

CHOOSING AN ATTACKER BY A LOCAL DERIVATION

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ABSTRACT. Within the ω -limit set of a point, a uniform method is given for such problems as choosing for each point in its kernel a recurrent point that attacks it. In the light of recent counter-examples, the method appears to be optimal. It is inspired by a lemma of Kunugui and forms a local derivation.

1. INTRODUCTION

Let \mathcal{X} be a Polish space with metric d and $f : \mathcal{X} \rightarrow \mathcal{X}$ a continuous function. We introduce the attacking relation $x \curvearrowright_f y$ as in [5, 6, 7]:

$$x \curvearrowright_f y \iff_{\text{df}} \forall \varepsilon > 0 \forall m \in \mathbb{N} \exists \ell [\ell \geq m \ \& \ d(f^\ell(x), y) < \varepsilon].$$

One may reduce the number of variables by saying

$$\forall m > 0 \exists \ell \geq m \ d(f^\ell(x), y) < \frac{1}{m}.$$

We write \mathcal{N} for *Baire space*, the space of infinite sequences of natural numbers: for each finite such sequence r we have the basic open set $\{\alpha \mid \alpha \upharpoonright \ell h(r) = r\}$. The (backward) *shift function* $\mathfrak{s} : \mathcal{N} \rightarrow \mathcal{N}$ is given by $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$.

In [7] two points a, b in Baire space \mathcal{N} were constructed such that for $f = \mathfrak{s}$, both the set $\{\delta \mid \exists \beta \in \mathcal{N} \ a \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \delta\}$ and the set $\{\delta \mid \exists \rho \in \mathcal{N} \ b \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \delta\}$ are complete analytic and therefore not Borel. These constructions formed counterexamples to conjectures made in [5], [6]. The second one in particular was contrary to the expectations of the author and of other researchers.

The purpose of this note is to describe, in the general setting of an arbitrary Polish space \mathcal{X} , continuous function $f : \mathcal{X} \rightarrow \mathcal{X}$ and point c in \mathcal{X} , a uniform method for choosing for each y such that $\exists x \ c \curvearrowright_f x \curvearrowright_f y$ a point x in $\omega_f(c)$ that attacks it, and for each y such that $\exists \rho \ c \curvearrowright_f \rho \curvearrowright_f \rho \curvearrowright_f y$ a recurrent point ρ in $\omega_f(c)$ that attacks it. The method was found in the course of attempting to prove the conjectures refuted by the examples of [7],

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and may still be of interest as providing a possible mechanism for proving of particular functions f and initial points c that the sets in question will, in their particular case, be indeed Borel.

1.0 HISTORICAL NOTE The method we shall describe was inspired by reading the discussion in [8] pp 254–259 of Kunugui’s Lemma, which we now state, and its applications.

1.1 KUNUGUI’S LEMMA *If A is closed and π is continuous and $\pi[A] \subseteq \bigcup_k K_k$ where each K_k is closed, then there is a basic open set O_r and a k such that $O_r \cap A$ is non-empty and $\pi[O_r \cap A] \subseteq K_k$.*

2. CHOOSING AN ATTACKER

We define closed sets $K_{y,m}$ for each $y \in \mathcal{X}$ and positive integer m :

2.0 DEFINITION $K_{y,m} =_{\text{df}} \bigcap_{\ell \geq m} \{x \mid d(f^\ell(x), y) \geq \frac{1}{m}\}$.

2.1 REMARK The F_σ set $\bigcup_m K_{y,m}$ is the set of points $x \in \mathcal{X}$ that fail to attack y .

Let N_r enumerate a countable basis of the space \mathcal{X} , each of finite diameter.

2.2 REMARK A Polish space is separable; therefore we suppose that we have fixed a countable dense subset $\langle p_i \mid i \in \mathbb{N} \rangle$ of \mathcal{X} and whenever we need to choose a member of a basic set N_r , can suppose that we choose the first point p_i to lie in N_r , and therefore avoid appeal to the axiom of choice.

For a closed subset B of \mathcal{X} we define

$$(2.3) \quad \bar{\delta}_{f,y}(B) =_{\text{df}} B \setminus \bigcup_r \{N_r \mid \exists m (N_r \cap B \subseteq K_{y,m})\}.$$

Whilst y and f are fixed for our discussion, we omit the subscripts from $\bar{\delta}$.

$\bar{\delta}(B)$ is a closed set and a subset of B ; the operation is monotone. So iterate, starting from the closed set $\{x \mid c \curvearrowright_f x\}$, often written $\omega_f(c)$, the ω -limit set of c :

$$(2.4) \quad \begin{aligned} B^0 &= \omega_f(c) \\ B^{\nu+1} &= \bar{\delta}(B^\nu) \\ B^\lambda &= \bigcap_{\nu < \lambda} B^\nu \quad \text{for limit } \lambda \end{aligned}$$

As we have a shrinking sequence of closed sets in a space with a countable basis, we shall reach in countably many steps a closed set B^∞ , with the property that

$$(2.5) \quad \bar{\delta}(B^\infty) = B^\infty.$$

Note that for any closed set B , if β is in B but not in $\bar{\delta}(B)$, then β is in some $K_{y,m}$ and therefore cannot attack y ; hence any $\beta \in \omega_f(c)$ that attacks y survives each successive application of $\bar{\delta}$, and therefore lies in B^∞ . Thus if y is attacked by some member of $\omega_f(c)$, B^∞ is not empty, and contains every member of B that attacks y .

By 2.4, B^∞ will have this property, where we write $\text{clos}(A)$ for the closure of a subset A of \mathcal{X} :

$$(2.6) \quad N_r \cap B^\infty \neq \emptyset \implies \forall m \exists s \text{ clos}(N_s) \subseteq N_r \ \& \ \text{diam}(N_s) \leq \frac{1}{2} \text{diam}(N_r) \ \& \\ \& \ N_s \cap K_{y,m} = \emptyset \ \& \ N_s \cap B^\infty \neq \emptyset$$

To see that, note that if $N_r \cap B^\infty \neq \emptyset$, then for each m there is a γ in $N_r \cap B^\infty \setminus K_{y,m}$, so for that γ we may choose s with $\gamma \in N_s$, $N_s \cap K_{y,m} = \emptyset$ and $\text{diam}(N_s)$ at most $\frac{1}{2} \text{diam}(N_r)$ and sufficiently small to ensure that $\text{clos}(N_s) \subseteq N_r$.

Given Property 2.6, we may easily find a shrinking sequence N_{s_i} of basic open sets, their diameters tending to 0, the closure of each being a subset of its predecessor, and each containing a point in B^∞ , with N_{s_0} any N_r not disjoint from B^∞ and for each $i \geq 1$, the set N_{s_i} disjoint from $K_{y,i}$. Let q_i be the canonically chosen (from $\langle p_i \mid i \in \mathbb{N} \rangle$) member of N_{s_i} . The sequence $(q_i)_i$ will be a Cauchy sequence, by the rapid shrinking of the diameters, and its limit point, x^∞ say, will be the sole member of $\bigcap_i \text{clos}(N_{s_i})$; that point x^∞ , being a limit of members of B^∞ , will itself be in B^∞ ; since $x^\infty \in \bigcap_i N_{s_i}$ and $N_{s_i} \cap K_{y,m} = \emptyset$, x^∞ will lie outside $K_{y,m}$ for each m and thus will attack y .

Indeed, our discussion has yielded this result:

2.7 PROPOSITION *B^∞ is the closure of the set of those points in $\omega_f(c)$ that attack y .*

Proof: the sequence of N_{s_i} can start inside an arbitrarily small neighbourhood of any given member of B^∞ . – (2.7)

2.8 Thus we have this entirely explicit procedure for finding a particular x in $\omega_f(c)$ attacking a given point y : first form the set $B_{f,y}^\infty$: if that is empty, no such x exists. Otherwise, form the set $\{r \in \mathbb{N} \mid N_r \cap B_{f,y}^\infty \neq \emptyset\}$; then using that set as an oracle, and repeatedly invoking Property 2.6, find a

sequence (s_i) as above, choosing the smallest appropriate s at each step; then the desired point x will be the limit of the Cauchy sequence $(q_i)_i$ of canonical members of N_{s_i} .

If there are many choices at each stage, one might build a tree of possibilities, arranging that each infinite branch through the tree will lead to a possible x ; in such a case the particular point x described in the previously paragraph will correspond to the left-most branch of that tree.

Thus we have a method for determining for each y an appropriate x without appeal to the axiom of choice.

We note some further properties of our definition.

2.9 PROPOSITION *If B is f -closed so is $\bar{\partial}_f(B)$.*

Proof : Let B be f -closed, and suppose that $x \in \bar{\partial}(B)$ but $f(x) \notin \bar{\partial}(B)$. Since $\bar{\partial}(B) \subseteq B$, $x \in B$ and therefore $f(x) \in B$ as B is f -closed. Thus there exist r, m such that $f(x) \in N_r$ and $N_r \cap B \subseteq K_{y,m}$. So $x \in f^{-1}[N_r]$ which is an open set by the continuity of f . So pick s with $x \in N_s \subseteq f^{-1}[N_r]$.

Now for any $z \in N_s \cap B$, $f(z) \in N_r \cap B \subseteq K_{y,m}$. So

$$\forall \ell \geq m(d(f^{\ell+1}(z), y) \geq \frac{1}{m} > \frac{1}{m+1}.$$

So for all $\ell \geq m+1$, $d(f^\ell, y) > \frac{1}{m+1}$, and thus $z \in K_{y,m+1}$. Hence $N_s \cap B \subseteq K_{y,m+1}$ and so $x \notin \bar{\partial}(B)$, a contradiction. (2.9)

2.10 COROLLARY *Starting from $B^0 = \omega_f(c)$, all B^ν and B^∞ are f -closed.*

2.11 REMARK If a set B is both closed and f -closed it is \curvearrowright_f -closed, in the sense that $x \in B$ and $x \curvearrowright_f z$ implies $z \in B$. Hence all the B^ν 's and B^∞ are \curvearrowright_f -closed.

3. CHOOSING A RECURRENT ATTACKER

This time we wish to start from $\{\rho \mid c \curvearrowright_f \rho \curvearrowright_f \rho\}$; but that, very probably, is not a closed set. It is however a G_δ . We appeal to a general result from descriptive set theory, originating in Lusin [4] and due perhaps jointly to Lusin and Souslin, which we take in the form stated as Exercise 13.10 of Kechris' treatise [3]:

3.0 PROPOSITION *If P is a Borel subset of a Polish space \mathcal{X} , then there is a closed subset C of $\mathcal{X} \times \mathcal{N}$ such that the projection $\pi : (x, \alpha) \mapsto x$ from $\mathcal{X} \times \mathcal{N}$ to \mathcal{X} is one-to-one on C and $\pi[C] = P$.*

For the proof of the general case, we refer the reader to [4] or [8]. However we note that when the Borel set is a G_δ set in a space, such as \mathcal{N} , with a countable basis of clopen sets, there is a very easy proof: represent

the G_δ set P as $\bigcap_m \mathcal{O}_m$ where each \mathcal{O}_m is open; write $\mathcal{O}_m = \bigcup_{k \in I_m} Q_k$, where Q_k is a basic clopen set, and arrange matters, by having repetitions if necessary, so that each I_m is an infinite set of positive integers. Let Ψ be $\prod_m I_m$; Ψ is evidently homœomorphic to \mathcal{N} . Then

$$\begin{aligned} x \in P &\iff \forall m \exists n : \in I_m \ x \in Q_n \\ &\iff \exists \psi : \in \Psi \ \forall m \ x \in Q_{\psi(m)} \\ &\iff \exists \psi : \in \Psi \ \forall m \ (x \in Q_{\psi(m)} \ \& \ \forall n : \in I_m \ (n < \psi(m) \implies x \notin Q_n)), \end{aligned}$$

the third line representing a minimization of the values of ψ . So if we take C to be the following closed subset of $\mathcal{X} \times \Psi$:

$$\{(x, \psi) \mid \forall m \ (x \in Q_{\psi(m)} \ \& \ \forall n : \in I_m \ (n < \psi(m) \implies x \notin Q_n)),\}$$

the projection $\pi : (x, \psi) \mapsto \rho$ will be injective on C and have image P .

So we start from a closed set C_0 of $\mathcal{X} \times \mathcal{N}$ with $\pi[C_0] = \{\rho \mid c \curvearrowright_f \rho \curvearrowright_f \rho\}$. We define $K_{y,m}$ as before but this time we enumerate a countable basis of $\mathcal{X} \times \mathcal{N}$ as O_r , (again regarding each as furnished with a canonical member of a fixed countable dense subset of $\mathcal{X}_t \text{imes } \mathcal{N}$) and define the π -*derivative* at y of a closed subset C of $\mathcal{X} \times \mathcal{N}$ thus:

$$3.1 \text{ DEFINITION } \quad \partial_{f,y}^\pi(C) =_{\text{df}} C \setminus \bigcup \{O_r \mid \exists m \ \pi[O_r \cap C] \subseteq K_{y,m}\}$$

3.2 REMARK $\partial^\pi(C)$ is closed.

3.3 REMARK If $\alpha \in C \setminus \partial^\pi(C)$ then for some r and m , $\alpha \in O_r \cap C$ and $\pi[O_r \cap C] \subseteq K_{y,m}$, so $\pi(\alpha)$ cannot attack y . Hence if $\alpha \in C$ and $\pi(\alpha) \curvearrowright_f y$ then $\alpha \in \partial^\pi(C)$.

Thus if we iterate, starting from C_0 ,

$$(3.4) \quad \begin{aligned} C^0 &= C_0 \\ C^{\nu+1} &= \partial^\pi(C^\nu) \\ C^\lambda &= \bigcap_{\nu < \lambda} C^\nu \quad \text{for limit } \lambda \end{aligned}$$

reaching a final closed set C^∞ , each $\alpha \in C_0$ with $\pi(\alpha)$ attacking y will be in C^∞ , and therefore its projection $\pi(\alpha)$ will be in $\pi[C^\infty]$, which thus contains all the recurrent points, attacked by c , that attack y .

In particular if y is in the abode, centre, or kernel of c , $C_{f,y}^\infty$ will be non-empty.

$C^\infty = \partial^\pi(C^\infty)$: so if $O_r \cap C^\infty$ is not empty, then for all m , $\pi[O_r \cap C^\infty]$ is not included in $K_{y,m}$, so for some $\gamma \in O_r \cap C^\infty$, $\pi(\gamma) \notin K_{y,m}$. For such m

and γ , choose \mathcal{O} open in \mathcal{X} with $\pi(\gamma) \in \mathcal{O}$ and $\mathcal{O} \cap K_{y,m}$ empty: possible as $K_{y,m}$ is closed. Then pick O_s with $\gamma \in O_s$, (so that $O_s \cap C^\infty$ is not empty), $\text{clos}(O_s) \subseteq O_r \cap \pi^{-1}[\mathcal{O}]$, and such that $\text{diam}(O_s) \leq \frac{1}{2}\text{diam}(O_r)$. Thus $\pi[O_s \cap C^\infty] \cap K_{y,m} = \emptyset$, and we have established the following property:

(3.5)

$$O_r \cap C^\infty \neq \emptyset \implies \forall m \exists s \text{ clos}(O_s) \subseteq O_r \ \& \ \text{diam}(O_s) \leq \frac{1}{2}\text{diam}(O_r) \ \& \\ \& \ \pi[O_s \cap C^\infty] \cap K_{y,m} = \emptyset \ \& \ O_s \cap C^\infty \neq \emptyset$$

Now given α_0 in C^∞ , find a sequence O_{s_i} each containing the closure of the next, with diameters finite and shrinking to 0, with $\pi[O_{s_i}] \cap K_{y,i} = \emptyset$ for $i \geq 1$, and with α_0 in O_{s_0} , the latter to have arbitrarily small diameter. The intersection will be non-empty, since it will contain the limit of the Cauchy sequence of canonical members of the sets O_{s_i} , and the intersection will be a single point, α^∞ , which will be in C^∞ . Further, for each m , $\pi(\alpha^\infty)$ will not be in $K_{y,m}$ and therefore $\pi(\alpha) \curvearrowright_f y$.

Thus we have proved the following

3.6 PROPOSITION $\pi[C_{f,y}^\infty]$ will be the closure of the set of recurrent points attacked by c and attacking y .

3.7 REMARK As before, we have a procedure for choosing a recurrent point in $\omega_f(c)$ attacking a given point $y \in \omega_f(c)$, when such exists: first form $C_{f,y}^\infty$. If it is empty, no such ρ exists. Otherwise form the set $\{r \mid O_r \cap C_{f,y}^\infty \neq \emptyset\}$ and use it with the Property 3.5 to build, entirely constructively, a tree all of whose infinite branches lead to recurrent points x attacking the given y ; and we may follow the left-most branch of that tree to obtain a definable such x .

4. FURTHER REMARKS AND PROBLEMS

Local derivations

Our functions δ and δ^π are *derivations* in the sense of Dellacherie's 1977 paper [1] and of Kechris' treatise [3], pp 270 et seq, where they are called *derivatives* and where numerous examples are discussed. For our purposes we may take a derivation in a space \mathcal{X} to be a map δ defined on the closed subsets of \mathcal{X} with the properties that for closed sets D and E , $\delta(D)$ is a closed subset of D , and $D \subseteq E \implies \delta(D) \subseteq \delta(E)$.

Dellacherie in his course [1] page 287 discusses the concept of a *local* derivation, which is one satisfying

$$\delta(X) = \{x \in X \mid \forall U : \in \mathcal{B} (x \in U \implies \delta(\text{clos}(U) \cap X) \neq \emptyset)\}$$

where \mathcal{B} is some countable basis of \mathcal{X} .

4.0 REMARK Our derivations $\bar{\partial}_y$ and $\bar{\partial}_y^\pi$ are local; hence if $B_{f,y}^\infty$ is empty, $B_{f,y}^0$ is the union of countably many closed sets B with $\bar{\partial}(B) = \emptyset$, namely those sets $B^0 \cap N_r$ which are subsets of some $K_{y,m}$. In the C case, if $C_{f,y}^\infty$ is empty, then C_0 will also be covered by countably many sets with $\bar{\partial}^\pi$ -derivate empty, namely the sets $C_0 \cap O_r$ with $\pi[C_0 \cap O_r]$ a subset of some $K_{y,m}$.

Some characterizations

In [6] we defined this sequence of sets:

$$\begin{aligned} A^0(c, f) &= \omega_f(c) \\ A^{\beta+1}(c, f) &= \{y \mid \exists x : \in A^\beta(c, f) \ x \curvearrowright_f y\} \\ A^\lambda(c, f) &= \bigcap_{\nu < \lambda} A^\nu(c, f) \quad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

A^0 is always a closed set; the others are all analytic, and the first construction in [7] gave a recursive $a \in \mathcal{N}$ with $A^1(a, \mathfrak{s})$ not Borel.

We also defined $\theta(a, f)$ as the least ordinal θ with $A^\theta(c, f) = A^{\theta+1}(c, f)$ and showed in [6] that θ never exceeds ω_1 , but, again by [7], for some recursive $b \in \mathcal{N}$, $\theta(b, \mathfrak{s}) = \omega_1$.

Hence although the sequence A^ν is shrinking, it is not composed of closed sets; nor in general need its terms be Borel sets, nor need it stop at a countable ordinal; so that attempts to represent its stages as projections of a shrinking sequence of closed sets are in general hopeless.

The arguments of §§2, 3 show the following:

4.1 PROPOSITION $B_{f,y}^\infty$ is non-empty if and only if $y \in A^1(c, f)$.

4.2 PROPOSITION $C_{f,y}^\infty$ is non-empty if and only if $y \in A^{\theta(c,f)}(c, f)$.

Two open problems

4.3 PROBLEM Is there a counterpart of 2.9 for $\bar{\partial}^\pi$?

4.4 PROBLEM To what more general problem of picking members of sets does this method apply ?

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A SCENARIO FOR TRANSFERRING HIGH SCORES

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1. INTRODUCTION

Let \mathcal{X} be a Polish space with metric d and $f : \mathcal{X} \rightarrow \mathcal{X}$ a continuous function. We recall the definition of the attacking relation $x \curvearrowright_f y$ studied in our papers [2, 3, 4]:

$$x \curvearrowright_f y \iff_{\text{df}} \forall \varepsilon > 0 \forall m \in \mathbb{N} \exists \ell [\ell \geq m \ \& \ d(f^\ell(x), y) < \varepsilon].$$

Our notation is related to a familiar one in dynamical systems: $x \curvearrowright_f y$ iff $y \in \omega_f(x)$.

An important example is *Baire space*, the space of infinite sequences of natural numbers, often denoted by \mathcal{N} or ${}^\omega\omega$, which for each finite such sequence r has the basic open set $\{\alpha \mid \alpha \upharpoonright \ell h(r) = r\}$. In that space the (backward) *shift function* $\mathfrak{s} : \mathcal{N} \rightarrow \mathcal{N}$ is given by $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$.

The *score*, $\theta(a, f)$, of a point a in \mathcal{X} with respect to the function f , is defined to be the least ordinal θ such that $A^\theta(a, f) = A^{\theta+1}(a, f)$, where we define recursively a shrinking sequence of sets by $A^0(a, f) = \omega_f(a)$, $A^{\nu+1}(a, f) = \{x \mid \exists y (y \in A^\nu(a, f) \ \& \ y \curvearrowright_f x)\}$ and for a limit ordinal λ , $A^\lambda(a, f) = \bigcap_{\nu < \lambda} A^\nu(a, f)$.

In [3] we showed, working always with the shift function \mathfrak{s} , that for Baire space ${}^\omega\omega$ or the Cantor space ${}^\omega 2$ there are points of score any given countable ordinal. In [4] we constructed a point in Baire space of score the first uncountable ordinal, which by results of [3] is the maximum possible. An unpublished transfer theorem of Christian Delhommé shows that a point of uncountable score and points of all countable scores will exist in Cantor space ${}^\omega 2$.

The question arises, which other spaces and functions will permit the existence of a point of uncountable score ?

We describe hypotheses on a dynamical system (\mathcal{X}, f) which will permit such a transfer. We then describe one setting in which one can establish all but the last hypothesis; then we show that an assumption of equicontinuity

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will yield that last hypothesis. However, experts on dynamical systems believe that, in the given setting, it is likely that f cannot be equicontinuous. So at present no dynamical system is known (to the author) to satisfy all the given hypotheses.

We work with DC , the mild form of the Axiom of Choice that implies that a relation is well-founded if and only if it admits no infinite descending sequences.

2. HYPOTHESES LEADING TO HIGH SCORES

2.0 THEOREM *Suppose that (\mathcal{X}, f) is a dynamical system such that there is a continuous surjection Ψ of \mathcal{X} onto either ${}^\omega\omega$ or some ${}^\omega m$ with $m \geq 2$, satisfying these properties, where we say that x is at α rather than $\Psi(x) = \alpha$:*

$$(2.0.0) \quad \text{for all } x \in \mathcal{X}, \Psi(f(x)) = \mathfrak{s}(\Psi(x))$$

$$(2.0.1) \quad \text{for all } x \text{ and } y \text{ in } \mathcal{X}, \text{ if } x \curvearrowright_f y \text{ then } \Psi(x) \curvearrowright_{\mathfrak{s}} \Psi(y)$$

$$(2.0.2) \quad \text{if } x \text{ is at } \alpha \text{ and } \alpha \curvearrowright_{\mathfrak{s}} \beta, \text{ then there is a } y \text{ at } \beta \text{ with } x \curvearrowright_f y.$$

$$(2.0.3) \quad \text{if } \alpha \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \gamma, a \text{ is at } \alpha, c \text{ is at } \gamma \text{ and } a \curvearrowright_f c, \text{ then there is a point } b \text{ at } \beta \text{ with } a \curvearrowright_f b \curvearrowright_f c.$$

Then for every x , the f -score of x equals the \mathfrak{s} -score of $\Psi(x)$, so that there are points in \mathcal{X} of all scores up to and including ω_1 .

REMARK My student Cédric Machefert has pointed out that (2.0.1) follows from (2.0.0); Christian Delhommé in unpublished work has obtained refinements of this theorem.

Proof: We recall the definition from [3], page 263, of the tree $T_y^x(f)$, where $x \curvearrowright_f y$. It is the set of finite sequences (y_0, y_1, \dots, y_n) such that $y_0 = y$, each $y_{i+1} \curvearrowright_f y_i$ and $x \curvearrowright_f y_n$.

Extend the definition of Ψ to such finite sequences in the natural way: $\Psi((y_0, \dots, y_n)) = (\Psi(y_0), \dots, \Psi(y_n))$. With this extended definition, Ψ preserves length and end-extension.

$$2.1 \text{ PROPOSITION} \quad \text{If } x \curvearrowright_f y, \Psi[T_y^x(f)] = T_{\Psi(y)}^{\Psi(x)}(\mathfrak{s}).$$

Proof: by (2.0.1), the restriction of Ψ to the first tree is into the second; by repeated use of (2.0.3), we see that it is onto. (2.1)

2.2 LEMMA *If x is at α , y is at β , and $x \curvearrowright_f y$, then $T_y^x(f)$ is well-founded iff $T_\beta^\alpha(\mathfrak{s})$ is.*

Proof: by repeated use of (2.0.1) and (2.0.3) to transfer infinite descending sequences from one tree to the other. \dashv (2.2)

2.3 DEFINITION If $a \curvearrowright_f b$, $T_b^a(f)$ is well-founded, and $s \in T_b^a(f)$, write $\varrho_{a,b,f}(s)$ for the rank of s in the tree $T_b^a(f)$, as given by the recursion

$$\varrho_{a,b,f}(s) = \sup\{\varrho_{a,b,f}(s \hat{\ } \langle y \rangle) + 1 \mid s \hat{\ } \langle y \rangle \in T_b^a(f)\}.$$

2.4 PROPOSITION If $a \curvearrowright_f b$, $\emptyset \neq s \in T_b^a(f)$ and $\varrho_{a,b,f}(s) = \xi$, then the last element $\ell(s)$ of the finite sequence s is in $A^\xi(a, f) \setminus A^{\xi+1}(a, f)$.

Proof: by induction on ξ , much as in the proof of Lemma 2.1 of [3]. \dashv (2.4)

2.5 COROLLARY $\theta(a, f)$, the f -score of a , equals $\sup\{\varrho_{a,b,f}(\emptyset) \mid a \curvearrowright_f b\}$.

Thus the score of a point a is computable from the ranks of the various trees $T_b^a(f)$.

2.6 PROPOSITION Suppose the two trees are both well-founded. Then they have the same rank: $\varrho_{a,b,f}(\emptyset) = \varrho_{\alpha,\beta,\mathfrak{s}}(\emptyset)$.

2.7 PROPOSITION If x is at α , then the f -score of x equals the \mathfrak{s} -score of α ; in particular if x is at α and α is of uncountable score, then so is x .

For points of uncountable score we can also argue as follows. A point is of uncountable score iff its abode is not a Borel set. Further the image under Ψ of the abode of x is the abode of α , and the image under Ψ of the escape of x is the escape of α .

Hence if the abode and escape of x were Borel, then both the abode and the escape of α would be analytic, and therefore by Souslin's celebrated result, would be Borel.

2.8 REMARK In this connection it might be worth looking again at the "original" construction of a point of uncountable score mentioned in the penultimate paragraph of [4], and there called c , where c "neatly" attacks α_T for every countable well-founded tree T . The nodes of α_T survive for $\zeta = \varrho(T)$ steps, and ζ can be arbitrarily large.

2.9 PROPOSITION Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho$. Let a be at α and x be at ρ with $a \curvearrowright_f x$. Then there are recurrent r and s such that $a \curvearrowright_f s \curvearrowright_f x \curvearrowright_f r$ and $\rho \curvearrowright_{\mathfrak{s}} \Psi(r) \curvearrowright_{\mathfrak{s}} \rho$.

Proof : by (2.0.2) there is a y at ρ such that $x \curvearrowright y$. Taking, by (2.0.3), an infinite backward sequence of ρ 's there are points y_i with $y_0 = y$, each $y_{i+1} \curvearrowright_f y_i$ and x attacking each y_i . By proposition 3.18 of [3] there is a recurrent point r with $x \curvearrowright_f r \curvearrowright_f y$, which by (2.0.1) gives $\rho \curvearrowright_{\mathfrak{s}} \Psi(r) \curvearrowright_{\mathfrak{s}} \rho$.

To find s , repeat the argument, inserting a chain of attacks $a \curvearrowright_f s_{i+1} \curvearrowright_f s_i \curvearrowright_f x$. ⊢ (2.9)

2.10 COROLLARY *If x is at β , a at α , $a \curvearrowright_f x$, and β is in the abode of α , then x is in the abode of a .*

Proof : for some ρ , $\alpha \curvearrowright_{\mathfrak{s}} \rho \curvearrowright_{\mathfrak{s}} \rho \curvearrowright \beta$; we can therefore find f -recurrent r with $a \curvearrowright_f r \curvearrowright_f x$. ⊢ (2.10)

Many things now fit well together. For example in [3] an operator Γ was introduced: $\Gamma_f(Z) = \{x \mid \exists y(y \in Z \ \& \ y \curvearrowright_f x)\}$.

2.11 LEMMA $\Psi[\omega_f(x)] = \omega_{\mathfrak{s}}(\Psi(x))$.

2.12 LEMMA $\Psi[\Gamma_f(Z)] = \Gamma_{\mathfrak{s}}(\Psi[Z])$.

2.13 PROPOSITION *For all ν , $\Psi[A^\nu(a, f)] = A^\nu(\Psi(a), \mathfrak{s})$.*

Proof : the previous two lemmata cover the case of 0 and successors. At limits, we must use the analysis of trees. ⊢ (2.13)

2.14 REMARK We have made little use of separability: it has been used only to show that any point that vanishes does so at a countable stage, thus giving the upper bound of ω_1 to the score. But for the sake of proving the existence of points of uncountable score, it shouldn't be necessary.

3. HORSESHOES

Following a lecture of Jozef Bobok we make a definition. The setting is a compact (Polish) space \mathcal{A} with metric d , a continuous function $f : \mathcal{A} \rightarrow \mathcal{A}$ and an integer $m \geq 2$. We write \emptyset for the sequence of length 0, and \emptyset for the empty set. Of course in many formal presentations of mathematics the two objects are the same.

3.0 DEFINITION An (m, f) -strong horseshoe is a sequence S_0, \dots, S_{m-1} of pairwise disjoint non-empty closed sets with the property that for all i and j less than m ,

$$S_i \subseteq f[S_j].$$

3.1 REMARK That is a stronger requirement than the condition met by Smale's original horseshoe, which was that $S_i \cap f[S_j]$ is non-empty for each i and j .

3.2 LEMMA $S_i \cap f^{-1}[S_j] \neq \emptyset$: indeed $f[S_i \cap f^{-1}[S_j]] = S_j$.

3.3 LEMMA $S_i \cap f^{-1}[S_j] \cap f^{-2}[S_k] \neq \emptyset$; indeed $f^2[S_i \cap f^{-1}[S_j] \cap f^{-2}[S_k]] = S_k$.

We shall be able to generalise the above, but must first adopt a less cumbersome notation.

3.4 DEFINITION Set $S^\emptyset =_{\text{df}} \bigcup_{i < m} S_i$, and for u a sequence of length $k + 1$ of numbers less than m , set $S^u =_{\text{df}} S^{u \uparrow k} \cap f^{-k}[S_{u(k)}]$. For $\alpha \in {}^\omega m$, set $S^\alpha = \bigcap_k S^{\alpha \uparrow k}$.

3.5 PROPOSITION $S^u \neq \emptyset$; indeed $f^k[S^u] = S_{u(k)}$.

3.6 PROPOSITION If $u = s \frown t$, the concatenation of s and t , and $\ell = lh(s)$, then $S^u = S^s \cap f^{-\ell}[S^t]$ and $f^\ell[S^u] = S^t$.

Proof : The first assertion holds by the identity $g^{-1}(C \cap D) = g^{-1}(C) \cap g^{-1}(D)$. For the second, the inclusion from left to right is evident, using the first assertion. Suppose that x is in S^t . Let $v = u \uparrow (\ell + 1)$. By Proposition 3.5, there is a y in S^v with $f^\ell(y) = x$; but then $y \in S^s \cap f^{-\ell}(S^t)$, and so, by the first assertion again, $x \in f^\ell[S^u]$. ⊢ (3.6)

3.7 PROPOSITION Each S^u is a non-empty closed subset of \mathcal{A} .

3.8 DEFINITION $\mathcal{X} =_{\text{df}} \{x \in \mathcal{A} \mid \forall k \geq 0, f^k(x) \in S^\emptyset\}$.

3.9 LEMMA Each S^α , the intersection along the path α , is non-empty.

Proof : by compactness. ⊢ (3.9)

3.10 LEMMA $\mathcal{X} = \bigcup_\alpha S^\alpha = \bigcap_k \bigcup \{S^u \mid u \in {}^k m\}$.

3.11 PROPOSITION \mathcal{X} is a closed non-empty subset of \mathcal{A} , and is therefore a compact Polish space.

3.12 LEMMA If $x \in \mathcal{X}$, then $f(x) \in \mathcal{X}$.

3.13 LEMMA If $x_k \rightarrow x$ as $k \rightarrow \infty$, and $x_k \in S^{\alpha \uparrow n_k}$, where $n_k \rightarrow \infty$ with k , then $x \in S^\alpha$.

Henceforth we work in the space \mathcal{X} .

A map to m -Cantor space

Let \mathcal{C} be the product space ${}^\omega m$, where m is given the discrete topology. \mathcal{C} is compact by Tychonoff. We shall use Greek letters $\alpha, \beta, \gamma, \rho$ for members of \mathcal{C} .

On \mathcal{C} we define the shift function \mathfrak{s} by $\mathfrak{s}(\alpha)(n) = \alpha(n+1)$.

3.14 DEFINITION For $x \in \mathcal{X}$, define $\Psi(x)(n)$ to be the $i < m$ such that $f^n(x)$ is in S_i .

3.15 PROPOSITION For $x \in \mathcal{X}$, $\Psi(f(x)) = \mathfrak{s}(\Psi(x))$.

3.16 REMARK This means that Ψ is an action map in the sense defined by Akin and Kolyada[1]

3.17 PROPOSITION If $x \in \bigcap_{k \in \omega} S^{\alpha^k}$, then $\Psi(x) = \alpha$; hence Ψ is surjective.

[The compactness ensures that the intersection along α is non-empty].

3.18 PROPOSITION Ψ is continuous.

We shall say that x is at α if $\Psi(x) = \alpha$.

Lifting an attack

3.19 PROPOSITION \mathcal{X} is \curvearrowright_f closed in the sense that if x is in \mathcal{X} and $x \curvearrowright_f y$ then $y \in \mathcal{X}$.

3.20 PROPOSITION If $x \curvearrowright_f y$ then $\Psi(x) \curvearrowright_{\mathfrak{s}} \Psi(y)$.

Proof : Let $\hat{\varepsilon} =_{\text{df}} \min_{i \neq j} d(S_i, S_j)$; thus $\hat{\varepsilon} > 0$ and has the property that two points in S^0 within distance $\hat{\varepsilon}$ of each other are in the same S_i .

Fix a natural number k . We seek ℓ such that $f^\ell(x)$ shadows y for k steps, in the sense that for $n < k$, $\Psi(f^\ell(x))(n) = \Psi(y)(n)$.

By the continuity of $f, f^2 \dots$ and f^{k-1} at y , there are $\delta_i > 0$ such that for each $i < k$

$$d(a, y) < \delta_i \implies d(f^i(a), f^i(y)) < \hat{\varepsilon}$$

Take δ to be the minimum of the δ_i 's. Pick ℓ exceeding k and large enough so that $d(f^\ell(x), y) < \delta$. -(3.20)

3.21 COROLLARY If r is f -recurrent then $\Psi(r)$ is \mathfrak{s} -recurrent.

Proof : Let $\Psi(r) = \rho$. $r \curvearrowright_f r$, so $\rho \curvearrowright_s \rho$. + (3.21)

3.22 PROPOSITION *Suppose that $\alpha \curvearrowright_s \beta$, and that x is at α . Then we can find y at β such that $x \curvearrowright_f y$.*

Proof : Let $\beta \upharpoonright k = s^{n_k}(\alpha) \upharpoonright k$. Then put $x_k = f^{n_k}(x)$. Each x_k is in $S^{\beta \upharpoonright k}$; by compactness some subsequence of the x 's converges, to y say. Then $y \in S_\beta$ and $x \curvearrowright_f y$. + (3.22)

3.23 COROLLARY *If x is at α and the abode of α is empty, so is the abode of x .*

Proof : the abode of a point is empty iff it attacks no recurrent points. + (3.23)

The clauses (2.0.0), (2.0.1) and (2.0.2) are established in the present setting by 3.15, 3.20 and 3.22.

4. THE EFFECT OF EQUICONTINUITY

Suppose now that at every point x of \mathcal{X} , f is equicontinuous in the sense that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall n \forall y [d(x, y) < \delta \implies d(f^n(x), f^n(y)) < \varepsilon].$$

We derive clause (2.0.3):

4.0 LEMMA *Suppose that $\alpha \curvearrowright_s \beta \curvearrowright_s \gamma$ and that a is at α , c at γ and $a \curvearrowright_f c$. Then there is a point b at β with $a \curvearrowright_f b \curvearrowright_f c$.*

Proof : Given k , a positive integer, there are (large) integers n_k and m_k such that for each $i < k$,

$$\gamma(i) = \beta(n_k + i) = \alpha(m_k + n_k + i)$$

and such that $f^{m_k+n_k}(a) \longrightarrow c$.

Set $b_k = f^{n_k}(a)$. Some subsequence, say for $k \in B \in [\omega]^\omega$, of those converges, to b , say. We know that $f^{n_k}(b_k) \longrightarrow d$. We shall use the equicontinuity of f at b to show that $f^{n_k}(b) \longrightarrow d$.

Given ε , we seek K such that for $k > K$, $k \in B$, $d(f^{n_k}(b), d) < \varepsilon$. We know that there is a K_0 such that for all $k \geq K_0$, $d(f^{n_k}(b_k), c) < \varepsilon/2$. Using the equicontinuity at b , we know that there is a δ such that if $d(z, b) < \delta$,

then for all k , $d(f^{n_k}(z), f^{n_k}(b)) < \varepsilon/2$. For large enough $k \in B$, we shall indeed have $d(b_k, b) < \delta$. \dashv (4.0)

4.1 REMARK Equicontinuity seems formally too strong, as for given k we are only interested in the point $z = b_k$.

Finally we mention a variant of (2.0.3), of which the proof has an interesting feature.

4.2 LEMMA *Suppose that $\alpha \curvearrowright_{\mathfrak{s}} \beta$ and that y is at β . Then there is an x at α with $x \curvearrowright_f y$.*

Proof : By Proposition 3.6, for given k and n_k with $\mathfrak{s}^{n_k}(\alpha) \upharpoonright k = \beta \upharpoonright k$, we may find $x_k \in S^{\alpha \upharpoonright n_k}$ with $f^{n_k}(x_k) = y$ (and not just near y !). Let x be the limit of some convergent subsequence of the x_k 's. x will be at α by Lemma 3.13. Then by the equicontinuity of f at x , given $\varepsilon > 0$ there is a $\delta > 0$ such that for each k and z $d(x, z) < \delta \implies d(f^{n_k}(x), f^{n_k}(z)) < \varepsilon$; taking z to be an x_k in the subsequence that is suitably near x , we see that $d(f^{n_k}(x), y) < \varepsilon$, and so $x \curvearrowright_f y$, as required. \dashv (4.2)

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