CONTRACTION GROUPS IN COMPLETE KAC-MOODY GROUPS

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ABSTRACT. Let G be an abstract Kac-Moody group over a finite field and $\overline{G}$ the closure of the image of G in the automorphism group of its positive building. We show that if the Dynkin diagram associated to G is irreducible and neither of spherical nor of affine type, then the contraction groups of elements in $\overline{G}$ which are not topologically periodic are not closed. (In those groups there always exist elements which are not topologically periodic.)

1. INTRODUCTION

Let $\mathfrak{g}$ be a continuous automorphism of a topological group G with continuous inverse. Its **contraction group** is the subgroup of G defined by

$$U_{\mathfrak{g}} := \left\{ x \in G \colon \mathfrak{g}^n(x) \longrightarrow e \quad \text{as } n \text{ goes to infinity} \right\}.$$

Interest in contraction groups has been stimulated by applications in the theory of probability measures and random walks on, and the representation theory of, locally compact groups. For these applications it is important to know whether a contraction group is closed. We refer the reader to the introduction in [2] and the references cited there for information about the applications of contraction groups and known results. Recent articles which treat contraction groups are [8] and [7].

The article [2] studied the contraction group $U_{\mathfrak{g}}$ and its supergroup

$$P_{\mathfrak{g}} := \left\{ x \in G \colon \{\mathfrak{g}^n(x) \colon n \in \mathbb{N}\} \text{ is relatively compact} \right\}$$

in the case where the ambient group is locally compact and totally disconnected, a case in which previously little was known. In contrast to $U_{\mathfrak{g}}$, the group $P_{\mathfrak{g}}$ is always closed if the ambient group G is totally disconnected [17, Proposition 3, parts (*iii*) and (*ii*)]. The group $P_{\mathfrak{g}}$ was named the **parabolic group** of the automorphism $\mathfrak{g}$ in [2] because for any inner automorphism of a semisimple algebraic group over a local field its parabolic group is the group of rational points of a rational parabolic subgroup (and every such group is of that form); the corresponding contraction

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group in that case is the group of rational points of the unipotent radical of the parabolic subgroup. In this algebraic group context, identifying parabolic subgroups (in the dynamical sense, introduced above) and their unipotent radicals with parabolic subgroups (in the algebraic group sense) and the corresponding contraction groups is a crucial technique used by G. Prasad to prove strong approximation for semisimple groups in positive characteristic [11]. This technique was later used again by G. Prasad to give a simple proof of Tits's theorem on cocompactness of open non-compact subgroups in simple algebraic groups over local fields [12], which can be proved also by appealing to Howe-Moore's property.

In this article we investigate which contraction groups of inner automorphisms in complete Kac-Moody groups are closed. Complete Kac-Moody groups (which we introduce in Section 2) are combinatorial generalizations of semisimple algebraic groups over local fields. In contrast to members of the latter class of groups, complete Kac-Moody groups are generically non-linear, totally disconnected, locally compact groups. These properties make them perfect test cases for the developing structure theory of totally disconnected, locally compact groups which was established in [17], and further advanced in [18] and [19].

Our main result is the following theorem, in whose statement the contraction group of a group element g is understood to be the contraction group of the inner automorphism $\mathfrak{g}\colon x \mapsto gxg^{-1}$.

Theorem 1 (Main Theorem)**.** *Let G be an abstract Kac-Moody group over a finite field and $\overline{G}$ be the closure of the image of G in the automorphism group of its positive building. Then the following are true:*

1. *The contraction group of any topologically periodic element in $\overline{G}$ is trivial.*
2. *If the type of G is irreducible and neither spherical nor affine, then the contraction group of any element that is not topologically periodic in $\overline{G}$ is not closed.*

Furthermore, the group $\overline{G}$ contains non-topologically periodic elements whenever G is not of spherical type.

The second assertion of Theorem 1 is in sharp contrast with the known results about contraction groups of elements in spherical and affine Kac-Moody groups. In particular, all contraction groups of inner automorphisms are closed for semisimple algebraic groups over local fields; this follows from the representation of contraction groups as rational points of unipotent radicals and we direct the reader to part 2 of Proposition 3 for a slightly more general statement.

Consequently, all contraction groups of inner automorphisms are closed for certain affine Kac-Moody groups, namely those that are geometric completions of Chevalley group schemes over the rings of Laurent polynomials over finite fields. For completions of Kac-Moody groups of any spherical type the same is seen to be true; see part 1 of Proposition 3.

Thus Theorem 1 and Proposition 3 provide another instance of the strong dichotomy between Euclidean and arbitrary non-Euclidean buildings with large automorphism groups which is already evident in results such as the Simplicity Theorem in [5] and the strong Tits alternative for infinite irreducible Coxeter groups by Margulis-Noskov-Vinberg [9, 10].

The groups covered by the second part of our Main Theorem are topologically simple [15], indeed in many cases algebraically simple [4] groups, whose flat rank assumes all positive integral values [1], and indeed are the first known groups who have non-closed contraction groups and whose flat rank can be larger than 2; we refer the reader to [19, 1] for the definition of flat rank. They are thus 'larger' but similar to the group of type-preserving isometries of a regular, locally finite tree, which is a simple, totally disconnected, locally compact group of flat rank 1, whose non-trivial contraction groups are non-closed. This follows from Example 3.13(2) in [2] and Remark 1.

The Main Theorem will be proved within the wider framework of groups with a locally finite twin root datum. Within this wider framework we need to impose the additional assumption that the root groups of the given root datum are contractive (a condition introduced in Subsection 4.2) in order to be able to prove the analogue of the second statement above. In the Kac-Moody case this condition is automatically fulfilled by a theorem of Caprace and Rémy. In all cases, the geometry of the underlying Coxeter complex will play a crucial role in the proof via the existence of 'a fundamental hyperbolic configuration', see Theorem 3.

2. FRAMEWORK

We study complete Kac-Moody groups; these were introduced in [14] under the name 'topological Kac-Moody groups'. A complete Kac-Moody group is a geometrically defined completion of an abstract Kac-Moody group over a finite field. Every Kac-Moody group is a group-valued functor, $\mathbf{G}$ say, on rings, which is defined by a Chevalley-Steinberg type presentation, whose main parameter is an integral matrix, a 'generalized Cartan matrix', which also defines a Coxeter system of finite rank; see [16, Subsection 3.6] and [13, Section 9] for details. For each ring R, the value $G := \mathbf{G}(R)$ of the functor $\mathbf{G}$ on R is an **abstract Kac-Moody group over** R.

For each field R the Chevalley-Steinberg presentation endows the abstract Kac-Moody group $\mathbf{G}(R)$ with the structure of a **group with a twin root datum**, which is the context in which our results are stated. A twin root datum is a collection $\big((U_\alpha)_{\alpha\in\Phi}, H\big)$ of subgroups of G indexed by the set Φ of roots of the associated Coxeter system (W,S) and satisfying certain axioms which ensure that the group G acts on a 'twinned' pair of buildings of type (W,S); see [13, 1.5.1]. See Subsection 0.3, respectively 0.4, in [5] for the list of axioms of a twin root datum and references to further literature on twin root data and twin buildings.

In order to define the **geometric completion** of $\mathbf{G}(R)$, assume that R is a finite field. Under this assumption all the groups which constitute the natural root datum of $\mathbf{G}(R)$ are finite; groups with a twin root datum having this property will be called **groups with a locally finite twin root datum**. The **Davis-realization** of the buildings defined by a locally finite twin root datum are locally finite, metric, $\mathsf{CAT}(0)$-complexes in the sense of [3] all of whose cells have finite diameter; see [1, Section 1.1] for a short explanation following M. Davis' exposition in [6]. The **geometric completion** of a group G with locally finite twin root datum is the closure of the image of G in the automorphism group of the Davis-realization of the positive building defined by the given root datum; if G is an abstract Kac-Moody group over a finite field that completion will be called the corresponding **complete Kac-Moody group** and denoted by $\overline{G}$.

The completion of an abstract Kac-Moody group is defined by its action on its building and our techniques rely on the $\mathsf{CAT}(0)$-geometry of the building, in particular the action of the group 'at infinity'. However, note that the topology and the completion of a group with locally finite twin root datum do not depend on the $\mathsf{CAT}(0)$-structure, only on the combinatorics of the action on the building; see Lemma 2 in [1]. Therefore one should be able to dispense with the use of the Davis-realization below.

We summarize the basic topological properties of automorphism groups of locally finite complexes in the following proposition.

Proposition 1. *Let X be a connected, locally finite cell complex. Then the compact-open topology on $\mathsf{Aut}(X)$ is a locally compact, totally disconnected (hence Hausdorff) group topology. This topology has a countable basis, hence is σ-compact and metrizable. Stabilizers and fixators of finite subcomplexes of X in $\mathsf{Aut}(X)$ are compact, open subgroups of $\mathsf{Aut}(X)$ and the collection of all fixators of finite subcomplexes form a neighborhood basis of the identity in $\mathsf{Aut}(X)$. These statements are also true for closed subgroups of $\mathsf{Aut}(X)$.*

Any closed subgroup, $\overline{G}$ say, of $\mathsf{Aut}(X)$, which admits a finite subcomplex whose $\overline{G}$-translates cover X, is compactly generated and cocompact in $\mathsf{Aut}(X)$.

Complete Kac-Moody groups hence have all the properties described above, including compact generation and co-compactness in the full automorphism group of its building even though we will not use the latter two properties in this paper.

3. Geometric reformulation of topological group concepts

In what follows, we reformulate topological group concepts in geometric terms, that is in terms of the action on the building. We begin with a geometric reformulation of relative compactness.

A closed subgroup $\overline{G}$ of the automorphism group of a connected, locally finite, metric complex X carries two natural structures of bornological group.

The first bornological group structure on $\overline{G}$ is the natural bornology induced by its topological group structure, and consists of the collection of all **relatively compact** subsets of the group $\overline{G}$.

The second bornological group structure on $\overline{G}$ is the bornology induced by the natural bornology on the metric space X, in which subsets of X are bounded if and only if they have finite diameter; this bornology on the group $\overline{G}$ consists of the collection of subsets M of $\overline{G}$ which have the property that for every bounded subset B of X the set $M.B$ is also bounded. One can verify that the latter condition on the subset M of $\overline{G}$ is equivalent to the condition that for some, and hence any, point x of X the set $M.x$ is bounded. We will call the sets in the second bornology on the group $\overline{G}$ **bounded** sets.

We now verify that these two bornologies coincide. For subsets Y, W of the metric space X define $\mathsf{Trans}_{\overline{G}}(Y,W) := \{g \in \overline{G} \colon g.Y \subseteq W\}$. Note that

$$\mathsf{Trans}_{\overline{G}}(\{y\},\{w\}) = \begin{cases} g_{wy}\overline{G}_y = \overline{G}_w g_{wy} = \overline{G}_w g_{wy} \overline{G}_y & \text{if } \exists g_{wy} \in \overline{G} \colon g_{wy}.y = w \\ \varnothing & \text{else} \end{cases}$$

Hence, whenever $\overline{G}$ is a closed subgroup of the automorphism group of a connected, locally finite complex X and y, w are points of X, the set $\mathsf{Trans}_{\overline{G}}(\{y\},\{w\})$ will be compact and open.

Lemma 1 (geometric reformulation of 'relatively compact')**.** *Let X be a connected, locally finite, metric complex, and assume that $\overline{G}$ is a closed subgroup of $\mathsf{Aut}(X)$ equipped with the compact-open topology. Then a subset of $\overline{G}$ is relatively compact if and only if it is bounded.*

Proof. We will use the criterion that a subset M of $\overline{G}$ is bounded if and only if, for some chosen vertex, x say, the set $M.x$ is bounded.

Assume first that M is a bounded subset of $\overline{G}$. This means that $M.x$ is a bounded, hence finite set of vertices. We conclude that

$$M \subseteq \bigcup_{y \in M.x} \mathsf{Trans}_{\overline{G}}(\{x\},\{y\}),$$

which shows that M is a relatively compact subset of G.

Conversely, assume that M is a relatively compact subset of $\overline{G}$. We have

$$M \subseteq \bigcup_{y \in X} \mathsf{Trans}_{\overline{G}}(\{x\},\{y\}).$$

and, since M is relatively compact, there is a finite subset $F(M,x)$ of X such that

$$M \subseteq \bigcup_{y \in F(M,x)} \mathsf{Trans}_{\overline{G}}(\{x\},\{y\}) =: T(M,x).$$

We conclude that $M.x \subseteq T(M,x).x \subseteq F(M,x)$ which shows that M is bounded. □

3.1. **Geometric reformulation of topological properties of isometries.** Under the additional condition that the complex X carries a $\mathsf{CAT}(0)$-structure, we use the previous result to reformulate the topological condition on a group element to be (topologically) periodic in dynamical terms.

Lemma 2 (weak geometric reformulation of 'topologically periodic')**.** *Let X be a connected, locally finite, metric $\mathsf{CAT}(0)$-complex. Equip $\mathsf{Aut}(X)$ with the compact-open topology and let g be an element of $\mathsf{Aut}(X)$. Then g is topologically periodic if and only if g has a fixed point.*

Proof. By Lemma 1, g is topologically periodic if and only if the group generated by g is bounded. Since a bounded group of automorphisms of a complete $\mathsf{CAT}(0)$-space has a fixed point, topologically periodic elements have fixed points.

Conversely, if g fixes the point x say, then g, and the group it generates, is contained in the compact set $\mathsf{Aut}(X)_x$. Hence g is topologically periodic. □

One can even detect the property of being topologically periodic in a purely geometric way: isometries of $\mathsf{CAT}(0)$-spaces which do not have fixed points are either parabolic or hyperbolic. If, in the previous lemma, we impose the additional condition that the complex X should have finitely many isometry classes of cells, then X is known to have no parabolic isometries and we obtain the following neat characterization.

Lemma 3 (strong geometric reformulation of 'topologically periodic')**.** *Let X be a connected, locally finite, metric $\mathsf{CAT}(0)$-complex with finitely many isometry classes of cells. Equip $\mathsf{Aut}(X)$ with the compact-open topology and let g be an element of $\mathsf{Aut}(X)$. Then the following properties are equivalent:*

(1) *g is topologically periodic;*
(2) *g has a fixed point;*
(3) *g is not hyperbolic.*

Proof. The assumption that the complex X has finitely many isometry classes of cells implies that no isometry of X is parabolic by a theorem of Bridson [3, II.6.6 Exercise (2) p. 231]. This shows that the second and third statement of the lemma are equivalent. The first and the second statement are equivalent by Lemma 2, which concludes the proof. □

In the case of interest to us, we can add a further characterization of 'topologically periodic' to those given above and we include it for completeness although we will not need to use it. The scale referred to in the statement is defined as in [17] and [18].

Lemma 4 (scale characterization of 'topologically periodic')**.** *If $\overline{G}$ is the geometric completion of a group with locally finite twin root datum (or the full automorphism group of its building) the following statements are also equivalent to the statements (1)–(3) of Lemma 3:*

(4) *the scale value $s_{\overline{G}}(g)$ is equal to 1;*
(5) *the scale value $s_{\overline{G}}(g^{-1})$ is equal to 1;*

Furthermore, $s_{\overline{G}}(g) = s_{\overline{G}}(g^{-1})$ for all g in $\overline{G}$.

Proof. This statement follows form Corollary 10 and Corollary 5 in [1]. □

3.2. **Geometric reformulation of the topological definition of a contraction group.** It follows from Lemma 4 and Proposition 3.24 in [2] that in the geometric completion of a group with locally finite twin root datum contraction groups of topologically periodic elements are bounded while the contraction groups of elements which are not topologically periodic are unbounded. In particular this observation applies to topological Kac-Moody groups.

The following lemma explains why in this paper we focus on contraction groups of non-topologically periodic elements. Note that we relax notation and denote the contraction group of inner conjugation with g by U_g.

Lemma 5 (contraction group of a topologically periodic element)**.** *Suppose that g is a topologically periodic element of a locally compact group. Then the contraction group U_g is trivial and hence closed.*

Proof. This is a special case of Lemma 3.5 in [2] where $v = g$ and $d = e$. □

Membership in contraction groups can be detected by examining the growth of fixed point sets while going to infinity. The precise formulation is as follows.

Lemma 6 (geometric reformulation of 'membership in a contraction group')**.** *Let X be a connected, locally finite, metric $\mathsf{CAT}(0)$-complex. Equip $\mathsf{Aut}(X)$ with the compact-open topology. Suppose that h is an hyperbolic isometry of X and let $-\xi$ be its repelling fixed point at infinity. Let $l\colon \mathbb{R} \to X$ be a geodesic line with $l(\infty) = -\xi$.*

Then an isometry g of X is in U_h if and only if for each $r > 0$ there is a real number $p(g,r)$ such that all points in X within distance r of the ray $l([p(g,r),\infty))$ are fixed by g.

Proof. The assumption $l(\infty) = -\xi$ implies that we may assume without loss of generality that l is an axis of h.

Suppose now that g is an isometry of X and let $r(g,n)$ be the radius of the ball around $P(g,n) := h^{-n}.l(0)$ that is fixed by g, with the convention that $r(g,n)$ equals $-\infty$ if g does not fix the point $P(g,n)$. By the definition of the contraction group U_h and the topology on $\mathsf{Aut}(X)$ the element g is contained in U_h if and only if $r(g,n)$ goes to infinity as n goes to infinity.

Since g is an isometry and l is an axis of h, the points $P(g,n)$ for n in $\mathbb{N}$ are equally spaced along $l(\mathbb{R})$. Therefore we may reformulate the condition for membership in U_h given at the end of the last paragraph as in the statement of the lemma. □

The results in Lemma5, Lemma 3 and Lemma 6 imply the following dichotomy for contraction groups.

Lemma 7 (dichotomy for contraction groups)**.** *If X is a connected, locally finite, metric $\mathsf{CAT}(0)$-complex with finitely many isometry classes of cells then we have the following dichotomy for contraction groups associated to isometries of X.*

- *Either the isometry is elliptic and its contraction group is trivial,*
- *or the isometry is hyperbolic and its contraction group is the set of isometries whose fixed point set grows without bounds when one approaches its repelling fixed point at infinity as described in Lemma 6.*

3.3. **Geometric reformulation of the topological definition of a parabolic group.** Using the compatibility result between the natural bornologies in Lemma 1 we can also prove a geometric characterization for membership in parabolic groups. We again relax notation and denote the parabolic group of inner conjugation with g by P_g.

Lemma 8 (geometric reformulation of 'membership in a parabolic group')**.** *Let X be a connected, locally finite, metric $\mathsf{CAT}(0)$-complex. Suppose that h is a hyperbolic isometry of X and let $-\xi$ be its repelling fixed point at infinity. Then P_h is the stabilizer of $-\xi$.*

Proof. Suppose first that g is an element of P_h. Let o be a point of X. By our assumption on g and by Lemma 1 there is a constant $M(g,o)$ such that

$$d(h^n g h^{-n}.o, o) = d(g.(h^{-n}.o), (h^{-n}.o)) < M(g,o) \text{ for all } n \in \mathbb{N}.$$

But the point $-\xi$ is the limit of the sequence $(h^{-n}.o)_{n\in\mathbb{N}}$ and thus by the definition of points at infinity of X we infer that g fixes $-\xi$.

Conversely, assume that g fixes the point $-\xi$. The above argument can be reversed and then shows that g is contained in P_h. □

There is a dichotomy for parabolic groups that is analogous to the dichotomy for contraction groups obtained in Lemma 7; the statement is as follows.

Lemma 9 (dichotomy for parabolic groups)**.** *If X is a connected, locally finite, metric $\mathsf{CAT}(0)$-complex with finitely many isometry classes of cells then we have the following dichotomy for parabolic groups associated to isometries of X.*

- *Either the isometry is elliptic and its parabolic group is the ambient group,*
- *or the isometry is hyperbolic and its parabolic group is the stabilizer of its repelling fixed point at infinity.*

Proof. Applying Lemma 3.5 in [2] in the case of parabolic groups with $v = g$ and $d = e$ one sees that parabolic groups defined by topologically periodic elements are equal to the ambient group; this settles the first possibility listed above. By Lemma 3 an isometry that is not elliptic must be hyperbolic and then the parabolic group has the claimed form by Lemma 8. □

We conclude this section with the following remark.

Remark 1. *Suppose G is a topological group, $\mathfrak{g} \in \mathsf{Aut}(G)$ and H is a $\mathfrak{g}$-stable subgroup of G. Then the contraction group of $\mathfrak{g}$ in H is the intersection of the contraction group of $\mathfrak{g}$ in G with H; an analogous statement is true for the parabolic groups of $\mathfrak{g}$ within H and G. Thus the geometric characterizations of contraction groups and parabolics given in Lemmas 6 and 8 and the dichotomies described in Lemma 7 and 9 also hold for subgroups of $\mathsf{Aut}(X)$ for the specified spaces X.*

4. Outline of the proof of the Main Theorem

We know from Lemma 5 that contraction groups of topologically periodic elements are trivial and hence closed. This proves statement 1 of our Main Theorem.

Under the additional condition on the type of the Weyl group given in statement 2, we will show that for any non-topologically periodic element, h say, of $\overline{G}$ the group $U_h \cap U_{h^{-1}}$ contains a $\overline{G}$-conjugate of a root group from the natural root datum for G.

4.1. The criterion implying non-closed contraction groups.

Theorem 3.32 in [2] gives 12 equivalent conditions for a contraction group in a metric totally disconnected, locally compact group to be closed. By the equivalence of conditions (1) and (4) from Theorem 3.32 in [2] the group U_h is not closed if and only if the group $\overline{U}_h \cap \overline{U}_{h^{-1}}$ is not trivial, hence the property whose verification we announced in the previous paragraph confirms statement 2 of our Main Theorem. The proof of this strengthening of statement 2 of Theorem 1 proceeds in three steps.

(1) Firstly, we show that any geodesic line, l say, can be moved to a line $l' = g.l$ with image in the standard apartment by a suitable element g of the completed group $\overline{G}$. In what follows we will be interested only in the case where the line l is an axis of a hyperbolic isometry $h \in \overline{G}$.
(2) Secondly, we use the assumption on the type of the Weyl group to show that for any geodesic line l' in the standard apartment there is a triple of roots (α, β, γ) in "fundamental hyperbolic configuration" with respect to l. By this we mean that α, β and γ are pairwise non-opposite pairwise disjoint roots, such that the two ends of l' are contained in the respective interiors of α and β.

(3) Thirdly and finally, we use that every split or almost split Kac-Moody group has (uniformly) contractive root groups, a notion introduced in Subsection 4.2 below, to arrive at the announced conclusion. More precisely, the geometric criterion for membership in contraction groups is used to show that whenever h' is a hyperbolic isometry in $\overline{G}$, the line l' is an axis of h' contained in the standard apartment and the fundamental hyperbolic configuration (α, β, γ) is chosen as mentioned in the previous item, then the root group $U_{-\gamma}$ is contained in the group $U_{h'} \cap U_{h'^{-1}}$.

In terms of the originally chosen hyperbolic isometry h and the element g of $\overline{G}$ found in step 1 above, the conclusion arrived at after step 3 is that $g^{-1}U_{-\gamma}g \subseteq U_h \cap U_{h^{-1}}$.

For our proof to work, we do not need to assume that our original group G is the abstract Kac-Moody group over a finite field. Step 1 uses that the group is a completion of a group with a locally finite twin root datum, Step 2 uses a property of the corresponding Coxeter complex and Step 3 works for groups with a locally finite twin root datum whose root groups are contractive, a notion which we introduce now.

4.2. **Contractive root groups.** As explained above, the following condition will play a central role in the proof of our Main Theorem. In the formulation of that condition, we denote the boundary wall of the half-apartment defined by a root α by $\partial\alpha$, as is customary.

Definition 1. *Let G be a group with twin root datum $(U_\alpha)_{\alpha\in\Phi}$. We say that G has **contractive root groups** if and only if for all α in Φ we have: If x is a point in the half-apartment defined by α, then the radius of the ball around x which is fixed pointwise by U_α goes to infinity as the distance of x to $\partial\alpha$ goes to infinity.*

The natural system of root groups of any split or almost split Kac-Moody group satisfies a stronger, uniform version of the condition of contractive root groups, which we introduce now. This latter condition was called condition (FPRS) in [5], where it was shown in Proposition 4 that any split or almost split Kac-Moody group satisfies it.

Definition 2. *Let G be a group with twin root datum $(U_\alpha)_{\alpha\in\Phi}$. We say that G has **uniformly contractive root groups** if and only if for each point x in the standard apartment of the positive building defined by the given root datum and all roots α in Φ whose corresponding half-apartment contains x, the radius of the ball which is fixed pointwise by U_α goes to infinity as the distance of $\partial\alpha$ to x goes to infinity.*

Remark 2. *By Lemma 6, for a group, G say, with twin root datum $(U_\alpha)_{\alpha\in\Phi}$, which has contractive root groups, for any root α the root group U_α is contained in the contraction group of any element g of G whose repelling point at infinity is defined* *by a geodesic ray contained in the interior of the half-apartment defined by α. The latter condition will be instrumental in showing our main theorem.*

Abramenko and Mühlherr constructed an example of a group with twin root datum that does not have uniformly contractive root groups. However, in that example the effect of fixed point sets staying bounded is obtained by going towards infinity along a non-periodic path of chambers. Therefore, it is not possible to find an automorphism of the building that translates in the direction of that path.

In discussions between the authors and Bernhard Mühlherr he asserted that a bound on the nilpotency degree of subgroups of the group with twin root datum would imply that fixed point sets always grow without bounds along periodic paths.

Remark 3. *It would be interesting to define and investigate quantitative versions of the notions of contractive and uniformly contractive root groups for groups with locally finite twin root datum. These quantitative versions would specify the growth of the radius of the ball fixed by a root group as a function of the distance of the center of that ball from the boundary hyperplane. We suspect that this growth might be linear in all situations if and only if all contraction groups of elements in the geometric completion of a group with locally finite twin root datum are closed.*

5. Proof of the Main Theorem

We will prove the following generalization of our Main Theorem.

Theorem 2 (strong version of the Main Theorem)**.** *Let G be a group with a locally finite twin root datum and $\overline{G}$ the closure of the image of G in the automorphism group of its positive building. Then the following are true:*

(1) *The contraction group of any topologically periodic element in $\overline{G}$ is trivial.*
(2) *If the root groups of G are contractive and the type of G is irreducible and neither spherical nor affine then the contraction group of any element that is not topologically periodic in $\overline{G}$ is not closed.*

Furthermore every element of infinite order in the Weyl group of G lifts to a non-topologically periodic element of $\overline{G}$; in particular, if the Weyl group of G is not of spherical type, then the group $\overline{G}$ contains non-topologically periodic elements.

The proof of this theorem will be obtained from several smaller results as outlined in Subsection 4.1 above. By Lemma 5, we only need to prove statement 2 and the existence statement for non-topologically periodic elements.

The first step towards the proof of statement 2 of Theorem 2 is provided by the following proposition.

Proposition 2 (geodesic lines can be moved to the standard apartment)**.** *Let G a group with locally finite twin root datum. Denote by $\overline{G}$ the geometric completion of*

G defined by the given root datum, by X the Davis-realization of the corresponding positive building and by $\mathbb{A}$ the corresponding standard apartment.

If l is a geodesic line in X, then there is an element g in $\overline{G}$ such that $g.l(\mathbb{R})$ is contained in $|\mathbb{A}|$ and intersects the fundamental chamber.

Proof. Since the group G acts transitively on chambers, there is an element g' in G such that $g'.l(\mathbb{R})$ intersects the fundamental chamber $c_0 \in \mathbb{A}$. We therefore may, and will, assume that $l(\mathbb{R})$ intersects c_0 from the outset.

Whenever l leaves $\mathbb{A}$, necessarily at a wall, use elements of the corresponding root group U_α which fixes c_0 to 'fold l into $\mathbb{A}$' . This needs to be done at increasing distance from c_0 along l 'on both sides', leading to an infinite product of elements from root groups. The sequence consisting of the partial products of that infinite product is contained in the stabilizer of c_0, which is a compact set. Hence that sequence has a convergent subsequence, which implies that the infinite product defined above is convergent, with limit g say. By construction, g attains the purpose of the element of the same name in the statement of the proposition and we are done. □

The second step in the proof of statment 2 of Theorem 2 consists of the following strengthening of Theorem 14 in [5].

Theorem 3 (a "fundamental hyperbolic configuration" exists w.r.t. any line)**.** *Let $\mathbb{A}$ be a Coxeter complex, whose type is irreducible and neither spherical nor affine. Suppose that $l\colon \mathbb{R} \to |\mathbb{A}|$ is a geodesic line. Then there is a triple of roots (α, β, γ) which are pairwise disjoint and pairwise non-opposite such that for suitably chosen real numbers a and b the rays $l(]-\infty, a])$ and $l([b, \infty[)$ are contained in the interior of the half-apartments defined by α and β respectively.*

Proof. The line $l(\mathbb{R})$ must cut some wall of $\mathbb{A}$, H say. One of the two roots whose boundary is H contains the ray $l(]-\infty, a])$ for sufficiently small a; we name that root α. Since the Coxeter complex is not of spherical type, there is another wall H' which cuts l, but not H. Call β the root whose boundary is H' and which contains the ray $l([b, \infty[)$ for sufficiently large b. The existence of a root γ as in the statement is then assured by Theorem 14 in [5], which completes the proof. □

The third and final step in the proof of statment 2 of Theorem 2 is an immediate consequence of our assumption that root groups are contractive and the geometric criterion for membership in contraction groups.

Lemma 10 (non-triviality of intersection of opposite contraction groups)**.** *Let $\overline{G}$ be a group which contains the root groups of a group with twin root datum all of whose root groups are contractive. Assume that $h \in \overline{G}$ is not topologically periodic and let l be an axis of h. If γ is a root whose position relative to l is as described in the previous lemma, then $U_{-\gamma} \subseteq U_h \cap U_{h^{-1}}$. Hence, U_h is not closed.*

Proof. Since the root group $U_{-\gamma}$ is contractive, Lemma 6 ensures that it is contained in any contraction group U_k with the property that the repelling fixed point of k at infinity is defined by a ray that is contained in the interior of the half-apartment defined by $-\gamma$. Both h and h^{-1} satisfy this condition on k, hence $U_{-\gamma} \subseteq U_h \cap U_{h^{-1}}$ as claimed. Since $U_{-\gamma}$ is not trivial, we infer from Theorem 3.32 in [2] that U_h is not closed. □

The following lemma provides the final statement of Theorem 2 and thereby concludes the proof of that theorem.

Lemma 11 (existence of non-topologically periodic elements)**.** *Let G be a group with a locally finite twin root datum and $\overline{G}$ the closure of the image of G in the automorphism group of its positive building. Then every element of infinite order in the Weyl group of G lifts to a non-topologically periodic element of $\overline{G}$; in particular, if the Weyl group of G is not of spherical type, then the group $\overline{G}$ contains non-topologically periodic elements.*

Proof. Since a Coxeter group is torsion if and only if it is of spherical type, the second claim follows from the first. In what follows, we will show that the lift of an element w in the Weyl group is topologically periodic if and only if w has finite order.

By Lemma 3, an element, n say, of $\overline{G}$ is topologically periodic if and only if its action on the building, X, has a fixed point. If that element n is obtained as an inverse image of an element, w say, of the Weyl group, it belongs to the stabilizer of the standard apartment $\mathbb{A}$. Since the Davis-realization $|\mathbb{A}|$ of the standard apartment is a complete, convex subspace of the complete $\mathsf{CAT}(0)$-space X, using the nearest-point projection from X onto $|\mathbb{A}|$, we see that the action of n on X has a fixed point if and only if its restricted action on $|\mathbb{A}|$ has a fixed point. The latter condition is equivalent to the condition that the natural action of w on $|\mathbb{A}|$ has a fixed point. Since this happens if and only if w has finite order, our claim is proved. □

6. THE CASE OF A DISCONNECTED DYNKIN DIAGRAM

The following two results may be used to reduce the determination of contraction groups for elements in arbitrary complete Kac-Moody groups to the determination of the contraction groups in the factors defined by the irreducible components. Their proofs are left to the reader.

Lemma 12 (product decomposition for root data with disconnected diagram)**.** *Let G be a group with a locally finite twin root datum such that the type of G is the product of irreducible factors whose restricted root data define groups $G_1, \ldots G_n$. Denote by $\underline{H}$ the quotient of a group H by its center. Then*

$$\underline{G} \cong \underline{G}_1 \times \cdots \times \underline{G}_n \qquad \text{and} \qquad \overline{G} \cong \overline{G}_1 \times \cdots \times \overline{G}_n\,.$$

as abstract, respectively topological, groups.

Lemma 13 (contraction groups of elements in products)**.** *Let $\overline{G}_1, \ldots, \overline{G}_n$ be locally compact groups and $(g_1, \ldots, g_1) \in \overline{G}_1 \times \ldots \times \overline{G}_n$. Then*

$$U_{(g_1,\ldots,g_n)} = U_{g_1} \times \cdots \times U_{g_n}\,.$$

We conjecture that the contraction groups for elements in a complete Kac-Moody group of spherical or affine type are always closed. Supporting evidence for that conjecture is provided by the following proposition.

Proposition 3 (contraction groups for spherical and known affine types)**.** *Let $\overline{G}$ be a totally disconnected, locally compact group. If*

(1) *either $\overline{G}$ is the geometric completion of an abstract Kac-Moody group of spherical type over a finite field,*
(2) *or $\overline{G}$ is a topological subgroup of the general linear group over a local field,*

then all contraction groups of elements in $\overline{G}$ are closed.

Proof. To show statement 1, observe that an abstract Kac-Moody group of spherical type over a finite field is a finite group. The associated complete group, $\overline{G}$, is then finite too and hence is a discrete group, because its topology is Hausdorff. Contraction groups in a discrete group are trivial, and it follows that all contraction groups of all elements in $\overline{G}$ are closed if G is of spherical type.

As noted in Remark 1, we obtain the contraction group of an element h with respect to a (topological) subgroup, H by intersecting the contraction group relative to the ambient group with H.

Thus to establish statement 2 it is enough to treat the special case of the general linear group over a local field, k say. Using the same observation again and noting that $\mathrm{GL}_n(k)$ can be realized as a closed subgroup of $\mathrm{SL}_{n+1}(k)$ via $g \mapsto \mathrm{diag}(g, \det(g)^{-1})$, it suffices to prove statement 2 in the special case of the group $\mathrm{SL}_n(k)$, where k is a local field. But contraction groups of elements in $\mathrm{SL}_n(k)$ have been shown to be k-rational points of unipotent radicals of k-parabolic subgroups in [12, Lemma 2] as explained in Example 3.13(1) in [2]; as such they are Zariski-closed and hence closed in the Hausdorff-topology induced by the field k. This proves statement 2 for the group $\mathrm{SL}_n(k)$, and, by the previous reductions, in all cases. □

There are complete Kac-Moody groups of affine type for which it is unknown whether the criterion listed under item 2 of Proposition 3 can be applied. For example, the complete Kac-Moody groups defined by the generalized Cartan-matrices $\begin{pmatrix} 2 & m \\ -1 & 2 \end{pmatrix}$ with integral $m < -4$ are of that kind.

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