

# HETEROCLINIC ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS ON RIEMANNIAN MANIFOLDS

FEI LIU<sup>1</sup>, JAUME LLIBRE<sup>2</sup> AND XIANG ZHANG<sup>1</sup>

ABSTRACT. Let  $\mathcal{M}$  be a smooth Riemannian manifold with the metric  $(g_{ij})$  of dimension  $n$ , and let  $H = \frac{1}{2}g^{ij}(q)p_i p_j + V(t, q)$  be a smooth Hamiltonian on  $\mathcal{M}$ , where  $(g^{ij})$  is the inverse matrix of  $(g_{ij})$ . Under suitable assumptions we prove the existence of heteroclinic orbits of the induced Hamiltonian systems.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The existence of homoclinic and heteroclinic orbits for Hamiltonian systems by using the variational methods and critical point theory has been studied by many authors (see for instance, [4, 5], [8, 9], [12]–[16] and [18]). We must say that Rabinowitz has given fundamental contributions to this field. Our present work is motivated by [12] and [9].

In [12] Rabinowitz studied the autonomous second order Hamiltonian system

$$(1) \quad \ddot{q} + V_q(q) = 0, \quad q = (q_1, \dots, q_n) \in \mathbb{R}^n,$$

with the function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the assumptions

( $R_1$ )  $V \in C^1(\mathbb{R}^n, \mathbb{R})$ , and  $V(q) \leq 0$  for all  $q \in \mathbb{R}^n$ .

( $R_2$ )  $V$  is periodic in  $q_i$  with a period  $T_i$ ,  $1 \leq i \leq n$ .

( $R_3$ ) The set  $\mathcal{U} = \{y \in \mathbb{R}^n; V(y) = 0\}$  consists only of isolated points.

Then for every  $x \in \mathcal{U}$ , there exist at least two heteroclinic orbits of (1) joining  $x$  to  $\mathcal{U} \setminus \{x\}$ . Moreover, at least one of these orbits emanates from  $x$  and at least one terminates at  $x$ . As mentioned in [9] Rabinowitz's proof strongly depends on the fact that the system is autonomous.

---

2000 *Mathematics Subject Classification.* 34C37, 37C10, 37C29, 37J45, 70H05.

*Key words and phrases.* heteroclinic orbit, Hamiltonian system, variational method, Riemannian manifold.

Izydorek and Janczewska [9] extended Rabinowitz's result to the non-autonomous second order Hamiltonian system

$$(2) \quad \ddot{q} + V_q(t, q) = 0, \quad t \in \mathbb{R}, \quad q = (q_1, \dots, q_n) \in \mathbb{R}^n,$$

with  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $\mathcal{W} \subset \mathbb{R}^n$  satisfying the assumptions

- (A<sub>1</sub>)  $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ , and  $V \leq 0$  on  $\mathbb{R} \times \mathbb{R}^n$ ,
- (A<sub>2</sub>)  $\#\mathcal{W} \geq 2$  and  $\gamma := \frac{1}{3} \inf\{|x - y|; x, y \in \mathcal{W}, x \neq y\} > 0$ ,
- (A<sub>3</sub>) for every  $0 < \varepsilon \leq \gamma$  there is a  $\delta > 0$  such that for all  $(t, z) \in \mathbb{R} \times \mathbb{R}^n$ , if  $d(z, \mathcal{W}) \geq \varepsilon$  then  $-V(t, z) \geq \delta$ ,
- (A<sub>4</sub>)  $-V(t, z) \rightarrow \infty$ , if  $|t| \rightarrow \infty$ , uniformly on every compact subset of  $\mathbb{R}^n \setminus \mathcal{W}$ ,
- (A<sub>5</sub>) for every  $x \in \mathcal{W}$ ,

$$\int_{-\infty}^{\infty} -V(t, x) dt \leq \gamma \sqrt{2\alpha},$$

where  $\alpha := \inf\{-V(t, z); t \in \mathbb{R}, d(z, \mathcal{W}) \geq \gamma\}$ .

They proved that for every  $x \in \mathcal{W}$  there exists at least one heteroclinic solution of (2) such that  $q(-\infty) = x$  and  $q(\infty) \in \mathcal{W} \setminus \{x\}$ , i.e.  $q$  emanates from  $x$  and terminates at a point in  $\mathcal{W} \setminus \{x\}$ .

We note that for  $n = 1$  the function  $V(t, q) =: V_1(t, q) = (1 + e^{-ct^2})(\cos q - 1)$  satisfies the assumptions (R<sub>1</sub>) – (R<sub>3</sub>) of [12] if  $c = 0$ , and satisfies the five conditions (A<sub>1</sub>) – (A<sub>5</sub>) if  $c < 0$ . We also note that Izydorek and Janczewska's result cannot be applied to the function  $V_1(t, q)$  with  $c > 0$ , because it does not satisfy the assumption (A<sub>4</sub>). In this paper we want to extend the results of [12, 9] to Hamiltonian systems on a manifold with the potential function  $V$  being in a broad class.

Let  $\mathcal{M}$  be a  $C^1$  smooth connected Riemannian manifold of dimension  $n$  with the metric  $g = (g_{ij}(q))$ , where  $q \in \mathcal{M}$ . Then on its cotangent bundle  $T^*\mathcal{M}$  there exists a natural symplectic structure. Assume that  $H$  is a Hamiltonian on  $\mathcal{M}$  given by

$$(3) \quad H = \frac{1}{2} g^{ij}(q) p_i p_j + V(t, q),$$

where  $(g^{ij}(q))$  is the inverse matrix of  $(g_{ij}(q))$  and  $V$  is a  $C^1$  smooth potential function. We note that the Hamiltonian function (3) on  $\mathcal{M}$  has been studied by several authors, see for instance [2] page 647. The aim of this paper is to study the existence of heteroclinic orbits of the following

Hamiltonian systems

$$(4) \quad \begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = g^{ij}(q)p_j, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = -\frac{1}{2} \frac{\partial g^{kl}}{\partial q_i} p_k p_l - \frac{\partial V}{\partial q_i}, \end{aligned} \quad i = 1, \dots, n$$

where we have used the Einstein summation. Suppose that

- (a)  $V \in C^1(\mathbb{R} \times \mathcal{M}, \mathbb{R})$ , and  $V(t, q) \leq 0$  in  $\mathbb{R} \times \mathcal{M}$ .
- (b) Let  $\mathcal{V} = \{q \in \mathcal{M} : V(t, q) = 0 \text{ for all } t \in \mathbb{R}\}$  and assume that
  - (b<sub>1</sub>)  $\#\mathcal{V} \geq 2$  and  $\sigma = \frac{1}{3} \min\{\rho(x, y) : x, y \in \mathcal{V}, x \neq y\} > 0$ , where  $\rho(\cdot, \cdot)$  denotes the Riemannian distance of two points on  $\mathcal{M}$ ;
  - (b<sub>2</sub>) for  $q \notin \mathcal{V}$ ,  $V(t, q) \neq 0$  for any  $t \in \mathbb{R}$ .
- (c) In any compact subset of  $\mathcal{M} \setminus \mathcal{V}$  we have  $\int_0^\infty V(t, q) dt = \int_{-\infty}^0 V(t, q) dt = -\infty$ .
- (d<sub>1</sub>) In any compact subset of  $\mathcal{M} \setminus \mathcal{V}$  we have  $-V(t, q) \rightarrow \infty$  if  $|t| \rightarrow \infty$ .
- (d<sub>2</sub>) For any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that if  $q \in B_\delta(\mathcal{V})$  then  $-V(t, q) \leq \varepsilon$ .

We remark that our condition (c) is weaker than (A<sub>3</sub>), because  $V(t, q) := V_2(t, q) = \frac{1}{1+t^{2/3}}(\sin q - 1)$  satisfies (c) but not (A<sub>3</sub>). We note that for  $n = 1$  the functions  $V_1(t, q)$  and  $V_2(t, q)$  both satisfy the conditions (a), (b) and (c). But  $V_1$  with  $c < 0$  satisfies (d<sub>1</sub>) but not (d<sub>2</sub>). The functions  $V_1$  with  $c > 0$  and  $V_2$  both satisfy (d<sub>2</sub>) but not (d<sub>1</sub>).

Our main result is the following.

**Theorem 1.** *Suppose that the conditions (a), (b), (c) and either (d<sub>1</sub>) or (d<sub>2</sub>) hold. Then for any  $x \in \mathcal{V}$  there exists  $y \in \mathcal{V} \setminus \{x\}$  for which the Hamiltonian system (4) has a heteroclinic orbit connecting  $x$  and  $y$ , i.e. there exists a complete solution  $q(t)$  of (4) satisfying*

$$q(-\infty) := \lim_{t \rightarrow -\infty} q(t) = x \text{ and } q(\infty) := \lim_{t \rightarrow \infty} q(t) = y.$$

## 2. PROOF OF THEOREM 1

From [1] the Hamiltonian system (4) is equivalent to the Lagrangian system

$$(5) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad L(t, q, \dot{q}) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - V(t, q), \quad i = 1, \dots, n$$

via the Legendre transformation

$$p^T = (g_{ij}(q)) \dot{q}^T,$$

where  $p = (p_1, \dots, p_n)$ ,  $\dot{q} = (\dot{q}^1, \dots, \dot{q}^n)$  and  $T$  denotes the transpose of a matrix. The solution of the Lagrangian system (5) is a critical point of the functional

$$(6) \quad I(q) := \int_{-\infty}^{\infty} \left( \frac{1}{2} g_{ij}(q(t)) \dot{q}^i(t) \dot{q}^j(t) - V(t, q(t)) \right) dt,$$

in the Hilbert space

$$\mathcal{H} := \left\{ q \in W_{\text{loc}}^{1,2}(\mathbb{R}, \mathcal{M}); \int_{-\infty}^{\infty} |\dot{q}|_1^2 dt < \infty \right\},$$

with the norm

$$(7) \quad \|q\|^2 = |q(0)|^2 + \int_{-\infty}^{\infty} |\dot{q}|_1^2 dt,$$

where  $|\dot{q}|_1^2 = g_{ij}(q) \dot{q}^i \dot{q}^j$  and  $|\cdot|$  is one of the equivalent norms in the  $n$ -dimensional space. Recall that  $W_{\text{loc}}^{1,2}(\mathbb{R}, \mathcal{M})$  consists of functions belonging to the Hilbert space  $W^{1,2}(J, \mathcal{M})$  where  $J \subset \mathbb{R}$  has compact closure in  $\mathbb{R}$  (see e.g. [7] page 154). We note that if  $q \in \mathcal{H}$ , then  $q \in C(\mathbb{R}, \mathcal{M})$ , the space of continuous functions having images in  $\mathcal{M}$ . In what follows we use the notation  $\rightarrow_w$  to denote the convergence of sequences in  $\mathcal{H}$  under the *weak topology* induced by the norm of  $\mathcal{H}$ . But the notation  $\rightarrow$  denotes the convergence of sequences in  $\mathcal{H}$  under the *strong topology* induced by the  $L^\infty$  norm of  $L^\infty(\mathbb{R}, \mathcal{M})$ . Recall that  $L^\infty(\mathbb{R}, \mathcal{M})$  denotes the Banach space of bounded functions (see e.g. [7] page 145).

In what follows we try to find some critical points of the functional  $I(q)$  by minimizing it in  $\mathcal{H}$ . The proof of Theorem 1 follows from a series of lemmas. We must say that we extend the ideas of the proof of [9] and [12] to our Hamiltonian systems.

For  $\varepsilon > 0$  let

$$\alpha_\varepsilon := \inf \{ -V(t, q); t \in \mathbb{R}, q \in \mathcal{M} \setminus B_\varepsilon(\mathcal{V}) \}.$$

Then  $\alpha_\varepsilon > 0$ , it follows from either the assumption  $(d_1)$  or  $(d_2)$ .

**Lemma 2.** *For  $0 < \varepsilon \leq \sigma$ . Assume that  $q \in \mathcal{H}$  and  $q(t) \in \mathcal{M} \setminus B_\varepsilon(\mathcal{V})$  for  $t \in \cup_{i=1}^k [a_i, b_i]$  with  $k \in \mathbb{N}$  and  $[a_i, b_i] \cap [a_j, b_j] = \emptyset$  for  $i \neq j$ . Then*

$$I(q) \geq \sqrt{2\alpha_\varepsilon} \sum_{i=1}^k l(q[a_i, b_i]),$$

where  $l(q[a_i, b_i])$  denotes the length of the curve  $q(t)$  when  $t$  arranges from  $a_i$  to  $b_i$ .

*Proof.* From the definition of the length of a curve on a Riemannian manifold (see e.g. [6] page 43 or [3] page 117), by direct calculations we get that

$$l := \sum_{i=1}^k l(q[a_i, b_i]) = \int_{\cup_{i=1}^k [a_i, b_i]} |\dot{q}(t)|_1 dt \leq s^{1/2} \left( \int_{\cup_{i=1}^k [a_i, b_i]} |\dot{q}(t)|_1^2 dt \right)^{1/2},$$

where  $s = \sum_{i=1}^k (b_i - a_i)$ , and we have used the Cauchy-Schwartz inequality. So we have

$$I(q) \geq \int_{\cup_{i=1}^k [a_i, b_i]} \left( \frac{1}{2} |\dot{q}(t)|_1^2 - V(t, q) \right) dt \geq \frac{l^2}{2s} + \alpha_\varepsilon s \geq l\sqrt{2\alpha_\varepsilon}.$$

□

Under the assumptions of Lemma 2, it follows that

$$(8) \quad I(q) \geq \sqrt{2\alpha_\varepsilon} \sum_{i=1}^k \rho(q(a_i), q(b_i)).$$

**Lemma 3.** *For any  $q \in \mathcal{H}$  we have*

$$(l(q[a, b]))^2 \leq 2(b - a)I(q), \quad \text{for all } a, b \in \mathbb{R}, a < b.$$

*Proof.* It follows from the fact that

$$I(q) \geq \int_{-\infty}^{\infty} \frac{1}{2} |\dot{q}(t)|_1^2 dt \geq \int_a^b \frac{1}{2} |\dot{q}(t)|_1^2 dt,$$

and

$$(l(q[a, b]))^2 = \left( \int_a^b |\dot{q}(t)|_1 dt \right)^2 \leq (b - a) \int_a^b |\dot{q}(t)|_1^2 dt.$$

□

In order to simplify notations we will use  $q(a, b)$  to denote the set  $\{q(t); t \in (a, b)\}$  for any  $a, b \in \mathbb{R}$  and  $a < b$ .

**Lemma 4.** *If  $q \in \mathcal{H}$  and  $I(q) < \infty$ , then  $q \in L^\infty(\mathbb{R}, \mathcal{M})$ .*

*Proof.* If the manifold  $\mathcal{M}$  is bounded, the conclusion follows easily from the definition of  $L^\infty(\mathbb{R}, \mathcal{M})$ . We assume that  $\mathcal{M}$  is unbounded.

Firstly we claim that the orbit  $q(\mathbb{R})$  intersects at most finitely many elements of  $\{\partial B_\sigma(x); x \in \mathcal{V}\}$ . Otherwise there exist sequences  $a_1 < b_2 \leq$

$a_2 < \dots < b_n \leq a_n < b_{n+1} \leq a_{n+1} < \dots$  and  $\{\xi_i\} \subset \mathcal{V}$  with  $\xi_i \neq \xi_j$  for  $i \neq j$  such that  $q(a_{i-1}, b_i) \cap \overline{B}_\sigma(\mathcal{V}) = \emptyset$ , and  $q(b_i), q(a_i) \in \partial B_\sigma(\xi_i)$ . It follows from Lemma 2 and (8) that for any  $m \in \mathbb{N}$

$$I(q) \geq \sqrt{2\alpha_\sigma} \sum_{i=1}^m \rho(q(a_i), q(b_{i+1})) \geq m\sigma\sqrt{2\alpha_\sigma}.$$

This means  $I(q) = \infty$ , a contradiction. The claim follows.

On the contrary if  $q \notin L^\infty(\mathbb{R}, \mathcal{M})$  we can choose a base point  $q_0 \in \mathcal{M}$  and a real number  $R > 0$  such that  $q(\mathbb{R}) \cap B_R(q_0) \neq \emptyset$  and  $q(t_1, \infty) \cap \overline{B}_\sigma(\mathcal{V}) = \emptyset$ , where  $t_1 := \sup\{q(t) \in \partial B_R(q_0)\}$ . For any  $n \in \mathbb{N}$  there exists a  $t_n \in \mathbb{R}$  such that  $\rho(q(t_n), q(t_1)) > n$  and  $t_1 < t_n < t_{n+1} \rightarrow \infty$  as  $n \rightarrow \infty$ . By Lemma 2 and (8) we have

$$I(q) \geq \sqrt{2\alpha_\sigma} l(q[t_1, t_n]) \geq \sqrt{2\alpha_\sigma} \rho(q(t_1), q(t_n)), \quad n \in \mathbb{N}.$$

This means that  $I(q) = \infty$ . The contradiction shows that  $q \in L^\infty(\mathbb{R}, \mathcal{M})$ .  $\square$

**Lemma 5.** *If  $q \in \mathcal{H}$  and  $I(q) < \infty$ , there exist  $\xi, \eta \in \mathcal{V}$  such that  $q(-\infty) = \xi$  and  $q(\infty) = \eta$ .*

*Proof.* Set  $\alpha(q(t)) := \{\alpha \in \mathcal{M}; \exists \{t_n\} \subset \mathbb{R} \text{ with } t_n \rightarrow -\infty \text{ as } n \rightarrow \infty \text{ such that } \lim_{n \rightarrow \infty} q(t_n) = \alpha\}$ ,  $\beta(q(t)) := \{\beta \in \mathcal{M}; \exists \{t_n\} \subset \mathbb{R} \text{ with } t_n \rightarrow \infty \text{ as } n \rightarrow \infty \text{ such that } \lim_{n \rightarrow \infty} q(t_n) = \beta\}$ . We call  $\alpha(q(t))$  and  $\beta(q(t))$  the  $\alpha$  and  $\omega$  limit sets of  $q(t)$ , respectively. By Lemma 4,  $\alpha(q(t)) \neq \emptyset$  and  $\beta(q(t)) \neq \emptyset$  because  $q(t)$  is bounded. We now prove that  $\beta(q) \subset \mathcal{V}$  and that it contains a unique point. The same conclusion holds for  $\alpha(q)$ .

On the contrary we assume that  $\beta(q) \not\subset \mathcal{V}$ , then there is a  $\zeta \in \beta(q)$  but  $\zeta \notin \mathcal{V}$ . So there exists  $\varepsilon > 0$  such that  $B_\varepsilon(\zeta) \cap B_\varepsilon(\mathcal{V}) = \emptyset$  and there exists  $\{t_n\} \subset \mathbb{R}$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $\lim_{n \rightarrow \infty} q(t_n) = \zeta$ . This implies that there is an  $N > 0$  such that for  $n > N$  we have  $q(t_n) \in B_\varepsilon(\zeta)$ . Set  $\mathcal{D}_\zeta := \{t \in [t_{N+1}, \infty); q(t) \in B_{\varepsilon/2}(\mathcal{V})\}$ , and  $T_\zeta := \sup \mathcal{D}_\zeta$ . In the case  $\mathcal{D}_\zeta = \emptyset$ , we set  $T_\zeta = t_{N+1}$ .

If  $B_{\varepsilon/2}(\mathcal{V})$  intersects at most finitely many elements of  $\{q[t_n, t_{n+1}]; n = N+1, N+2, \dots\}$ , then  $T_\zeta < \infty$ . Hence by (c) it follows from the boundedness of  $q(t)$  that

$$I(q) \geq \int_{T_\zeta}^{\infty} -V(t, q(t)) dt = \infty,$$

a contradiction.

If  $B_{\varepsilon/2}(\mathcal{V})$  intersects infinitely many elements of  $\{q[t_n, t_{n+1}]; n = N + 1, N + 2, \dots\}$ , there exist  $[a_{n_k}, b_{n_k}] \subset [t_{n_k}, t_{n_k+1}]$ ,  $k = 1, 2, \dots$ , for which

$$q(a_{n_k}) \in \partial B_\varepsilon(\zeta), \quad q(b_{n_k}) \in \partial B_{\varepsilon/2}(\mathcal{V}), \quad q(a_{n_k}, b_{n_k}) \cap B_{\varepsilon/2}(\mathcal{V}) = \emptyset.$$

We get from Lemma 2 that for any  $k \in \mathbb{N}$

$$I(q) \geq \frac{\varepsilon}{2} k \sqrt{2\alpha_{\varepsilon/2}}.$$

This means that  $I(q) = \infty$ , a contradiction. Hence we should have  $\beta(q) \subseteq \mathcal{V}$ .

If  $\xi, \eta \in \beta(q)$  and  $\xi \neq \eta$ . For  $0 < \varepsilon \leq \sigma$ , there exist  $[a_n, b_n] \subset \mathbb{R}$ ,  $n = 1, 2, \dots$ , with  $[a_i, b_i] \cap [a_j, b_j] = \emptyset$  for  $i \neq j$  such that  $q(a_n) \in \partial B_\varepsilon(\xi)$ ,  $q(b_n) \in \partial B_\varepsilon(\eta)$  and  $q(a_n, b_n) \cap B_\varepsilon(\mathcal{V}) = \emptyset$ . Then an analogous argument as in the last paragraph implies that  $I(q) = \infty$ . This contradiction forces that  $\beta(q)$  has a unique element.

We denote by  $q(\infty)$  (resp.  $q(-\infty)$ ) the unique point of the  $\omega$  (resp.  $\alpha$ ) limit set of  $q(t)$ . Then  $q(\infty), q(-\infty) \in \mathcal{V}$ .  $\square$

For any given  $0 < \varepsilon \leq \sigma$  and fixed  $x \in \mathcal{V}$ , choose any  $y \in \mathcal{V} \setminus \{x\}$ . Set

$$\mathcal{H}_\varepsilon(y) := \{q(t) \in \mathcal{H}; q(-\infty) = x, q(\infty) = y, q(\mathbb{R}) \cap B_\varepsilon(\mathcal{V} \setminus \{x, y\}) = \emptyset\}.$$

The set  $\mathcal{H}_\varepsilon(y)$  is non-empty, because it contains the function:  $q(t) = x$  for  $t \in (-\infty, 0)$ ,  $q(t) = y$  for  $t \in (1, \infty)$ , and  $q[0, 1]$  consists of a piecewise smooth curve having a finite length, connecting  $x$  and  $y$ , and with  $q(0, 1) \cap B_\varepsilon(\mathcal{V} \setminus \{x, y\}) = \emptyset$ . Define

$$(9) \quad \mu_\varepsilon(y) := \inf_{q \in \mathcal{H}_\varepsilon(y)} I(q).$$

Then  $0 < \mu_\varepsilon(y) < \infty$  and  $\mu_\varepsilon(y) \rightarrow \infty$  if  $y \rightarrow \infty$  by Lemma 4.

**Lemma 6.** *There exists a  $q_{\varepsilon, y} \in \mathcal{H}_\varepsilon(y)$  such that  $I(q_{\varepsilon, y}) = \mu_\varepsilon(y)$ .*

*Proof.* Set

$$\mathcal{H}_{1, \varepsilon}(y) := \{q \in \mathcal{H}_\varepsilon(y); I(q) \leq \mu_\varepsilon(y) + 1\}.$$

First we claim that  $\mathcal{H}_{1, \varepsilon}(y)$  is bounded in  $\mathcal{H}$ . Indeed, for any  $q \in \mathcal{H}_{1, \varepsilon}(y)$

$$\mu_\varepsilon(y) + 1 \geq I(q) \geq \int_{-\infty}^{\infty} \frac{1}{2} |\dot{q}(t)|_1^2 dt.$$

In addition there exists a  $c > 0$  such that  $\rho(q(0), x) \leq c$  for all  $q \in \mathcal{H}_{1, \varepsilon}(y)$ . Otherwise for any  $k \in \mathbb{N}$  there is a  $q_k \in \mathcal{H}_{1, \varepsilon}(y)$  such that  $\rho(q_k(0), x) \geq k$ . Set  $t_k := \max\{t < 0; q_k(t) \in \partial B_\varepsilon(x)\}$ . By Lemma 2 and (8) we have

$$I(q_k) \geq \sqrt{2\alpha_\varepsilon} \rho(q_k(0), q_k(t_k)) \rightarrow \infty, \text{ as } k \rightarrow \infty,$$

a contradiction. This implies that there exists a  $c^*$  such that  $|q(0)|^2 \leq c^*$  for all  $q \in \mathcal{H}_{1,\varepsilon}$ . Hence from (7) we have

$$\|q\|^2 \leq 2(\mu_\varepsilon(y) + 1) + c^* := M_1.$$

This proves the claim.

Second the set  $\mathcal{H}_{1,\varepsilon}(y)$  is bounded in the strong topology. It follows from the norm  $\|q - x\|_{L^\infty} := \max\{\rho(q(t), x), t \in \mathbb{R}\}$  and the fact that

$$\rho(q(t), x) \leq \frac{I(q)}{\sqrt{2\alpha_\varepsilon}} + \varepsilon + \rho(x, y).$$

We now prove this last inequality. If  $q(t) \in \overline{B_\varepsilon}(x)$  then  $\rho(q(t), x) \leq \varepsilon$ . If  $q(t) \in \overline{B_\varepsilon}(y)$  then  $\rho(q(t), x) \leq \varepsilon + \rho(x, y)$ . If  $q(t) \notin \overline{B_\varepsilon}(x) \cup \overline{B_\varepsilon}(y)$  and  $\{q(s), s \leq t\} \cap \overline{B_\varepsilon}(y) = \emptyset$ , we have  $\rho(q(t), x) \leq \rho(q(t), q(t_x)) + \rho(q(t_x), x) \leq \frac{I(q)}{\sqrt{2\alpha_\varepsilon}} + \varepsilon$ , where  $t_x = \max\{s < t; q(s) \in \partial B_\varepsilon(x)\}$ . If  $q(t) \notin \overline{B_\varepsilon}(x) \cup \overline{B_\varepsilon}(y)$  and  $\{q(s), s \leq t\} \cap \overline{B_\varepsilon}(y) \neq \emptyset$ , then  $\rho(q(t), x) \leq \rho(q(t), q(t_y)) + \rho(q(t_y), y) + \rho(x, y) \leq \frac{I(q)}{\sqrt{2\alpha_\varepsilon}} + \varepsilon + \rho(x, y)$ , where  $t_y = \max\{s < t; q(s) \in \partial B_\varepsilon(y)\}$ . This completes the proof of the inequality.

Third we claim that any sequence of  $\mathcal{H}_{1,\varepsilon}(y)$  are equicontinuous. Indeed, for any  $q \in \mathcal{H}_{1,\varepsilon}(y)$ , and any  $r, s \in \mathbb{R}$ , Lemma 3 implies

$$\rho(q(s), q(r)) \leq \sqrt{2I(q)|s - r|} \leq \sqrt{2(\mu_\varepsilon(y) + 1)}\sqrt{|r - s|}.$$

The claim follows.

Let  $\{q_m\}$  be a minimizing sequence for (9). Then from the definition of  $H_{1,\varepsilon}(y)$  and the fact that  $\lim_{m \rightarrow \infty} I(q_m) = \mu_\varepsilon(y)$ , we can assume without loss of generality that  $\{q_m\} \subset \mathcal{H}_{1,\varepsilon}(y)$ . Then  $\{q_m\}$  is bounded in  $\mathcal{H}$ , and consequently there is a subsequence which is convergent to some element of the Hilbert space  $\mathcal{H}$ , denoted by  $q_{\varepsilon,y}$ . In order to simplify the notation we suppose without loss of generality that  $q_m \rightarrow_w q_{\varepsilon,y}$ , otherwise instead of  $\{q_m\}$  we must have a subsequence. From the last two claims, the sequence  $\{q_m\}$  is uniformly bounded and equicontinuous. It follows from the Arzela-Ascoli Theorem (see e.g. [17] page 245) that  $q_m \rightarrow q_{\varepsilon,y}$  in  $L_{loc}^\infty(\mathbb{R}, \mathcal{M})$ .

Next we prove that  $I(q_{\varepsilon,y}) \leq \mu_\varepsilon(y)$ . For any fixed  $a, b \in \mathbb{R}$  and  $a < b$ , define

$$(10) \quad I_{a,b}(q) := \int_a^b \left( \frac{1}{2} |\dot{q}(t)|_1^2 - V(t, q(t)) \right) dt.$$

Then it follows from Lemma 11 of the Appendix that  $I_{a,b}(q)$  is lower semi-continuous in the weak topology. Since

$$I_{a,b}(q_m) \leq I(q_m) \leq \mu_\varepsilon(y) + 1,$$

we have

$$I_{a,b}(q_{\varepsilon,y}) \leq \liminf_{m \rightarrow \infty} I_{a,b}(q_m) \leq \lim_{m \rightarrow \infty} I(q_m) = \mu_\varepsilon(y) \leq \mu_\varepsilon(y) + 1.$$

Since  $q_{\varepsilon,y} \in \mathcal{H}$ , and so is continuous in  $\mathbb{R}$ . By the arbitrary of  $a, b \in \mathbb{R}$ , we should have  $I(q_{\varepsilon,y}) \leq \mu_\varepsilon(y)$  by letting  $a \rightarrow -\infty$  and  $b \rightarrow \infty$ .

Finally we prove that  $q_{\varepsilon,y} \in \mathcal{H}_\varepsilon(y)$ . We claim first that  $\{q_{\varepsilon,y}(t); t \in \mathbb{R}\} \cap B_\varepsilon(\mathcal{V} \setminus \{x, y\}) = \emptyset$ . Otherwise there exist  $\xi \in \mathcal{V} \setminus \{x, y\}$  and  $t_0 \in \mathbb{R}$  such that  $q_{\varepsilon,y}(t_0) \in B_\varepsilon(\xi)$ , then there exists  $M > 0$  so that for  $m > M$  we have  $q_m(t_0) \in B_\varepsilon(\xi)$  because  $q_m \rightarrow q_{\varepsilon,y}$  uniformly on any given neighborhood of  $t_0$ . This is in contradiction with  $q_m \in \mathcal{H}_\varepsilon(y)$ . Hence the claim follows.

Next we only need to prove that  $q_{\varepsilon,y}(-\infty) = x$  and  $q_{\varepsilon,y}(\infty) = y$ . Lemma 5 shows that  $q_{\varepsilon,y}(-\infty)$  and  $q_{\varepsilon,y}(\infty)$  both exist and belong to  $\mathcal{V}$ . Moreover we get from the last claim that  $q_{\varepsilon,y}(-\infty), q_{\varepsilon,y}(\infty) \in \{x, y\}$ . On the contrary we assume that  $q_{\varepsilon,y}(\infty) = x$ .

*Case 1.* The condition  $(d_1)$  holds. Since  $q_{\varepsilon,y}(\infty) = x$ , there is a  $t_* \in \mathbb{R}$  such that if  $t \geq t_*$  we have  $q_{\varepsilon,y}(t) \in B_{\varepsilon/2}(x)$ . For any  $s > t_*$ , since  $q_m \rightarrow q_{\varepsilon,y}$  uniformly on  $[t_*, s]$ , there exists  $M(s) > 0$  such that  $q_m(t) \in B_\varepsilon(x)$  for  $m > M(s)$  and  $t \in [t_*, s]$ . We choose one of such  $m$ 's, and denote it by  $m(s)$ . By the definition of  $H_\varepsilon(y)$  we have  $q_{m(s)}(\infty) = y$ , so we can set  $t_1(s) = \max\{t \in \mathbb{R}; q_{m(s)}(t) \in \partial B_\varepsilon(x)\}$  and  $t_2(s) = \min\{t \in \mathbb{R}; q_{m(s)}(t) \in \partial B_\varepsilon(y)\}$ . Then we have  $s < t_1(s) < t_2(s)$ . Now we obtain

$$t_2(s) - t_1(s) \geq \frac{(l(q[t_1(s), t_2(s)]))^2}{2(\mu_\varepsilon(y) + 1)} \geq \frac{\sigma^2}{2(\mu_\varepsilon(y) + 1)} := c > 0,$$

the first inequality follows from Lemma 3 and  $I(q_m) \leq \mu_{\varepsilon,y} + 1$ , and the second from  $(b_1)$ .

So we get from the Mean Value Theorem for integration that

$$\begin{aligned} I(q_{m(s)}) &\geq \int_{t_1(s)}^{t_2(s)} -V(t, q_{m(s)}(t)) dt \\ &= -V(t_s, q_{m(s)}(t_s))(t_2(s) - t_1(s)) \geq -cV(t_s, q_{m(s)}(t_s)), \end{aligned}$$

where  $t_s \in [t_1(s), t_2(s)]$ . Since  $q_{m(s)}(t_s) \notin B_\varepsilon(\mathcal{V})$  and  $t_s > s$ , by the boundedness of  $\{q_m\} \subset \mathcal{H}_{1,\varepsilon}$  in the strong topology we obtain from the condition  $(d_1)$  that  $I(q_{m(s)}) \rightarrow \infty$  as  $s \rightarrow \infty$ . This is in contradiction with the fact that  $I(q_m) \leq \mu_{\varepsilon,y} + 1$  for all  $m \in \mathbb{N}$ . Hence we have  $q_{\varepsilon,y}(\infty) = y$ . Working in a similar way we can prove that  $q_{\varepsilon,y}(-\infty) = x$ .

*Case 2.* The condition  $(d_2)$  holds. First it is easy to check that the case  $q_{\varepsilon,y}(t) \equiv x$  does not happen, because each  $q_m$  connects  $x$  and  $y$ , and  $q_m \rightarrow_w q_{\varepsilon,y}$  in  $\mathcal{H}$ . Thus we have a  $t_0$  such that  $q_{\varepsilon,y}(t_0) \neq x$ . Then there exist  $k \in \mathbb{N}$ ,

$k \geq 2$  and  $t_1^* \in \mathbb{R}$  such that  $q_{\varepsilon,y}(t_1^*) \in \partial B_{\varepsilon/(k-1)}(x)$ . By the assumption  $(d_2)$  we can choose a  $\delta > 0$  satisfying  $\delta < \varepsilon/(4k)$  and

$$(11) \quad 2\delta^2 + \max_{\rho(q,x) \leq 2\delta} (-V(t,q)) < \frac{\varepsilon}{4k} \sqrt{2\alpha_{\varepsilon/(2k)}}.$$

Since  $q_{\varepsilon,y}(\infty) = x$  there exists a  $t_2^* > t_1^*$  such that  $q_{\varepsilon,y}(t) \in B_\delta(x)$  for all  $t \geq t_2^*$ . The sequence  $q_m \rightarrow q_{\varepsilon,y}$  uniformly on  $[t_1^*, t_2^*]$  implies that there is a  $M > 0$  such that for  $m > M$  we have  $q_m(t_1^*) \in \mathcal{M} \setminus B_{\varepsilon/k}(x)$  and  $q_m(t_2^*) \in B_{2\delta}(x)$ . Consequently, for  $m > M$

$$I(q_m) \geq \sqrt{2\alpha_{\varepsilon/(2k)}} \frac{\varepsilon}{2k} + \int_{t_2^*}^{\infty} L(t, q_m(t), \dot{q}_m(t)) dt.$$

Set

$$Q_m(t) = \begin{cases} x & \text{if } t < t_2^* - 1, \\ g_m(t) & \text{if } t \in [t_2^* - 1, t_2^*], \\ q_m(t) & \text{if } t > t_2^*, \end{cases}$$

where  $g_m(t)$  is the minimal geodesic connecting  $x$  to  $q_m(t_2^*)$ , whose existence follows from Corollary 10.8 of [10] via the  $\delta$  being chosen sufficiently small. Clearly  $Q_m \in \mathcal{H}_\varepsilon(y)$ . Since  $|\dot{g}_m(t)| \equiv \text{constant}$  and  $\rho(q_m(t_2^*), x) < 2\delta$ , we have

$$\int_{t_2^*-1}^{t_2^*} g_{ij}(g_m(t)) \dot{g}_m^i(t) \dot{g}_m^j(t) dt \leq (2\delta)^2.$$

Since  $L(t, Q_m, \dot{Q}_m) \equiv 0$  on  $(-\infty, t_2^* - 1]$ , we get

$$\begin{aligned} I(Q_m) &= \int_{t_2^*-1}^{t_2^*} \left( \frac{1}{2} g_{ij}(g_m(t)) \dot{g}_m^i(t) \dot{g}_m^j(t) - V(t, g_m(t)) \right) dt \\ &\quad + \int_{t_2^*}^{\infty} L(t, q_m(t), \dot{q}_m(t)) dt \\ &\leq \frac{1}{2} (2\delta)^2 + \max_{\rho(q,x) \leq 2\delta} \{-V(t,q)\} + I(q_m) - \sqrt{2\alpha_{\varepsilon/(2k)}} \frac{\varepsilon}{2k} \\ &\leq I(q_m) - \frac{\varepsilon}{4k} \sqrt{2\alpha_{\varepsilon/(2k)}}. \end{aligned}$$

This implies

$$\begin{aligned} \inf_{q \in \mathcal{H}_\varepsilon(y)} I(q) &\leq \liminf_{m \rightarrow \infty} I(Q_m) \leq \lim_{m \rightarrow \infty} I(q_m) - \frac{\varepsilon}{4k} \sqrt{2\alpha_{\varepsilon/(2k)}} \\ &= \inf_{q \in \mathcal{H}_\varepsilon(y)} I(q) - \frac{\varepsilon}{4k} \sqrt{2\alpha_{\varepsilon/(2k)}}, \end{aligned}$$

a contradiction. So we have  $q_{\varepsilon,y}(\infty) = y$ .

Summarizing the above proof we get that  $q_{\varepsilon,y} \in \mathcal{H}_{1,\varepsilon} \subset \mathcal{H}_\varepsilon$ . Consequently  $I(q_{\varepsilon,y}) = \mu_\varepsilon(y)$ . This proves that  $q_{\varepsilon,y}$  is a minimal of  $I$  in  $\mathcal{H}_\varepsilon(y)$ .  $\square$

We now consider the regularity of the minimal.

**Lemma 7.** *For every  $y \in \mathcal{V} \setminus \{x\}$ , the minimal  $q_{\varepsilon,y}$  of  $I$  in  $\mathcal{H}_\varepsilon(y)$  given in Lemma 6 is a  $C^2$  solution of system (4) in  $\mathbb{R} \setminus R(\varepsilon,y)$ , where  $R(\varepsilon,y) = \{t \in \mathbb{R}; q_{\varepsilon,y}(t) \in \partial B_\varepsilon(\mathcal{V} \setminus \{x,y\})\}$ .*

*Proof.* : We only need to prove that  $q_{\varepsilon,y}$  is  $C^2$  in any interval  $(r,s) \subset \mathbb{R} \setminus R(\varepsilon,y)$ . For  $\phi \in \mathcal{H}$  with  $\text{supp } \phi \subset (r,s)$ , and  $|\delta|$  sufficiently small, we have  $q_{\varepsilon,y} + \delta\phi \in \mathcal{H}_\varepsilon$ . Hence

$$I(q_{\varepsilon,y}) \leq I(q_{\varepsilon,y} + \delta\phi),$$

i.e.  $q_\varepsilon(y)$  is a local minimum of  $I(q)$ . This implies that if the first variation exists, it must vanish.

In the next step we prove the first variation exists. Indeed,

$$\begin{aligned} \frac{I(q_{\varepsilon,y} + \delta\phi) - I(q_{\varepsilon,y})}{\delta} &= \int_{-\infty}^{\infty} \frac{L(t, q_{\varepsilon,y} + \delta\phi, \dot{q}_{\varepsilon,y} + \delta\dot{\phi}) - L(t, q_{\varepsilon,y}, \dot{q}_{\varepsilon,y})}{\delta} dt \\ &= \int_r^s \left( f(t, \delta)\dot{\phi} + g(t, \delta)\phi \right) dt, \end{aligned}$$

where  $f(t, \delta) = \int_0^1 L_{\dot{q}}(t, q_{\varepsilon,y} + \delta\phi, \dot{q}_{\varepsilon,y} + s\delta\dot{\phi}) ds$  and  $g(t, \delta) = \int_0^1 L_q(t, q_{\varepsilon,y} + s\delta\phi, \dot{q}_{\varepsilon,y}) ds$ . Recall that  $L(t, q, \dot{q})$  is the Lagrangian function defined at the beginning of this section, and  $L_q$  and  $L_{\dot{q}}$  denote the partial derivatives of  $L$  with respect to  $q$  and  $\dot{q}$ , respectively. For the second equality we have used the fact that  $\phi$  has the compact support belonging to  $(r, s)$ .

From the form of  $L$  we get for  $|\delta| \ll 1$  and  $s \in [0, 1]$  that on  $\text{supp } \phi$

$$|f(t, \delta)| \leq c_1(1 + |\dot{q}| + |\dot{\phi}|), \quad |g(t, \delta)| \leq c_1(1 + |\dot{q}|^2 + |\dot{\phi}|^2),$$

where  $c_1$  is a constant depending only on  $q_{\varepsilon,y}$ ,  $\phi$  and the given interval  $(r, s)$ . This implies that  $f(t, \delta)\dot{\phi}$ ,  $g(t, \delta)\phi \in L^1[r, s]$ , because the majorants  $c_1(1 + |\dot{x}| + |\dot{\phi}|)\dot{\phi}$  and  $c_1(1 + |\dot{x}|^2 + |\dot{\phi}|^2)\phi$  are Lebesgue integrable. Moreover by the Lebesgue's Dominated Convergence Theorem (see e.g. [17] page 26) the limit  $\lim_{\delta \rightarrow 0} (I(q_{\varepsilon,y} + \delta\phi) - I(q_{\varepsilon,y})) / \delta$  exists. Consequently the first variation vanishes.

The above proof yields

$$\begin{aligned}
& \lim_{\delta \rightarrow 0} \frac{I(q_{\varepsilon, y} + \delta \phi) - I(q_{\varepsilon, y})}{\delta} \\
&= \int_r^s \left( L_p(t, q_{\varepsilon, y}, \dot{q}_{\varepsilon, y}) \dot{\phi} + L_q(t, q_{\varepsilon, y}, \dot{q}_{\varepsilon, y}) \phi \right) dt \\
&= \int_r^s \left( L_p(t, q_{\varepsilon, y}, \dot{q}_{\varepsilon, y}) - \int_r^t L_q(\nu, q_{\varepsilon, y}, \dot{q}_{\varepsilon, y}) d\nu + c \right) \dot{\phi} dt \\
&= 0
\end{aligned}$$

where  $c$  is a constant of integration. By the arbitrariness of  $\phi$  we have

$$(12) \quad L_p(t, q_{\varepsilon, y}, \dot{q}_{\varepsilon, y}) = \int_r^t L_q(\nu, q_{\varepsilon, y}, \dot{q}_{\varepsilon, y}) d\nu - c, \text{ for } t \in [r, s],$$

which is called the integrated Euler equation. We can check easily that the function on the right hand side of (12) is absolutely continuous. Since  $L_{\dot{q}\dot{q}}$  is positively definite we get from the Implicit Function Theorem that  $\dot{q}_{\varepsilon, y}(t)$  is  $C^0$  on  $[r, s]$ , and hence  $q_{\varepsilon, y} \in C^1[r, s]$ . It follows that the function on the right hand side of (12) is  $C^1$ . Again by the Implicit Function Theorem we have  $\dot{q}_{\varepsilon, y} \in C^1[r, s]$ , and consequently  $q_{\varepsilon, y} \in C^2[r, s]$ .

From the relation between the Euler equation and the Hamiltonian equation we get that  $q_{\varepsilon, y}$  is a  $C^2$  solution of system (4) on the interval  $[r, s]$ , and hence on  $\mathbb{R} \setminus R(\varepsilon, y)$ . This proves the lemma.  $\square$

**Lemma 8.** *For any  $M > 0$  and  $y \in \mathcal{V} \setminus \{x\}$ , if  $q \in \{q \in \mathcal{H}; q(-\infty) = x, q(\infty) = y \text{ and } I(q) \leq M\}$ , there exists constant  $\mathcal{K}$  depending only on  $M, \sigma$  and  $\alpha_\sigma$  such that*

$$\rho(q(t), x) \leq \mathcal{K} := \frac{3M}{\sqrt{2\alpha_\sigma}} + 3\sigma.$$

*Proof.* For any given  $t \in \mathbb{R}$  we assume without loss of generality that  $q(t) \notin \overline{B_\sigma(\mathcal{V})}$ . Otherwise there is a  $\tau \in \mathbb{R}$  such that  $q(\tau) \notin \overline{B_\sigma(\mathcal{V})}$  and  $\rho(q(t), q(\tau)) \leq 2\sigma$ . Then we can choose  $\tau$  instead of the  $t$  in the following proof, because  $\rho(q(t), x) \leq \rho(q(\tau), x) + 2\sigma$ .

Set  $t_0 = \max\{s \in \mathbb{R}; q(s) \in \partial B_\sigma(x) \text{ and } q(-\infty, s) \cap \overline{B_\sigma(\mathcal{V} \setminus \{x\})} = \emptyset\}$ . If  $t_0 > t$  there exists  $s_0$  satisfying  $t < s_0 \leq t_0$  such that  $s_0 \in \partial B_\sigma(x)$  and  $q[t, s_0] \cap B_\sigma(x) = \emptyset$ . So we have  $\rho(q(t), x) \leq \rho(q(t), q(s_0)) + \sigma \leq M/\sqrt{2\alpha_\sigma} + \sigma$ , because by Lemma 2 we have  $M \geq I(q) \geq \sqrt{2\alpha_\sigma} \rho(q(t), q(s_0))$ .

If  $t_0 < t$  it follows from the continuity of  $q$  that  $q[t_0, s] \subset \mathcal{M}$  is compact. So there exist finitely many points  $s_0 = t_0 < s_1 \leq t_1 < s_2 \leq t_2 < \dots < s_{k-1} \leq t_{k-1} < s_k = t$  with  $1 \leq k \in \mathbb{N}$  and  $\xi_i \in \mathcal{V}$  with  $\xi_i \neq \xi_j$  for  $1 \leq i \neq j \leq k-1$  such that  $q(s_i), q(t_i) \in \partial B_\sigma(\xi_i)$  for  $i = 1, \dots, k-1$  and

$q(t_i, s_{i+1}) \cap B_\sigma(\mathcal{V}) = \emptyset$  for  $k = 0, 1, \dots, k-1$ . By Lemma 2 we have

$$I(q) \geq \sqrt{2\alpha_\sigma} \sum_{i=1}^k \rho(q(t_{i-1}), q(s_i)) \geq \sqrt{2\alpha_\sigma}(k-1)\sigma.$$

It yields that

$$\sum_{i=1}^k \rho(q(t_{i-1}), q(s_i)) \leq \frac{M}{\sqrt{2\alpha_\sigma}}, \quad k \leq \frac{M}{\sigma\sqrt{2\alpha_\sigma}} + 1.$$

By the triangular inequality we get

$$\rho(q(t), x) \leq \sum_{i=1}^k \rho(q(t_{i-1}), q(s_i)) + (2k-1)\sigma \leq \frac{3M}{\sqrt{2\alpha_\sigma}} + \sigma.$$

This proves the lemma.  $\square$

We note that the bound  $\mathcal{K}$  in Lemma 8 is independent of the choice of  $y$ .

**Lemma 9.** For  $0 < \varepsilon \leq \sigma$ , set

$$\mu_\varepsilon = \inf \{ \mu_\varepsilon(y); y \in \mathcal{V} \setminus \{x\} \}.$$

Then there exist  $y \in \mathcal{V} \setminus \{x\}$  and a sequence  $\{\varepsilon_j\}$  satisfying  $0 < \varepsilon_{j+1} < \varepsilon_j \leq \sigma$  and  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$  for which  $I(q_{\varepsilon_j, y}) = \mu_{\varepsilon_j}(y) = \mu_{\varepsilon_j}$ .

*Proof.* Since  $\mu_\varepsilon$  is a constant and  $0 < \mu_\varepsilon(y) \rightarrow \infty$  as  $y \rightarrow \infty$ , there is a constant  $R_\varepsilon > 0$  and a  $y_\varepsilon \in B_{R_\varepsilon}(x) \cap (\mathcal{V} \setminus \{x\})$  such that  $\mu_\varepsilon = \mu_\varepsilon(y_\varepsilon)$ . Recall that  $\mu_\varepsilon(y)$  was defined in (9).

Choose a sequence  $\{\varepsilon_j\}$  satisfying  $\varepsilon_1 \leq \sigma$ ,  $\varepsilon_j > \varepsilon_{j+1}$  and  $\varepsilon_j \rightarrow 0$  as  $j \rightarrow \infty$ . Then  $\mu_{\varepsilon_j} \geq \mu_{\varepsilon_{j+1}}$  by  $\mu_{\varepsilon_j}(y) \geq \mu_{\varepsilon_{j+1}}(y)$  and there exists  $y_{\varepsilon_j} \in \mathcal{V} \setminus \{x\}$  such that  $\mu_{\varepsilon_j} = \mu_{\varepsilon_j}(y_{\varepsilon_j})$ . Let  $q_{\varepsilon_j, y_{\varepsilon_j}}$  be the element of  $\mathcal{H}_{\varepsilon_j}(y_{\varepsilon_j})$  for which  $I(q_{\varepsilon_j, y_{\varepsilon_j}}) = \mu_{\varepsilon_j}(y_{\varepsilon_j})$ . Recall that the existence of  $q_{\varepsilon_j, y_{\varepsilon_j}}$  follows from Lemma 6. Since  $I(q_{\varepsilon_j, y_{\varepsilon_j}}) \leq \mu_{\varepsilon_1}$  for all  $j \in \mathbb{N}$ , it follows from Lemma 8 that the sequence  $\{q_{\varepsilon_j, y_{\varepsilon_j}}\}_{j=1}^\infty$  is uniformly bounded. Consequently the sequence  $\{y_{\varepsilon_j}\} \subset \mathcal{V} \setminus \{x\}$  is bounded, and so it possesses a constant subsequence. Without loss of generality we assume that  $y_{\varepsilon_j} = y \in \mathcal{V} \setminus \{x\}$  for all  $j$ , otherwise instead of  $y_{\varepsilon_j}$  we must have a constant subsequence. Then we have  $I(q_{\varepsilon_j, y}) = \mu_{\varepsilon_j}(y) = \mu_{\varepsilon_j}$ . This proves the lemma.  $\square$

**Lemma 10.** For  $j \in \mathbb{N}$  large enough  $q_{\varepsilon_j, y}$  is a heteroclinic solution of system (4) connecting  $x$  and  $y$ .

*Proof.* For simplifying the notation, we set  $q_j := q_{\varepsilon_j, y}$ . From Lemmas 6 and 7 it is sufficient to prove that  $q_j(\mathbb{R}) \cap \partial B_{\varepsilon_j}(\mathcal{V} \setminus \{x, y\}) = \emptyset$  for  $j \gg 1$ . If not there is a subsequence  $\{j_m\}$  of  $\{j\}_{j=1}^\infty$  with  $j_m \rightarrow \infty$  as  $m \rightarrow \infty$ , a time sequence  $\{t_{j_m}\}$ , and  $\{\eta_{j_m}\} \subset \mathcal{V} \setminus \{x, y\}$  such that  $q_{j_m}(t_{j_m}) \in \partial B_{\varepsilon_{j_m}}(\eta_{j_m})$

and  $q_{j_m}(-\infty, t_{j_m}) \cap \partial B_{\varepsilon_{j_m}}(\mathcal{V} \setminus \{x, y\}) = \emptyset$ . Since  $I(q_{j_m}) = \mu_{\varepsilon_{j_m}} \leq \mu_{\varepsilon_1}$ , by Lemma 2 the sequence  $\{\eta_{j_m}\}$  is bounded. Otherwise  $\limsup_{m \rightarrow \infty} I(q_{j_m}) = \infty$ , a contradiction. So the sequence  $\{\eta_{j_m}\}$  possesses a constant subsequence. Without loss of generality we assume that  $\{\eta_{j_m}\}$  is a constant sequence, and set  $\eta_{j_m} = \eta$  for all  $m \in \mathbb{N}$ . Now we distinguish two cases.

Case (i): there exists a subsequence of  $\{j_m\}$ , for simplicity to notation we denote it by  $\{j_m\}$  too, such that  $q_{j_m}(t) \notin \partial B_{\varepsilon_{j_m}}(y)$  for all  $t < t_{j_m}$ . Case (ii): there is a  $K \in \mathbb{N}$  such that for each  $m > K$  there exists  $\tau_{j_m} < t_{j_m}$  for which  $q_{j_m}(\tau_{j_m}) \in \partial B_{\varepsilon_{j_m}}(y)$ . We assume without loss of generality that  $q_{j_m}(-\infty, \tau_{j_m}) \cap \partial B_{\varepsilon_{j_m}}(\mathcal{V} \setminus \{x\}) = \emptyset$ .

In case (i) we set for  $m$  large enough

$$Q_{j_m} = \begin{cases} q_{j_m}(t) & \text{if } t \leq t_{j_m}, \\ g_{j_m}(t) & \text{if } t_{j_m} < t \leq t_{j_m} + \varepsilon_{j_m}, \\ \eta & \text{if } t > t_{j_m} + \varepsilon_{j_m}, \end{cases}$$

where  $g_{j_m}$  is the minimal geodesic connecting  $\eta$  and  $q_{j_m}(t_{j_m})$ , whose existence follows from Corollary 10.8 of [10] and the fact that  $q_{j_m}(t_{j_m}) \in \partial B_{\varepsilon_{j_m}}(\eta)$  and  $\varepsilon_{j_m} \rightarrow 0$  as  $m \rightarrow \infty$ . Then  $Q_{j_m} \in \mathcal{H}_{\varepsilon_{j_m}}(\eta)$ . Moreover we have

$$\begin{aligned} & I(q_{j_m}) - I(Q_{j_m}) \\ &= \int_{t_{j_m}}^{\infty} L(t, q_{j_m}, \dot{q}_{j_m}) dt - \int_{t_{j_m}}^{t_{j_m} + \varepsilon_{j_m}} \left[ \frac{1}{2} |\dot{Q}_{j_m}(t)|_1^2 - V(t, Q_{j_m}(t)) \right] dt \\ (13) \quad & \geq \sigma \sqrt{2\alpha_\sigma} - \int_{t_{j_m}}^{t_{j_m} + \varepsilon_{j_m}} \frac{1}{2} |\dot{g}_{j_m}(t)|_1^2 dt + \int_{t_{j_m}}^{t_{j_m} + \varepsilon_{j_m}} V(t, g_{j_m}(t)) dt, \end{aligned}$$

where in the first term we have used Lemma 2 and the fact that  $q_{j_m}$  intersects  $\partial B_\sigma(y)$  and  $\partial B_\sigma(\eta)$ . Since  $|\dot{g}_{j_m}(t)|_1 \equiv \text{constant}$  for  $t \in [t_{j_m}, t_{j_m} + \varepsilon_{j_m}]$  and  $\varepsilon_{j_m} \rightarrow 0$  as  $m \rightarrow \infty$ , it follows that the second term of (13) tends to zero as  $m \rightarrow \infty$ . We now prove that the third term of (13) also goes to zero as  $m \rightarrow \infty$ .

If condition  $(d_2)$  holds, by the Mean Value Theorem for integration and the fact that  $g_{j_m}[t_{j_m}, t_{j_m} + \varepsilon_{j_m}] \subset \bar{B}_{\varepsilon_{j_m}}(\eta)$  is a geodesic, it follows that the third term of (13) goes to zero as  $m \rightarrow \infty$ .

If condition  $(d_1)$  holds, we first prove that  $\{t_{j_m}\}$  is bounded. Otherwise we assume without loss of generality that it is unbounded from below. Set  $s_{j_m} = \max\{t \in \mathbb{R}; q_{j_m}(t) \in \partial B_\sigma(x)\}$  and  $r_{j_m} = \min\{t > s_{j_m}; q_{j_m}(t) \in \partial B_\sigma(\eta)\}$ . Then  $s_{j_m} < r_{j_m} \leq t_{j_m}$ . By Lemma 3 we have  $r_{j_m} - s_{j_m} \geq$

$\sigma^2/(2\mu_{j_m}) \geq \sigma^2/(2\mu_{j_1}) := c > 0$ . In addition

$$\begin{aligned} \mu_{j_1} \geq \mu_{j_m} &= I(q_{j_m}) \geq \int_{s_{j_m}}^{r_{j_m}} -V(t, q_{j_m}(t)) dt \\ &= -V(t_{j_m}^*, q_{j_m}(t_{j_m}^*))(r_{j_m} - s_{j_m}) \geq -cV(t_{j_m}^*, q_{j_m}(t_{j_m}^*)), \end{aligned}$$

where  $t_{j_m}^* \in [s_{j_m}, r_{j_m}]$  comes from the Mean Value Theorem for integration. By Lemma 8  $\{q_{j_m}\}$  is bounded. So it follows from the fact  $q_{j_m}(t_{j_m}^*) \notin B_{\varepsilon_{j_m}}(\mathcal{V})$  and the condition  $(d_1)$  that

$$\mu_{j_1} \geq \limsup_{m \rightarrow \infty} (-cV(t_{j_m}^*, q_{j_m}(t_{j_m}^*))) = \infty,$$

a contradiction, where we have used the fact that  $\liminf_{m \rightarrow \infty} t_{j_m}^* = -\infty$ . This proves the boundedness of  $\{t_{j_m}\}$ .

Since  $\{q_{j_m}(t)\}$  and  $\{t_{j_m}\}$  are both bounded, we get from the continuity of  $V(t, q)$  that  $\{V(t, q_{j_m}(t)); t \in [t_{j_m}, t_{j_m} + \varepsilon_{j_m}], m \in \mathbb{N}\}$  is uniformly bounded. This implies that the third term goes to zero as  $m \rightarrow \infty$ .

The above proofs show that for all  $m$  sufficiently large  $I(Q_{j_m}) < I(q_{j_m}) = \mu_{\varepsilon_{j_m}}$ . But since  $Q_{j_m} \in \mathcal{H}_{\varepsilon_{j_m}}(\eta)$ , it follows from Lemma 9 that  $I(Q_{j_m}) \geq \mu_{\varepsilon_{j_m}}(\eta) \geq \mu_{\varepsilon_{j_m}} = I(q_{j_m})$ , a contradiction.

In case  $(ii)$  we can define a sequence of functions  $\{Q_{j_m}(t)\}$  as those given in case  $(i)$ , but instead of  $t_{j_m}$  by  $\tau_{j_m}$ , and of  $\eta$  by  $y$ . Then analogous arguments as those in the proof of case  $(i)$  show that  $I(Q_{j_m}) < I(q_{j_m}) = \mu_{\varepsilon_{j_m}}$  for  $m$  large enough, a contradiction. We note that in this case  $\{Q_{j_m}(t)\} \subset \mathcal{H}_{\varepsilon_{j_m}}(y)$ .

Summarizing the above proofs, we have obtained that  $q_j(\mathbb{R}) \cap \partial B_{\varepsilon_j}(\mathcal{V} \setminus \{x, y\}) = \emptyset$  for  $j$  large enough. This proves the lemma.  $\square$

Following Lemmas 9 and 10 we have finished the proof of the theorem.

### 3. APPENDIX

**Lemma 11.** *For any given  $a, b \in \mathbb{R}$  with  $a < b$ , the functional  $I_{a,b}(q)$  defined in (10) is lower semi-continuous in the weak topology restricted to  $\mathcal{H}_{1,\varepsilon}$ .*

*Proof.* The main idea of the proof follows from [11]. We claim that for  $\{q_m\} \subset \mathcal{H}_{1,\varepsilon}$ , if  $q_m \rightarrow_w q$  in  $\mathcal{H}$  as  $m \rightarrow \infty$  then  $\int_a^b \phi(q_m - q) dt \rightarrow 0$  for  $\phi \in L[a, b]$  and  $\int_a^b \phi(\dot{q}_m - \dot{q}) dt \rightarrow 0$  for  $\phi \in L^2[a, b]$ .

First  $q_m \rightarrow q$  in  $C[a, b]$ . Indeed from the proof of Lemma 6 the sequence  $\{q_m\}$  is an equicontinuous family of uniformly bounded functions in  $C[a, b]$ .

So combining the Arzela–Ascoli Theorem and the assumption  $q_m \rightarrow_w q$ , we obtain that  $q_m \rightarrow q$  uniformly in  $C[a, b]$ .

The first claim follows from

$$\left| \int_a^b \phi(q_m - q) dt \right| \leq \|q_m - q\|_{L^\infty} \left| \int_a^b |\phi| dt \right|,$$

and  $\|q_m - q\|_{L^\infty} \rightarrow 0$  uniformly as  $m \rightarrow \infty$ .

For the second claim if  $\phi \in C^1[a, b]$  the claim follows easily from the integration by parts, the Schwartz inequality and the fact that  $q_m \rightarrow q$  uniformly in  $C[a, b]$ . Since  $C^1[a, b]$  is dense in  $L^2[a, b]$ , for any  $\varepsilon > 0$  there exists  $\psi \in C^1[a, b]$  such that  $\|\phi - \psi\|_{L^2} < \varepsilon$ . Then we have

$$\left| \int_a^b \phi(\dot{q}_m - \dot{q}) dt \right| \leq \left| \int_a^b \psi(\dot{q}_m - \dot{q}) dt \right| + 4\varepsilon(\mu_\varepsilon(y) + 1),$$

where we have used the fact that  $q_m, q \in \mathcal{H}_{1,\varepsilon}$  and the Cauchy–Schwartz inequality. Taking the supremum limit on the last inequality as  $m \rightarrow \infty$ , then the claim follows from the arbitrariness of  $\varepsilon$ .

Recall that  $L(t, q, \dot{q}) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - V(t, q)$ . By the Intermediate Value Theorem we have

$$\begin{aligned} & L(t, q, p_1) - L(t, q, p_2) - L_p(t, q, p_2)(p_1 - p_2) \\ &= \frac{1}{2} \sum_{i,j=1}^n L_{p_i p_j}(t, q, \bar{p})(p_{1,i} - p_{2,i})(p_{1,j} - p_{2,j}) \geq 0, \end{aligned}$$

where  $\bar{p}$  is located between  $p_1$  and  $p_2$ . This yields

$$\begin{aligned} & I_{a,b}(q_m) - I_{a,b}(q) \\ &= \int_a^b (L(t, q_m, \dot{q}_m) - L(t, q, \dot{q}_m) + L(t, q, \dot{q}_m) - L(t, q, \dot{q})) dt \\ &\geq \int_a^b L_q(t, \tilde{q}, \dot{q}_m)(q_m - q) dt + \int_a^b L_{\dot{q}}(t, q, \dot{q})(\dot{q}_m - \dot{q}) dt, \end{aligned}$$

where  $\tilde{q}(t)$  is located between  $q_m(t)$  and  $q(t)$ . By our assumption and the form of  $L$  there exists a  $c > 0$  such that  $|L_q| \leq c(1 + |\dot{q}|^2)$  and  $|L_{\dot{q}}| \leq c(1 + |\dot{q}|)$  on the interval  $[a, b]$ . This implies that  $L_q \in L^1[a, b]$  and  $L_{\dot{q}} \in L^2[a, b]$ . Hence from the claim at the beginning of the proof of this lemma we get that

$$\liminf_{m \rightarrow \infty} (I_{a,b}(q_m) - I_{a,b}(q)) \geq 0.$$

This proves the lemma.  $\square$

#### ACKNOWLEDGEMENTS

The second author is partially supported by a MCYT/FEDER grant No. MTM2005-06098-C02-01 and by a CICYT grant No. 2005SGR 00550. The first and third authors are partially supported by NNSF of China grant No. 10671123. The third author is also partially supported by NCET of China grant No. 050391 and by SAB2006-0098 of Spain. He thanks the Centre de Recerca Matemàtica for their hospitality and the financial support.

#### REFERENCES

- [1] V. I. ARNOLD, *Mathematical Methods of Classical Mechanics* (second edition), Springer, New York, 1989.
- [2] A. V. BOLSINOV AND A. T. FOMENKO, *Integrable Hamiltonian Systems: Geometry, Topology and Classification*, Chapman & Hall/CRC, Boca Raton, FL, 2004.
- [3] K. BURNS AND M. GIDEA, *Differential geometry and topology: with a view to dynamical systems*, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2005.
- [4] P. C. CARRIÀ AND O. H. MIYAGAKI, *Existence of homoclinic solutions for a class of time dependent Hamiltonian systems*, J. Math. Anal. Appl. **230** (1999), 157–172.
- [5] CH. N. CHEN AND S. Y. TZENG, *Periodic solutions and their connecting orbits of Hamiltonian systems*, J. Diff. Eqns. **177** (2001), 121–145.
- [6] M. DO CARMO, *Riemannian Geometry*, Birkhaser, Boston, 1992.
- [7] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second order* (second edition), Springer, Berlin, 1983.
- [8] M. IZYDOREK AND J. JANCZEWSKA, *Homoclinic solutions for a class of the second order Hamiltonian systems*, J. Diff. Eqns. **219** (2005), 375–389.
- [9] M. IZYDOREK AND J. JANCZEWSKA, *Heteroclinic solutions for a class of the second order Hamiltonian systems*, J. Diff. Eqns. **238** (2007), 381–393.
- [10] J. MILNOR, *Morse Theory*, Princenton University Press, Princenton, 1969.
- [11] J. MOSER, *Selected Chapters in the Calculus of Variations*, Birkhäuser, Basel, 2003.
- [12] P. H. RABINOWITZ, *Periodic and heteroclinic orbits for a periodic Hamiltonian system*, Ann. Inst. H. Poincaré **6(5)** (1989), 311–346.
- [13] P. H. RABINOWITZ, *Homoclinic orbits for a class of Hamiltonian systems*, Proc. Roy. Soc. Edinburgh Sect. A **114** (1990), 33–38.
- [14] P. H. RABINOWITZ, *Connecting orbits for a reversible Hamiltonian systems*, Ergodic Theory Dynam. Systems **20** (2000), 1767–1784.
- [15] P. H. RABINOWITZ, *Variational methods for Hamiltonian systems*, in Handbood of Dynamical Systems, Vol 1A, Elsevier, Amsterdam, 2002, pp.1091–1127.
- [16] P. H. RABINOWITZ AND K. TANAKA, *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z. **206** (1991), 472–499.
- [17] W. RUDIN, *Real and complex analysis* (Third edition), McGraw-Hill Book Co., New York, 1987.
- [18] A. SZULKIN AND W. ZOU, *Homoclinic orbits for asymptotically linear Hamiltonian systems*, J. Funct. Anal. **187** (2001), 25–41.

<sup>1</sup> DEPARTMENT OF MATHEMATICS, SHANGHAI JIAOTONG UNIVERSITY, SHANGHAI, 200240,  
P. R. CHINA

*E-mail address:* `limliufei@sjtu.edu.cn`, `xzhang@sjtu.edu.cn`

<sup>2</sup> DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193  
BELLATERRA, BARCELONA, SPAIN

*E-mail address:* `jllibre@mat.uab.cat`