

# ON THE SET OF SUBRINGS WHICH ARE DIRECTED UNIONS OF ARTINIAN SUBRINGS

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ABSTRACT. The paper contributes to the investigation of zero-dimensional rings which can be written as a directed union of Artinian subrings. We give conditions on  $\mathcal{DU}(R)$  in order to be nonempty.

## INTRODUCTION

Let  $R$  be a commutative ring with identity. We recall that  $R$  is reduced if  $\bigcap_{P \in \text{Min}(R)} P = (0)$ , where  $\text{Min}(R)$  is the set of minimal prime ideals of  $R$ , and zero-dimensional if all prime ideals are maximal. The ring  $R$  is said to be von Neumann regular if for each element  $r \in R$ , there exists  $r' \in R$  such that  $r = r^2 r'$ . It is easily seen that  $R$  is a von Neumann regular ring if and only if  $R$  is reduced and zero-dimensional. Initially we note that if  $R$  is the directed union of a family  $\{R_\alpha\}_{\alpha \in A}$  of subrings, each of dimension at most  $n$ , then  $\dim(R) \leq n$ ; this follows from the fact that any chain  $P_0 \subset P_1 \subset \dots \subset P_k$  of prime ideals of  $R$  contracts to a chain of distinct primes on some  $R_\alpha$ . Thus, a ring that is a directed union of Artinian subrings is zero-dimensional. Recently many authors have been interested in zero-dimensionality (see [6, 7, 9]). The purpose of this paper is to pursue the study of the problem of whether a von Neumann regular ring  $R$  is expressible as a directed union of Artinian subrings, raised by Gilmer and Heinzer in 1992 [3, Problem 42].

This fact leads us to consider the set of subrings of  $R$  which are directed unions of Artinian subrings of  $R$  denoted  $\mathcal{DU}(R)$  and the family of Artinian subrings  $\mathcal{A}(R)$ . Also we use  $\text{Idem}(R)$ ,  $\text{Spec}(R)$ , and  $\mathcal{C}(R)$ , respectively, to denote the set of idempotent elements of  $R$ , the set of prime ideals of  $R$ , and the set  $\{\text{char}(R/M) : M \text{ ranges over the maximal ideals of } R\}$ .

The present paper considers the following questions:

( $q_1$ ) If  $\mathcal{A}(R)$  is a nonempty set is it true that  $\mathcal{DU}(R) \neq \emptyset$  ?

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( $q_2$ ) If  $\mathcal{A}(R) \neq \emptyset$  under what conditions is the set  $\mathcal{DU}(R)$  nonempty?

( $q_3$ ) Let  $R$  be a von Neumann regular ring, under what conditions is  $R$  expressible as a directed union of Artinian subrings?

It is worthwhile recalling that any zero-dimensional ring  $R$  with finite spectrum is expressible as a directed union of Artinian subrings [7, Theorem 5.4]. This leads us to examine the case where spectrum is infinite, i.e., we are interested with rings which are expressible as a directed unions of Artinian subrings but are not Artinians.

We give a short overview of the paper. The first section is concerned with questions ( $q_1$ ) and ( $q_2$ ). We give a characterization of the set  $\mathcal{DU}(R)$ . Precisely, our main results of this section are Proposition 1.3, which gives the relationship between  $\mathcal{A}(R)$  and  $\mathcal{DU}(R)$ , and Theorem 1.12 which shows that  $\mathcal{DU}(R)$  can be a nonempty set and  $R$  is not a directed union of Artinian subrings. In the second section we give a necessary and sufficient condition for a von Neumann regular ring to be a directed union of Artinian subrings that is an answer to question ( $q_3$ ). We probe this problem in connection with their families of residue fields say  $\mathcal{F}(R) = \{\frac{R}{M} : M \text{ a maximal ideal of } R\}$ .

## 1. PRELIMINARIES AND GENERAL RESULTS

Throughout this paper, all rings are assumed to be commutative with unit. Further, all ring-homomorphisms are unital. If  $R$  is a subring of a ring  $S$ , we assume that the unity of  $S$  belongs to  $R$ . Turning to the Artinian case, two key properties of an Artinian ring  $R$  that come into play are that  $\text{Spec}(R)$  is finite and that  $R$  has only finitely many idempotents. The existence of Artinian subrings was among the primary focus of many mathematicians [6, 9, 10]. They were particularly interested in the question of whether a zero-dimensional ring  $R$  is expressible as a directed union of Artinian subrings. Naturally hereditarily zero-dimensional rings<sup>1</sup> are example. More generally each zero-dimensional ring with finite spectrum is a directed union of Artinian subrings. However not every zero-dimensional ring is expressible as a directed union of Artinian subrings. For instance, if  $R = \prod_{i=1}^{\infty} \mathbb{Q}$  denotes the infinite product of copies  $\mathbb{Q}$ , the ring  $R$  is a zero-dimensional ring that is not a directed union of Artinian subrings.

This section was inspired by a desire to understand the relationship between  $\mathcal{A}(R)$  and  $\mathcal{DU}(R)$ .

Let  $(R_j, f_{jk})$  be a directed system of rings, indexed by a directed set  $(I, \leq)$ . Let  $R = \bigcup_{j \in I} R_j$ , together with the canonical maps  $f_j : R_j \rightarrow R$ . The ring  $R$  is said to be a directed union of the  $R_j$ 's if the  $f_{jk}$ 's are inclusion maps.

<sup>1</sup>A ring  $R$  is hereditarily zero-dimensional if all subrings of  $R$  are zero-dimensional.

Thus, directed unions can be treated by assuming all  $f_{jk}$  to be monomorphisms. Notice that  $R$  need not be Artinian even if each  $R_j$  is Artinian. Next, we use the fact that the Krull dimension is preserved under integral extensions (cf.[4, (11.8)]). In particular, an integral extension ring of a zero-dimensional ring is zero-dimensional.

**Lemma 1.1.** *If  $S$  is integral over an Artinian ring  $A$ , then  $S$  is written as a directed union of Artinian subrings, and hence  $\mathcal{DU}(S) \neq \emptyset$ .*

**Proof.** we can write

$$S = \bigcup \{A[F] : F \text{ is a finite subset of } S\}.$$

The ring  $A[F]$  is intermediate between  $A$  and  $S$ , so  $\dim(A[F]) = 0$  and  $A[F]$  is a finitely generated  $A$ -module. It follows that  $A[F]$  is a Noetherian ring, and hence  $A[F]$  is Artinian (see [2, Theorem 8.5]). Now, let  $F_1$  and  $F_2$  be two finite subsets of  $S$ , then  $F = F_1 \cup F_2 \subset S$  and  $A[F_1] \subseteq A[F]$  and  $A[F_2] \subseteq A[F]$ . It follows that  $\{A[F] : F \text{ is a finite subset of } S\}$  is a directed family of Artinian subrings and hence  $S = \bigcup \{A[F] : F \text{ is a finite subset of } S\}$  is a directed union of Artinian subrings. Thus,  $\mathcal{DU}(S) \neq \emptyset$ .  $\square$

In general, if  $R$  is von Neumann regular which is integral over a Noetherian subring, then  $R$  is a directed union of Artinian subrings.

**Lemma 1.2.** *Let  $R$  be a ring. Then*

- (i) *If  $\mathcal{DU}(R) \neq \emptyset$  then  $\mathcal{C}(R)$  is finite.*
- (ii) *If  $\mathcal{C}(R)$  is infinite then  $\mathcal{A}(R) = \emptyset$ .*

**Proof.** (i) Let  $S \in \mathcal{DU}(R)$ , then  $S = \bigcup_{i \in I} S_i$  is a directed union of Artinian subrings  $S_i$ . Then  $S_i \in \mathcal{A}(R)$  and hence  $\mathcal{A}(R) \neq \emptyset$ . Therefore  $\mathcal{C}(R)$  is finite [5, Proposition 1].

(ii) Assume that  $\mathcal{A}(R) \neq \emptyset$ , let  $S \in \mathcal{A}(R)$ , then  $\mathcal{C}(R) \subseteq \mathcal{C}(S)$  and hence  $\mathcal{C}(S)$  is infinite, a contradiction with  $S$  is Artinian ( $\mathcal{C}(R)$  is finite).  $\square$

For the next results, we need the following definition.

Let  $\{R_\alpha\}_{\alpha \in A}$  be a nonempty family of rings and  $\prod_{\alpha \in A} R_\alpha$  their product. We frequently consider  $\prod_{\alpha \in A} R_\alpha$  as the set of all functions  $f : A \rightarrow \bigcup_{\alpha \in A} R_\alpha$ , such that  $f(\alpha) \in R_\alpha$  for each  $\alpha \in A$ , with addition and multiplication defined pointwise:  $(f+g)(\alpha) = f(\alpha) + g(\alpha)$  and  $(fg)(\alpha) = f(\alpha)g(\alpha)$ . In this perspective, the direct sum ideal of  $\prod_{\alpha \in A} R_\alpha$ , denoted  $\bigoplus_{\alpha \in A} R_\alpha$ , is the set of functions  $f \in \prod_{\alpha \in A} R_\alpha$  that are finitely nonzero (i.e.,  $\{\alpha \in A : f(\alpha) \neq 0 \text{ in } R_\alpha\}$  is finite). We use  $\mathbb{Z}$  and  $\mathbb{N}$ , respectively, to denote the set of integers and the set of natural numbers.

**Proposition 1.3.** *If  $R = \bigoplus_{j=1}^n R_j$  is the direct product of zero-dimensional rings  $R_j$ , then the following conditions are equivalent.*

- (i)  $\mathcal{A}(R) \neq \emptyset$ ;
- (ii)  $\mathcal{A}(R_j) \neq \emptyset$ , for each  $j = 1, \dots, n$ ;
- (iii)  $\mathcal{DU}(R_j) \neq \emptyset$ , for each  $j = 1, \dots, n$ ;
- (iv)  $\mathcal{DU}(R) \neq \emptyset$ .

We use the following lemmas to prove Proposition 1.3.

**Lemma 1.4.** *Let  $R$  be a zero-dimensional ring with only finitely many idempotent elements, then  $R$  is expressible as a directed union of Artinian subrings.*

In order to prove this result, we need the following lemmas.

**Lemma 1.5.** *Any zero-dimensional ring  $R$  with only finitely many idempotents is semi-quasilocal.*

**Proof.** Let

$M_1, \dots, M_{r+1}$  be distinct maximal ideals of  $R$ . Let  $x \in M_{r+1} \setminus (\bigcup_{i=1}^r M_i)$ , since  $\dim R = 0$  by [8, Theorem 3.1], there exists  $t \in \mathbb{Z}^+$  and  $e$  an idempotent element of  $R$  such that  $x^t R = eR$ . Hence  $e \in M_{r+1} \setminus (\bigcup_{i=1}^r M_i)$ . It follows that if  $R$  has  $n$  maximal ideals, it has at least  $n - 1$  idempotents. Therefore,  $R$  is necessarily semi-quasilocal.  $\square$

**Lemma 1.6.** [7, Corollary 5.5] *If  $R$  is a zero-dimensional semi-quasilocal ring then there exists an Artinian subring of  $R$ .*

**Proof of Lemma 1.4.** An immediate consequence of Lemma 1.5 and Lemma 1.6.  $\square$

**Proof of Proposition 1.3.**

(i)  $\Leftrightarrow$  (ii). Let  $S \in \mathcal{A}(R)$  and  $e_j$  be the idempotent element of  $R$  associated with  $\{j\}$ . Then  $S[e_1, \dots, e_n]$  is an integral extension of  $S$ . Moreover,  $S[e_1, \dots, e_n]$  is finitely generated over  $S$  and hence  $S[e_1, \dots, e_n] = Se_1 \oplus \dots \oplus Se_n$  is Artinian. Therefore,  $Se_j$  is an Artinian subring of  $R_j$ , for each  $j$ . Conversely, if  $S_j \in \mathcal{A}(R_j)$  for each  $j$ , then  $\bigoplus_{j=1}^n S_j$  is an Artinian subring of  $R$ .

(ii)  $\Leftrightarrow$  (iii). Let  $j_o \in \{1, \dots, n\}$  and suppose that  $\mathcal{A}(R_{j_o}) \neq \emptyset$ . Let  $A \in \mathcal{A}(R_{j_o})$  and  $\text{Idem}(R_{j_o}) = \{e_l\}_{l \in L}$  be the set of all idempotent elements of  $R_{j_o}$ . Two cases are then possible:

Case 1: If  $\{e_l\}_{l \in L}$  is a finite set, by Lemma 1.4,  $R_{j_o}$  is a directed union of Artinian subrings and hence  $\mathcal{DU}(R) \neq \emptyset$ .

Case 2: If  $\{e_l\}_{l \in L}$  is infinite, then  $A[\{e_l\}_{l \in L}]$  is integral over  $A$ . According to Lemma 1.1,  $A[\{e_l\}_{l \in L}]$  is a directed union of Artinian subrings. Consequently,  $\mathcal{DU}(R) \neq \emptyset$ . Conversely, if  $S = \bigcup_{i \in I} S_i \in \mathcal{DU}(R_{j_o})$  for some

$j_o \in \{1, \dots, n\}$ , then  $S_i \in \mathcal{A}(R_{j_o})$  and hence  $\mathcal{A}(R_{j_o}) \neq \emptyset$ .

(iii)  $\Leftrightarrow$  (iv). Let  $S_j \in \mathcal{DU}(R_j)$  for each  $j = 1, \dots, n$  and  $S = \bigoplus_{j=1}^n S_j$ . To show that  $S \in \mathcal{DU}(R)$  it suffices to show it for  $n = 2$ . Now suppose that  $S = S_1 \oplus S_2$ , where  $S_1 = \bigcup_{i \in J} V_i$  and  $S_2 = \bigcup_{k \in K} W_k$  are directed unions of Artinian subrings. It is easy to see that  $S = S_1 \oplus S_2 = \bigcup_{(i,k) \in J \times K} (V_i \oplus W_k)$  and  $\{V_i \oplus W_k\}_{(i,k) \in J \times K}$  is a directed family. We notice also that  $V_i \oplus W_k$  is Artinian for each  $(i, k) \in J \times K$ . It follows that  $S = \bigcup_{(i,k) \in J \times K} (V_i \oplus W_k)$  is a directed union of Artinian subrings. Conversely, let  $W = \bigcup_{j \in I} W_j \in \mathcal{DU}(R)$  and  $e_i$  be the idempotent element of  $R$  associated with  $\{i\}$ . Then  $We_i = \bigcup_{j \in I} W_j e_i$ . From (i)  $\Leftrightarrow$  (ii)  $W_j e_i \in \mathcal{A}(R_i)$  and if  $W_l \subseteq W_s$  then  $W_l e_i \subseteq W_s e_i$  and hence the family  $\{W_j e_i\}_{j \in I}$  is directed. Thus,  $We_i \in \mathcal{DU}(R_i)$ .  $\square$

Proposition 1.3 does not hold for infinite direct products as the following example shows.

**Example 1.7.** Let  $R = \prod_{i=1}^{\infty} \frac{\mathbb{Z}}{p_i \mathbb{Z}}$ , where  $\{p_i\}_{i=1}^{\infty}$  is an infinite family of distinct prime integers. Clearly,  $\mathcal{A}(\frac{\mathbb{Z}}{p_i \mathbb{Z}}) \neq \emptyset$  (respectively,  $\mathcal{DU}(R) \neq \emptyset$ ) for each  $i$ . However, since  $\{p_i : i = 1, 2, \dots\} \subseteq \mathcal{C}(R)$ , we have that  $\mathcal{C}(R)$  is infinite. Thus, by [10, Theorem 2.1],  $\mathcal{A}(R) = \emptyset$  (respectively,  $\mathcal{DU}(R) = \emptyset$ ). This means that  $R$  has no Artinian subring.

**Proposition 1.8.** Let  $R$  be a ring and  $\{L_\lambda\}_{\lambda \in \Lambda}$  its residue fields. Then  $\mathcal{A}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$  if and only if  $\mathcal{DU}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ .

**Proof.** Suppose that  $\mathcal{DU}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ , let  $S \in \mathcal{DU}(\prod_{\lambda \in \Lambda} L_\lambda)$  then  $S = \bigcup_{i \in I} S_i$  is a directed union of Artinian subrings, and hence  $S_i \in \mathcal{A}(S) \subseteq \mathcal{A}(\prod_{\lambda \in \Lambda} L_\lambda)$ . Thus,  $\mathcal{A}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ . Conversely, assume that  $\mathcal{A}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ , then  $\bigcup_{\lambda \in \Lambda} \mathcal{C}(L_\lambda) \subseteq \mathcal{C}(\prod_{\lambda \in \Lambda} L_\lambda)$  is a finite set of prime numbers. In other words,  $\bigcup_{\lambda \in \Lambda} \mathcal{C}(L_\lambda) = \{p_1, \dots, p_k\}$ , where  $p_i$  is a prime number for each  $i = 1, \dots, k$ . We can write  $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$  as a finite partition, where  $\Lambda_i = \{\lambda \in \Lambda / \text{char}(L_\lambda) = p_i\}$ , for each  $i = 1, \dots, k$ , and hence  $\prod_{\lambda \in \Lambda} L_\lambda = \prod_{\lambda \in \Lambda_1} L_\lambda \oplus \prod_{\lambda \in \Lambda_2} L_\lambda \oplus \dots \oplus \prod_{\lambda \in \Lambda_k} L_\lambda$ . Without loss of generality, we assume that  $\Lambda = \Lambda_i$ , for some  $i \in \{1, \dots, k\}$ . In other words, we suppose that  $\text{char}(L_\lambda) = p_i$  for each  $\lambda \in \Lambda$ . In this case,  $\mathbb{Z}_{p_i}^w \subseteq \prod_{\lambda \in \Lambda} L_\lambda$ , where  $w$  is the cardinality of  $\Lambda$ ,  $\mathbb{Z}_{p_i}^w$  is the infinite direct product of copies of  $\mathbb{Z}_{p_i}$ , and  $\mathbb{Z}_{p_i}$  is the prime field of characteristic  $p_i$ . Since  $\lim_{\rightarrow} \mathbb{Z}_{p_i}^{(i)} \subsetneq \mathbb{Z}_{p_i}^w$ , where  $\mathbb{Z}_{p_i}^{(i)} = \{\{x_j\}_{j=1}^{\infty} \in \mathbb{Z}_{p_i}^{w_0} : x_{i-1} = x_i = \dots\}$  is a subring of  $R$ , and  $\mathbb{Z}_{p_i}^{(i)} \simeq \mathbb{Z}_{p_i}^i$ , the product of  $i$  copies of  $\mathbb{Z}_{p_i}$ , an Artinian von Neumann regular ring. Therefore, the direct limit  $\lim_{\rightarrow} \mathbb{Z}_{p_i}^{(i)} \in \mathcal{DU}(\mathbb{Z}_{p_i}^w)$  and  $\mathcal{DU}(\mathbb{Z}_{p_i}^w) \subseteq \mathcal{DU}(\prod_{\lambda \in \Lambda} L_\lambda)$ , we must have  $\mathcal{DU}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ .  $\square$

**Remark 1.9.** If  $\mathcal{C}(R)$  is finite, then  $\mathcal{DU}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ . This follows from the fact that the finiteness of  $\mathcal{C}(R)$  implies that  $\mathcal{A}(\prod_{\lambda \in \Lambda} L_\lambda) \neq \emptyset$ .

**Proposition 1.10.** *Let  $R$  be a zero-dimensional ring and  $N(R)$  its nilradical. Then the following conditions are equivalent:*

- (i)  $\mathcal{DU}(R) \neq \emptyset$ ;
- (ii)  $\mathcal{DU}(\frac{R}{N(R)}) \neq \emptyset$ .

**Proof.** (i)  $\Rightarrow$  (ii). Let  $\varphi : R \rightarrow \frac{R}{N(R)}$  be the canonical projection, and  $T \in \mathcal{DU}(R)$ . Then  $\varphi(T)$  is a directed union of Artinian subrings of  $\frac{R}{N(R)}$ . Indeed, if  $T = \bigcup_{i \in I} T_i$  is a directed union of Artinian subrings, then it is not difficult to see that  $\varphi(T) = \bigcup_{i \in I} \varphi(T_i)$ , where  $\varphi(T_i) = \frac{T_i}{T_i \cap N(R)} = \frac{T_i}{N(T_i)}$ . Notice that if  $T_i \subseteq T_j$  then  $\frac{T_i}{N(T_i)} \subseteq \frac{T_j}{N(T_j)}$ , in other words,  $\varphi(T_i) \subseteq \varphi(T_j)$  and it is easy to see that  $\frac{T_i}{N(T_i)} \in \mathcal{A}(\frac{R}{N(R)})$  for each  $i \in I$ . Therefore,  $\varphi(T) = \bigcup_{i \in I} \varphi(T_i)$  is a directed union of Artinian subrings. Thus,  $\mathcal{DU}(\frac{R}{N(R)}) \neq \emptyset$ .  
(ii)  $\Rightarrow$  (i). Suppose that  $\mathcal{DU}(\frac{R}{N(R)}) \neq \emptyset$  and let  $S = \bigcup_{i \in I} S_i \in \mathcal{DU}(\frac{R}{N(R)})$  then  $S_i \in \mathcal{A}(\frac{R}{N(R)})$ . Let  $\sigma : R \rightarrow \frac{R}{N(R)}$  the canonical projection. Let  $i_0 \in I$  and  $R_{i_0} = \sigma^{-1}(S_{i_0})$ , the inverse image of  $S_{i_0}$ . It is not difficult to show that  $\frac{R_{i_0}}{N(R_{i_0})} \simeq S_{i_0}$  and hence  $R_{i_0}$  is a semi-quasilocal zero-dimensional subring of  $R$ . By [7, Corollary 5.5],  $R_{i_0}$  is a directed union of Artinian subrings. In other words,  $R_{i_0} \in \mathcal{DU}(R)$ .  $\square$

Now we observe that if  $R$  is a directed union of Artinian subrings, then  $\mathcal{DU}(R)$  is a nonempty set. This raises the question of what the relationship is between  $\mathcal{DU}(R) \neq \emptyset$  and the property that  $R$  is a directed union of Artinian subrings.

**Example 1.11.** Let  $\Omega = \mathbb{Q}(X)$  be a simple transcendental extension of  $\mathbb{Q}$ , where  $\mathbb{Q}$  denotes the field of rational numbers. Let  $R = \Omega^{\mathbb{N}}$  be a countable direct product of copies of  $\Omega$ . We consider  $\Omega^{(i)} = \{\{x_j\}_{j=1}^{\infty} \in \Omega^{\mathbb{N}} : x_{i-1} = x_i = \dots\}$  a subring of  $R$ . It is easy to see that  $\Omega^{(i)} \simeq \Omega^i$ , the finite product of  $i$  copies of  $\Omega$ , an Artinian von Neumann regular ring. It follows that  $\mathcal{A}(R) \neq \emptyset$ , and  $\varinjlim \Omega^{(i)} = T \subseteq \Omega^{\mathbb{N}}$ . The ring  $T$  is a directed union of Artinian subrings, in other words,  $T \in \mathcal{DU}(R)$ . However,  $R$  is not a directed union of Artinian subrings: indeed, consider  $y = (y_i)_{i \in \mathbb{N}^*} \in R$  such that  $y_i \neq y_j$  for  $i \neq j$ . Then, for each  $i \in \mathbb{N}^*$ ,  $y \notin \Omega^{(i)}$ . It follows that  $\varinjlim \Omega^{(i)} \subsetneq \Omega^{\mathbb{N}}$ . Hence  $\mathcal{A}(R) \neq \emptyset$  does not imply that  $R$  is a directed union of Artinian subrings.

Let  $R$  be a ring and  $\{R_\alpha\}_{\alpha \in A}$  an infinite family of nonzero rings such that  $R$  is, up to isomorphism, a subring of each  $R_\alpha$ . We use  $R^*$  to denote

the diagonal imbedding of  $R$  in  $\prod_{\alpha \in A} R_\alpha$ , that is  $R^* = \varphi(R)$ , where  $\varphi : R \hookrightarrow \prod_{\alpha \in A} R_\alpha$  is a monomorphism defined by  $\varphi(x) = \{x_\alpha\}_{\alpha \in A}$  such that  $x_\alpha = x$  for each  $\alpha \in A$ .

**Theorem 1.12.** *Let  $R$  be a zero-dimensional ring and  $\mathcal{F}(R) = \{L_i\}_{i \in I}$  its set of residue fields. Assume that for each pair  $(i, j) \in I^2$  we have  $L_j \cap L_k = L \notin \mathcal{F}(R)$ . Then*

- (1)  $\mathcal{DU}(R) \neq \emptyset$ .
- (2)  $R$  is not a directed union of Artinian subrings.

To prove this result, we need the following Lemma.

**Lemma 1.13.** [11, Proposition 2.1] *Let  $R$  be a Von Neumann regular ring and  $R \subset T \subset \prod_{\lambda \in \Lambda} \frac{R}{M_\lambda}$  such that  $T = \bigcup_{i \in I} T_i$  is a directed union of Artinian subrings and  $\{M_\lambda\}_{\lambda \in \Lambda} = \text{Spec}(R)$ . Then  $\mathcal{F}(R) = \mathcal{F}(T)$ .*

**Proof of Theorem 1.12.** (1) According to Lemma 1.2, we assume that  $\mathcal{C}(R)$  is finite, otherwise,  $\mathcal{DU}(R) = \emptyset$ . Since  $\mathcal{C}(R) = \{p_1, \dots, p_l\}$  is a finite set of prime integers, we can write  $I = I_1 \cup \dots \cup I_l$  as a partition of  $I$  where  $I_j = \{i \in I / \text{char}(L_i) = p_j\}$  for  $j = 1, \dots, l$ . Let  $T_j = \prod_{i \in I_j} L_i$  for each  $j = 1, \dots, l$ . Thus  $\prod_{i=1}^l T_i$  is isomorphic to  $\prod_{i \in I} L_i$ . For  $j = 1, \dots, l$ , let  $\mathbb{Z}_{p_j}$  be the prime subfield of characteristic  $p_j$  and  $\mathbb{Z}_{p_j}^{I_j}$  the direct product of copies  $\mathbb{Z}_{p_j}$ . Then  $\prod_{j=1}^l \mathbb{Z}_{p_j}^{I_j}$  is a subring of  $\prod_{i \in I} L_i$ . By [7, Theorem 6.7],  $\prod_{j=1}^l \mathbb{Z}_{p_j}^{I_j} = \bigcup_{k \in A} S_k$  is a directed union of Artinian subrings. Let  $R_k = R \cap S_k$  is a von Neumann regular subring of  $R$ , for each  $k \in I$ . Since each  $R_k$  is a subring of  $S_k$  and  $|\text{Idem}(R_k)| \leq |\text{Idem}(S_k)|$ , it follows that each  $R_k$  is Artinian. The family  $\{R_k\}_{k \in A}$  is directed because the family  $\{S_k\}_{k \in A}$  is so. It follows that  $\bigcup_{k \in A} R_k \subseteq R$  is a directed union of Artinian subrings. Thus  $\mathcal{DU}(R) \neq \emptyset$ , and hence  $\mathcal{DU}(R) \neq \emptyset$ .

(2) Since  $\mathcal{C}(R) = \{p_1, \dots, p_l\}$  and  $\mathcal{F}(R) = \mathcal{F}_1(R) \cup \mathcal{F}_2(R) \cup \dots \cup \mathcal{F}_l(R)$ , where  $\mathcal{F}_j(R) = \{F \in \mathcal{F}(R) / \text{char}(F) = p_j\}$ , to prove that the condition of theorem 1.12 is satisfied for  $\mathcal{F}(R)$  it suffices to show that it is satisfied for each  $\mathcal{F}_j(R)$ . In this case we suppose that  $\mathcal{F}(R) = \mathcal{F}_j(R)$  for some  $j \in \{1, \dots, l\}$ . Now, assume that  $L \notin \mathcal{F}_j(R)$ . Let  $\mathcal{S}$  be the subring of  $\prod_{j \in I} L_j$  consisting of eventually constant sequences. Thus  $\mathcal{S} \simeq L^* + J$ , the  $L$ -subalgebra of  $\prod_{j \in I} L_j$  generated by the direct sum ideal  $J = \bigoplus_{i \in I} L_i$ , where  $L^*$  is the diagonal imbedding of  $L$  in  $\prod_{j \in I} L_j$ . First, we claim that  $\mathcal{S}$  is the maximal subring of  $\prod_{j \in I} L_j$  with respect to being a directed union of Artinian subrings. Let  $T = \bigcup_{j \in J} T_j$  be a subring of  $\prod_{\alpha \in A} L_\alpha$  that is a directed union of Artinian subrings, and let  $t = \{t_\alpha\}_{\alpha \in A} \in T$ . There exists  $j_o \in J$  such that  $t \in T_{j_o}$  and  $T_{j_o}$  is a finite product of fields. Hence

$t \in \mathcal{S}$ . We claim that  $\mathcal{F}(\mathcal{S}) = \{L\} \cup \{L_j\}_{j \in I}$ . Let  $p_i : T = \prod_{j \in I} L_j \rightarrow L_i$  be the canonical projection and  $p_{i|\mathcal{S}}$  its restriction on  $\mathcal{S}$ , which is a surjective homomorphism. We have that  $\text{Ker} p_{i|\mathcal{S}} = (1 - e_i)T \cap \mathcal{S} = (1 - e_i)\mathcal{S} = M_i$ , with  $e_i$  the primitive idempotent with support  $\{i\}$ , and  $\mathcal{S}/M_i \simeq L_i$  for each  $i \in I$ . Also,  $J$  is a maximal ideal of  $\mathcal{S}$  and  $\mathcal{S}/J \simeq (L)^* \simeq L$ . Thus  $\{J\} \cup \{M_i\}_{i \in I} \subseteq \text{Max}(\mathcal{S})$ . Let  $P \in \text{Spec}(\mathcal{S})$ , if  $J \subseteq P$ , then  $J = P$ . If  $J \not\subseteq P$  then  $e_i \notin P$ , for some  $i \in I$ , and hence  $1 - e_i \in P$ . Therefore,  $M_i \subseteq P$  and  $P = M_i$ . Consequently,  $\text{Max}(\mathcal{S}) = \{J\} \cup \{M_i\}_{i \in I}$ . Thus  $\mathcal{F}(\mathcal{S}) = \{L\} \cup \{L_j\}_{j \in I}$ . If  $R$  is a directed union of Artinian subrings, then  $R \subset \mathcal{S}$ . By Lemma 1.13,  $\mathcal{F}(R) = \mathcal{F}(\mathcal{S})$ , a contradiction to the fact that  $L \notin \mathcal{F}(R)$ .  $\square$

**Corollary 1.14.** *Let  $R$  be a ring and  $S$  is a multiplicative closed subset of  $R$ .*

- (i) *If  $R$  is a directed union of Artinian subrings, then  $S^{-1}R$  so is.*
- (ii) *If  $\mathcal{DU}(R) \neq \emptyset$ , then  $\mathcal{DU}(S^{-1}R) \neq \emptyset$ .*
- (iii) *We have  $\mathcal{DU}(R(X)) \neq \emptyset$  if  $\mathcal{DU}(R) \neq \emptyset$ , where  $R(X)$  is the Nagata ring and  $X$  is an indeterminate over  $R$ .*

**Proof.** (i) Suppose that  $R = \bigcup_{\alpha \in A} R_\alpha$  is a directed union of Artinian subrings and  $S_\alpha = S \cap R_\alpha$  be a multiplicative closed subset of  $R_\alpha$ . It is not difficult to show that  $S^{-1}R = \bigcup_{\alpha \in A} S_\alpha R_\alpha$ . Since  $R_\alpha$  is Artinian, the localization  $S_\alpha^{-1}R_\alpha$  is also Artinian. The family  $\{S_\alpha^{-1}R_\alpha\}_{\alpha \in A}$  is directed since  $\{R_\alpha\}_{\alpha \in A}$  so is. It follows that  $S^{-1}R = \bigcup_{\alpha \in A} S_\alpha R_\alpha$  is a directed union of Artinian subrings.

(ii) Suppose that  $\mathcal{DU}(R) \neq \emptyset$ , let  $T = \bigcup_{j \in J} T_j \in \mathcal{DU}(R)$  and  $U = S \cap T$  be a multiplicative closed subset of  $T$ . By (i),  $U^{-1}T \subseteq S^{-1}R$  is a directed union of Artinian subrings. It follows that  $\mathcal{DU}(S^{-1}R) \neq \emptyset$ . As  $R(X) = S^{-1}R[X]$ , where  $S = R[X] \setminus \bigcup \{MR[X] : M \text{ is a maximal ideal of } R\}$ , we have  $\mathcal{DU}(R(X)) \neq \emptyset$  if  $\mathcal{DU}(R) \neq \emptyset$ .

(iii) If  $R = \bigcup_{i \in I} R_i$  is a directed union of Artinian subrings, then  $R(X) = \bigcup_{i \in I} R_i(X)$ . Since each  $R_i$  is Noetherian, by [12, (6.17)],  $R_i(X)$  is also Noetherian and each  $R_i(X)$  is zero-dimensional as each  $R_i$  is zero-dimensional (cf. [1, Proposition 1.21]). By [2, Theorem 8.5],  $R_i(X)$  is an Artinian ring for each  $i \in I$ . The family  $\{R_i(X)\}_{i \in I}$  is directed because so is the family  $\{R_i\}_{i \in I}$ . Then  $R(X)$  is a directed union of Artinian subrings.  $\square$

## 2. VON NEUMANN REGULAR RINGS AS A DIRECTED UNION OF ARTINIAN SUBRINGS

Let  $R$  be a von Neumann regular ring and  $\{M_i\}_{i \in I}$  its spectrum. Since  $R$  is reduced, we have  $\bigcap_{i \in I} M_i = (0)$  and hence the homomorphism  $\varphi : R \rightarrow \prod_{i \in I} \frac{R}{M_i}$ , defined by  $\varphi(x) = x + M_i$ , is injective. This allows us to view  $R$  as a subring of  $\prod_{i \in I} \frac{R}{M_i}$ . We identify  $x$  with its image  $\{x_i\}_{i \in I} \in \prod_{i \in I} \frac{R}{M_i}$ . Finally, we denote  $F_i = \frac{R}{M_i}$  for each  $i \in I$ . We assume that there is a field  $\Omega$  containing each  $F_i$ . We can always make this assumption if the  $F_i$  have the same characteristic.

Given  $x \in R$ , let  $x_i \in \Omega$  be the coset  $x + M_i \in F_i$ , but viewed as an element of  $\Omega$ . We identify  $x$  with its image  $\{x_i\}_{i \in I} \in \Omega^I$ , the infinite direct product of copies of  $\Omega$ . Given  $y = \{y_i\}_{i \in I} \in \Omega^I$ , let  $\|y\| = \{y_i : i \in I\} \subseteq \Omega$ . Finally, we put

$$\mathcal{S} = \{y \in \Omega^I : \|y\| \text{ is finite}\}$$

**Proposition 2.1.** *With the notation and assumptions above, we have the following:*

- (1)  $\mathcal{S}$  is von Neumann regular.
- (2)  $\mathcal{S}$  is a directed union of Artinian subrings.
- (3) If  $R \subset \mathcal{S}$  then  $R$  is a directed union of Artinian subrings.

To prove this result, we require the following two lemmas.

**Lemma 2.2.** *Let  $\{S_\alpha\}_{\alpha \in A}$  be a family of von Neumann regular subrings of a von Neumann regular ring  $T$ . Then  $R := \bigcap_{\alpha \in A} S_\alpha$  is von Neumann regular.*

**Proof.** Given  $x \in R$ , let  $y$  be the unique element of  $T$  satisfying  $xyx = x$  and  $xyy = y$ . For the existence, choose any  $z$  such that  $xzx = x$  and put  $y := zxz$ . For the uniqueness, just note that uniqueness is clear for fields, so it is true locally and therefore globally. The existence and uniqueness, applied to each  $S_\alpha$ , shows that  $y \in S_\alpha$  for each  $\alpha$ . Thus  $y \in R$  as desired.  $\square$

**Lemma 2.3.** *Let  $R$  and  $S$  be von Neumann regular rings, with  $R$  a subring of  $S$ .*

- (i) *If  $S$  is a directed union of Artinian subrings, so is  $R$ .*
- (ii) *If  $\mathcal{DU}(S) \neq \emptyset$ , then  $\mathcal{DU}(R) \neq \emptyset$ .*

**Proof.** (i) Write  $S = \bigcup_{i \in I} S_i$  as the directed union of Artinian rings  $S_i$ . Since  $R = \bigcup_{i \in I} R_i$  is the directed limit of the rings  $R_i = R \cap S_i$ , it will suffice to show that each  $R_i$  is Artinian. By Lemma 2.2, we know at least that  $R_i$  is von Neumann regular. Since  $R_i$  has only finitely many idempotents (as  $R_i \subseteq S_i$ ),  $R_i$  is a direct product of finitely many fields.  $\square$

**Proof of Proposition 2.1.** (1) Given  $x \in \mathcal{S}$ , define  $y \in \Omega^I$  by letting  $y_i = x_i^{-1}$  if  $x_i \neq 0$ , and  $y_i = 0$  if  $x_i = 0$ . Then  $y \in \mathcal{S}$  and  $xyx = x$ .  
(2) Given a partition  $\mathcal{P}$  of  $I$  into pairwise disjoint sets  $J_1, \dots, J_n$ , let  $\mathcal{A}_{\mathcal{P}}$  be the subring of  $\Omega^I$  consisting of those elements  $\{y_i\}_{i \in I}$  that are constant on each set  $J_r$  in the partition, that is,  $i, j \in J_r$  implies  $y_i = y_j$ . Then  $\mathcal{A}_{\mathcal{P}} \simeq K_1 \times \dots \times K_n$ , where  $K_r = \bigcap_{i \in J_r} F_i$ . Thus each  $\mathcal{A}_{\mathcal{P}}$  is Artinian. The family  $\{\mathcal{A}_{\mathcal{P}}, \mathcal{P} \text{ is a partition of } I\}$  is directed. Indeed, If  $\mathcal{P} = \{J_i\}_{i=1}^l$  and  $\mathcal{Q} = \{I_j\}_{j=1}^k$  are finite partitions of  $I$ , then  $\mathcal{P} \cap \mathcal{Q} = \{J_i \cap I_k / 1 \leq j \leq l, 1 \leq i \leq k\}$  is a finite partition of  $I$ , and  $\mathcal{A}_{\mathcal{P}} \cup \mathcal{A}_{\mathcal{Q}} \subseteq \mathcal{A}_{\mathcal{P} \cap \mathcal{Q}}$ . Since  $\mathcal{S}$  is the union of the rings  $\mathcal{A}_{\mathcal{P}}$ , (2) is proved.  
(3) This follows from (1), (2) and Lemma 2.3.  $\square$

Here is a verification that  $\mathbb{Q}^{\mathbb{N}}$  is not a direct limit of Artinian rings.

**Lemma 2.4.** *Let  $K$  be a field, let  $I$  be a non-empty index set, and put  $\Pi = K^I$ . We identify  $K$  with the subfield of  $\Pi$  consisting of constant functions in  $\Pi$ . Then  $K$  is a maximal subfield of  $\Pi$ .*

**Proof.** Suppose  $L$  is a subfield of  $\Pi$  that properly contains  $K$ . Choose two elements  $x, y \in L$  that are linearly independent over  $K$ . Choose  $i \in I$ , and put  $x_i = x(i)$  and  $y(i) = y_i$ . Then the  $i^{\text{th}}$  coordinate of  $z := y_i x - x_i y$  is 0, whence  $z$  is a non-unit of  $\Pi$ . Since  $L$  is a field and  $z \in L$ ,  $z = 0$ . This forces  $y_i = 0$  (since  $x$  and  $y$  are linearly independent). But then  $y$  is not a unit, and this means that  $y = 0$ , again contradicting linear independence.  $\square$

Now let  $A$  be an Artinian subring of  $\Pi := \mathbb{Q}^I$ , where  $I$  is any index set. Let  $e_1, \dots, e_n$  be the primitive idempotents of  $A$ , with  $e_1 + \dots + e_n = 1$ . These idempotents give a corresponding partition  $\mathcal{P} = S_1 \cup \dots \cup S_n$  of  $\mathbb{N}$ , where  $S_i$  is the support of  $e_i$ . The field  $Ae_i$  is a subfield of  $\Pi e_i \cong \mathbb{Q}^{S_i}$ . Therefore  $Ae_i$  contains the prime subfield  $\mathbb{Q}$  of  $\mathbb{Q}^{S_i}$  (consisting of functions that are constant on  $S_i$ ), and by the Lemma  $Ae_i$  consists precisely of the constant functions on  $S_i$ . It follows that  $A$  consists of the functions in  $\Pi$  that are constant on each set  $S_i$ .

Given a partition  $\mathcal{P}$  of  $I$  into a finite number of pairwise disjoint sets, let  $\mathcal{A}_{\mathcal{P}}$  be the subring of  $\mathbb{Q}^I$  consisting of functions that are constant on each set in  $\mathcal{P}$ . Of course  $\mathcal{A}_{\mathcal{P}}$  is isomorphic to  $\mathbb{Q}^n$ , where  $n$  is the number of sets in the partition. In particular,  $\mathcal{A}_{\mathcal{P}}$  is Artinian. We have proved the following:

**Proposition 2.5.** *The Artinian subrings of  $\mathbb{Q}^I$  are precisely the rings of the form  $\mathcal{A}_{\mathcal{P}}$ , where  $\mathcal{P}$  is some finite partition of  $I$ .*

**Corollary 2.6.** *If  $I$  is infinite, then  $\mathbb{Q}^I$  is not a direct limit of Artinian subrings.*

**Lemma 2.7.** *Let  $R$  be a zero-dimensional ring with finite spectrum, then  $R$  is expressible as a finite product of zero-dimensional quasi-local subrings.*

**Proof.** Let  $\text{Spec}(R) = \{M_i\}_{i=1}^n$  be the spectrum of  $R$ . Let  $S_{M_i}(0)$  to denote  $\text{Ker}\varphi_i$  for each  $i = 1, \dots, n$ , where  $\varphi_i : R \rightarrow R_{M_i}$  and  $\varphi_i(r) = \frac{r}{1}$ , is the canonical homomorphism. Since  $\text{Rad}(S_{M_i}(0)) = M_i$ ,  $S_{M_i}(0)$  is a primary ideal. Note that  $\bigcap_{i=1}^n S_{M_i}(0) = (0)$  and  $S_{M_i}(0) + S_{M_j}(0) = R$  for each  $i \neq j$  in  $\{1, \dots, n\}$ . Therefore,  $R \simeq \frac{R}{\bigcap_{i=1}^n S_{M_i}(0)}$ . By the Chinese remainder Theorem,  $R \simeq \prod_{i=1}^n \frac{R}{S_{M_i}(0)}$ , where  $\frac{R}{S_{M_i}(0)}$  is quasi-local and zero-dimensional, for  $i = 1, \dots, n$ .  $\square$

**Lemma 2.8.** *Let  $R$  be a von Neumann regular ring. Then the following conditions are equivalent.*

- (i)  $R$  is Artinian;
- (ii)  $R$  is a finite product of fields;
- (iii)  $R$  is Noetherian.

**Proof.** We suppose that  $R$  is von Neumann regular and Artinian then by [2, Corollary 8.2],  $\text{Spec}(R)$  is finite and hence  $R = R_1 \oplus \dots \oplus R_n$ , where each  $R_i$  is a quasi-local and zero-dimensional ring, for  $i = 1, \dots, n$ . Since  $R$  is von Neumann regular, each  $R_i$  is a von Neumann regular ring. As each  $R_i$  is quasi-local, [9, Theorem 3.1],  $R_i$  is a field for each  $i = 1, \dots, n$ . It follows that  $R$  is a finite product of fields. The rest of the proof is straightforward since every subring of a reduced ring is reduced.  $\square$

**Proposition 2.9.** *Let  $R$  be a von Neumann regular ring and  $\mathcal{F}(R) = \{F_\lambda\}_{\lambda \in \Lambda}$  its residue fields. If there exists  $k \in \mathbb{N} \setminus \{0\}$  such that the set  $\{\lambda \in \Lambda : |F_\lambda| > k\}$  is finite, then  $\mathcal{DU}(R) \neq \emptyset$ .*

**Proof.** Since  $R$  is a reduced ring, we have  $\bigcap_{\lambda \in \Lambda} M_\lambda = (0)$ . Then the homomorphism  $\varphi : R \rightarrow \prod_{\lambda \in \Lambda} R/M_\lambda$  defined by  $\varphi(r) = \{r_\lambda\}_{\lambda \in \Lambda}$ , where  $r_\lambda \equiv \bar{r} \text{ Mod } M_\lambda$ , is injective. In this case we regard  $R$  as a subring of  $\prod_{\lambda \in \Lambda} F_\lambda$ , where  $F_\lambda = \frac{R}{M_\lambda}$  for each  $\lambda \in \Lambda$ . As  $\{\lambda \in \Lambda : |F_\lambda| > k\}$  is finite for some  $k \in \mathbb{N}^*$ , by [7, Theorem 6.7]  $\prod_{\lambda \in \Lambda} F_\lambda$  is a directed union of Artinian subrings. By Proposition 1.3,  $R$  is also a directed union of Artinian subrings and hence  $\mathcal{DU}(R) \neq \emptyset$ .  $\square$

Throughout this paper, we assume that for each  $k \in \mathbb{N}^*$  the set  $\{\lambda \in \Lambda : |F_\lambda| > k\}$  is infinite.

**Remark 2.10.** (1) From Proposition 2.9, if  $F$  is a field, then the infinite direct product  $\prod_{\alpha \in A} F$  is a directed union of Artinian subrings if and only if  $F$  is finite.

(2) Let  $L$  be a field, then  $\mathcal{DU}(\prod_{\alpha \in A} L) \neq \emptyset$ .

**Theorem 2.11.** *Let  $R$  be a von Neumann regular ring with  $\mathcal{F}(R) = \{L_\alpha\}_{\alpha \in A}$ . Then  $\mathcal{S} = \{r = \{r_\alpha\}_{\alpha \in A} \in \prod_{\alpha \in A} L_\alpha : \|r\| \text{ is finite}\}$  is a directed union of Artinian subrings if and only if for each subset  $B$  of  $A$ , there exists a partition  $B = \bigcup_{i=1}^s B_i$ , where each family  $\{L_\alpha : \alpha \in B_i\}$  has a common subfield, up to isomorphism.*

**Proof.** Assume that  $\mathcal{S}$  is a directed union of Artinian subrings and let  $B$  be a subset of  $A$ . Let  $f \in \mathcal{S}$ , then  $f$  belongs to a finite product of fields and hence it has only finitely many distinct components  $f_1, \dots, f_t$ . Now, let  $A_i = \{\alpha \in A : f(\alpha) = f_i\}$ , for each  $i = 1, \dots, t$ , so  $A = \bigcup_{i=1}^t A_i$  is a partition of  $A$  which satisfies that  $\bigcap_{\alpha \in A_j} L_\alpha$  is a field, up to isomorphism, for each  $j = 1, \dots, t$ . So we only consider  $B = \bigcup \{A_i \cap B : 1 \leq i \leq t\}$ , with  $A_k \cap B \neq \emptyset$  for each  $k = 1, \dots, t$ . Furthermore,  $\{L_\alpha : \alpha \in A_k \cap B\}$  has a common subfield, up to isomorphism. Conversely, to show that  $\mathcal{S}$  is a directed union of Artinian subrings, we need only to show that for each  $f$  in  $\mathcal{S}$  the set  $\{f(\alpha) : \alpha \in A\} = \{f_1, \dots, f_t\}$  is contained in a finite product of fields. For, let  $f \in \mathcal{S}$ , and consider the partition  $A_f = \bigcup_{i=1}^n A_i$  associated with  $f$ , i.e.,  $A_i = \{\alpha \in A : f(\alpha) = f_i\}$  and  $i = 1, \dots, t$ . Let  $f_i^* = (f_i, \dots, f_i, \dots) \in \prod_{\alpha \in A_i} L_\alpha$ , by hypothesis, there exists a partition  $A_i = \bigcup_{k=1}^{s_i} I_k$ ,  $i = 1, \dots, k$ , such that each family of fields  $\mathcal{F}_k = \{L_\alpha : \alpha \in I_k\}$  has a common subfield  $K_k^i$  (which need not belong to  $\mathcal{F}_k$ ). Thus,  $f_i^* \in \prod_{k=1}^{s_i} K_k^{i*}$ , where  $K_k^{i*}$  denotes the diagonal imbedding of  $K_k^i$  in  $\prod_{\alpha \in I_k} L_\alpha$ . Consequently,  $f \in \prod_{i=1}^t (\prod_{k=1}^{s_i} K_k^{i*})$  and hence  $f$  belongs to a finite product of fields.  $\square$

**Corollary 2.12.** *A von Neumann regular ring  $R$  is a directed union of Artinian subrings if and only if  $R \subseteq \mathcal{S}$ .*

**Corollary 2.13.** *A von Neumann regular ring  $R$  is imbeddable in a directed union of Artinian subrings if and only if  $R$  has the same property.*

**Example 2.14.** (1) Let  $\{p\} \cup \{q_i\}_{i \in \mathbb{N}^*}$  be an infinite family of distinct positive prime integers. Let  $\mathcal{F} = \{GF(p)\} \cup \{GF(p^{q_i})\}_{i=1}^\infty$  be a family of finite Galois fields, and  $\alpha$  an element such that  $\alpha^2 = 1$  with  $\alpha \notin GF(p)$ . Let  $\mathcal{L} = GF(p)(\alpha)$  be a simple algebraic extension of  $GF(p)$ . We denote  $\varphi_i$  the imbedding of  $GF(p)$  in  $GF(p^{q_i})$  for each  $i \in \mathbb{N}^*$ . Let  $\varphi = \prod_{i=1}^\infty \varphi_i$ ,  $T = \prod_{i=1}^\infty GF(p^{q_i})$  and  $I = \bigoplus_{i=1}^\infty GF(p^{q_i})$  the direct sum ideal of  $T$ . We denote  $GF(p)^* = \varphi(GF(p)) = R_o$  the diagonal imbedding of  $GF(p)$  in  $T$ . Let  $\mathcal{S}_1 = R_o + I$ , since  $\mathcal{S}_1$  is a subring of  $T$  and  $\dim(\mathcal{S}_1) = 0$ , the ring  $\mathcal{S}_1$  is a von Neumann regular ring. We have shown in the proof of Theorem 2.11 that  $\mathcal{F}(\mathcal{S}_1) = \{GF(p)\} \cup \{GF(p^{q_i}) : i \in \mathbb{N}^*\}$ . From  $\mathcal{F}(\mathcal{S}_1)$  we construct a direct union of Artinian subrings. Let  $S_1 = GF(p^{q_1}) \times GF(p)^* \simeq GF(p^{q_1}) \times GF(p)$ ;  $\dots$ ;  $S_i = GF(p^{q_1}) \times \dots \times GF(p^{q_i}) \times GF(p)^* \simeq GF(p^{q_1}) \times \dots \times GF(p^{q_i}) \times$

$GF(p)$ . We have that  $S_j \subset S_{j+1}$ , for each positive integer  $j \in \mathbb{N}^*$ . Therefore,  $S_1 = \bigcup_{i \in \mathbb{N}^*} S_i$  is an increasing union of Artinian subrings. We can generalize this example to the case where  $\mathcal{F}(R) = \mathcal{F}$  has a partition  $\mathcal{F} = \bigcup_{i=1}^n \mathcal{F}_i$  into finite nonempty subfamilies  $\mathcal{F}_i$  where all elements of  $\mathcal{F}_i$  have the same characteristic, and every element of  $\mathcal{F}_i$  has a common subfield.

(2) Let  $R = GF(p)(\alpha) + \mathcal{J}$ , the  $\mathcal{L}$ -subalgebra of  $T_1 = \prod_{i=1}^{\infty} \mathcal{L}(\zeta_i)$  generated by the direct sum ideal  $\mathcal{J} = \bigoplus_{i=1}^{\infty} \mathcal{L}(\zeta_i)$ , where  $\zeta_i$  is a  $p^q$ -primitive root of unity and  $q$  is a prime integer. For each  $i \in \mathbb{Z}^+$ , let  $\phi_i : \mathcal{L} \rightarrow \mathcal{L}(\zeta_i)$  be the field-homomorphism taking  $\alpha$  to  $\zeta_i$ . Let  $\phi = \{\phi_i\}_{i=1}^{\infty} : T_1 \rightarrow T_1$ , a ring-homomorphism. Let  $R_1 = \phi(R)$ . Being isomorphic to  $R$ , the ring  $R_1$  is a directed union of Artinian subrings. We observe that the element  $\zeta = \{\zeta_i\}_{i=1}^{\infty} \in R_1$  which is not in  $\mathcal{S}$  but is contained in  $\phi(\mathcal{S})$  ( $\mathcal{S}$  is the maximum among all subrings of  $T_1$  that are directed union of Artinian subrings).

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