O\left(\frac{\log n}{n}\right) \right) \right) \right) \\
&= 1 + \frac{\log(\alpha(n+1))}{n+1} + O\left(\frac{(\log n)^2}{n^2}\right),
\end{aligned}$$

and certainly that

$$\phi(n+1)^{\frac{1}{n+1}} = \exp\left(\frac{\log \phi(n+1)}{n+1}\right) = 1 + \frac{\log \phi(n+1)}{n+1} + O\left(\frac{(\log n)^2}{n^2}\right),$$

which together give

$$\begin{aligned} g_{n+1} - g_n &= \left(\alpha \log\left(\frac{\phi(n+1)}{\alpha(n+1)}\right) + O\left(\frac{(\log n)^2}{n}\right)\right) \left(1 + O\left(\frac{\log n}{n}\right)\right) \\ &= \alpha \log\left(\frac{\phi(n+1)}{\alpha(n+1)}\right) + o(1), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

To prove part (iii) of our theorem, it is enough to show that for any $\varepsilon > 0$, one can find $M = \lfloor 5/\varepsilon \rfloor + 1$ consecutive integers $n+1, \dots, n+M$ such that for every $j \in \{1, 2, \dots, M\}$, one has

$$(12) \quad \varepsilon/5 \leq \{g_{n+j} - g_{n+j-1}\} \leq \varepsilon.$$

We shall first build a family of integers n_1, \dots, n_M such that for every $j \leq M$ we have $j \mid n_j$ and

$$(13) \quad \varepsilon/4 \leq \left\{ \alpha \log\left(\frac{\phi(n_j)}{\alpha n_j}\right) \right\} \leq \varepsilon/2.$$

Then, we shall use again the Chinese Remainder Lemma as in (i) to prove that there exists n such that for any j , the integer $n+j \leq 2x^2$ is the product of n_j by a number of at most $O(\log x / \log \log x)$ prime factors each exceeding $\log x$, so that (13) will imply (12).

In a way which bears similarity with our treatment of part (i), we build finite disjoint sets \mathcal{P}_j of primes $p > M$ such that

$$\left\{ \alpha \log\left(\frac{\phi(j)}{\alpha j} \prod_{p \in \mathcal{P}_j} \left(1 - \frac{1}{p}\right)\right) \right\} \in \left[\frac{\varepsilon}{4}, \frac{\varepsilon}{2}\right].$$

We then let $P_j = \prod_{p \in \mathcal{P}_j} p$ and we put $n_j = jP_j$. Let again x be a positive real number whose logarithm exceeds all the members in $\bigcup_{j=1}^M \mathcal{P}_j$, and let again \mathcal{Q} be the set of primes $p \leq \log x$ not in $\bigcup_{j=1}^M \mathcal{P}_j$. Put $Q = \prod_{q \in \mathcal{Q}} q$ and let $N = M!Q \prod_{j=1}^M P_j$. By the Chinese Remainder Lemma, there exists an arithmetic progression modulo N , say $n_0 + N\ell$ such that $n \equiv n_0 \pmod{N}$ is equivalent to $n \equiv 0 \pmod{M!Q}$ and $n \equiv -j \pmod{P_j}$ for all $j = 1, \dots, M$. If x is large, then $N < x^2$, therefore one can choose such a value for n such that $n + M \in [x^2, 2x^2]$. Clearly, all prime factors of $n+j$ which are $\leq \log x$ are precisely the prime divisors of n_j and certainly there are at most $O(\log x / \log \log x)$ of them. This takes care of part (iii).

(iv) This is trivial. Indeed, $h_n \gg n/\log n$, while

$$h_{n+1} - h_n = \left(\sum_{m \leq n} \frac{1}{\phi(m)} - \frac{n}{\phi(n+1)} \right) \left(\sum_{m \leq n} \frac{1}{\phi(m)} \right)^{-1} \left(\sum_{m \leq n+1} \frac{1}{\phi(m)} \right)^{-1}.$$

The first factor above is $\asymp \log n$, while the next two are $\asymp (\log n)^{-1}$ showing that

$$h_{n+1} - h_n = O\left(\frac{1}{\log n}\right) = o(1) \quad \text{as } n \rightarrow \infty.$$

These estimates trivially imply that $\{h_n\}$ is dense in $[0, 1]$.

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REFERENCES

- [1] R. Balasubramanian and F. Luca, *On the sum of the first n values of the Euler function*, Preprint, 2007.
- [2] J. Cilleruelo and F. Luca, *On sum of the first n primes*, Quart. J. Math. (Oxford), to appear.
- [3] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, The Clarendon Press, Oxford University Press, 1979, xvi+426 pp.
- [4] I. Kátai, *On a paper of F. Luca and A. Sankaranarayanan*, Preprint, 2007.
- [5] F. Luca, *On the sum of the first n primes being a square*, Lithuanian Math. J. **47**(2007), 243–247.
- [6] F. Luca and A. Sankaranarayanan, *On positive integers n such that $\phi(1) + \dots + \phi(n)$ is a square*, Bol. Soc. Mat. Mexicana, to appear.

JEAN-MARC DESHOUILERS
 IMB, UNIVERSITÉS BORDEAUX 1, 2 ET CNRS
 33405 TALENCE CEDEX, FRANCE
E-mail address: jean-marc.deshouillers@math.u-bordeaux1.fr

FLORIAN LUCA
 INSTITUTO DE MATEMÁTICAS
 UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
 C.P. 58089, MORELIA, MICHOACÁN, MÉXICO
E-mail address: fluca@matmor.unam.mx