A NOTE ON THE SCHEMES OF REPLACEMENT AND COLLECTION

A. R. D. M ATHIAS*

ABSTRACT. We derive the schemes of ZF from certain weak forms of the same.

0. INTRODUCTION

We denote by Repcoll the axiom scheme that says of each formula $\Phi(\vec{y}, z, \vec{t})$ of the language of set theory that

$$\overrightarrow{\forall y} :\in u \; \exists! z \Phi(\vec{y}, z, \vec{t}) \Longrightarrow \exists w \, \overrightarrow{\forall y} :\in u \; \forall z \big(\Phi(\vec{y}, z, \vec{t}) \Longrightarrow z \in w\big).$$

The instances of this scheme have hypotheses that are stronger than those of Collection and conclusions that are weaker than those of Replacement. In its formulation $\overrightarrow{\forall y} :\in u$, for $\vec{y}$ the finite sequence $y_1, y_2, \ldots y_n$ of variables, denotes the corresponding finite sequence of restricted quantifiers $\forall y_1 :\in u \; \forall y_2 :\in u \; \ldots \forall y_n :\in u$.

Let M_0, M_1 be the systems described in *The Strength of Mac Lane set theory* [M]: the axioms of M_0 are those of Extensionality, Null Set, Pairing, Sumset, Power Set, and the scheme of Δ_0 Separation; and M_1 is M_0 with the addition of the axioms of Foundation—the assertion that every non-empty set x has a member y with $x \cap y$ empty—and the principle TCo of Transitive Containment, the assertion that every set is a member of a transitive set.

We answer a question of A. K. Simpson by showing that all axioms of ZF except the axiom of infinity are provable, classically, in M_1 + Repcoll, but not in certain weakenings of that system.

1. ESTABLISHING RANKS

We first prove that every set has a rank; in doing so we shall use the fact, proved in [M] as Theorem 6·9, part (i), that all instances of the scheme of $\Delta_O^{\mathcal{P}}$ separation are provable in the theory M_0. A formula is $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $\mathcal{Q}x :\subseteq y$ or $\mathcal{Q}x :\in y$ where $\mathcal{Q}$ is either $\forall$ or $\exists$, and x and y are distinct variables.

* Centre de Recerca Matemàtica, Bellaterra, Catalonia and ERMIT, Université de la Réunion.

16·*iv*·2002, revised 29·*ii*·2004, further revised 29·*viii*·2004.

1.0 DEFINITION A *rank function* is a function f with domain a transitive set u such that for each x in u,

$$f(x) = \bigcup\{f(y) + 1 \mid y \in x\}.$$

1.1 LEMMA (M_1) *For each transitive set u there is at most one function, which we denote by ϱ_u when it exists, with domain u and satisfying the recurrence*

$$\varrho_u(z) = \bigcup\{\varrho_u(y) + 1 \mid y \in z\}.$$

Proof : The recurrence is of the form $f(x) = G(f \restriction x)$. Consider, using Foundation, a minimal member $\bar{w}$ of $\{w \mid \varrho_1(w) \neq \varrho_2(w)\}$, for distinct solutions ϱ_1 and ϱ_2 to the recursion equation. Then $\varrho_1(\bar{w}) = G(\varrho_1 \restriction \bar{w}) = G(\varrho_2 \restriction \bar{w}) = \varrho_2(\bar{w})$, a contradiction. ⊣ (1.1)

1.2 REMARK If ϱ_u exists and w is a transitive subset of u, then ϱ_w exists, and equals the restriction $\varrho_u \restriction w$ of ϱ_u to w.

1.3 COROLLARY *If $a \in \mathrm{Dom}\,\varrho_u \cap \mathrm{Dom}\,\varrho_v$, then $\varrho_u(a) = \varrho_v(a)$.*

Proof : Both are equal to $\varrho_{u\cap v}$. ⊣ (1.3)

1.4 REMARK $f = \varrho_u$ is $\Delta_0^{\mathsf{M}_1}$, being equivalent in M_1 to

$$\bigcup u \subseteq u \;\&\; Fn(f) \;\&\; \mathrm{Dom}\,(f) = u \;\&\; \forall x {:}{\in} u\; f(x) \in ON \;\&$$
$$\forall x {:}{\in} u\; \forall y {:}{\in} x\; f(y) < f(x) mand \forall x {:}{\in} u\; \forall \zeta {:}{\in} f(x)\; \exists y {:}{\in} x\; \zeta \leqslant f(y).$$

By the *transitive closure* of a set x is meant a transitive set u of which x is a subset and such that for any transitive set $v \supseteq x$, $u \subseteq v$. Evidently there can be at most one such u for given x.

1.5 PROPOSITION (M_1) *Every set has a transitive closure.*

Proof : Given x, and applying TCo, let u be a transitive set of which x is a member and therefore a subset. Using Power Set and Δ_0 Separation, form the set T of all transitive subsets v of u of which x is a subset. T is non-empty, so again using Δ_0 separation, its intersection, c, say, will be a set. Then certainly c is a transitive set of which x is a subset; and if b is any other, so is $b \cap u$, which latter is a member of T and hence a superset of c, so that $c \subseteq b$, as required. ⊣ (1.5)

1.6 DEFINITION We write $\mathrm{tcl}(x)$ for the transitive closure of x.

The smallest transitive set of which x is a member is of course $\mathrm{tcl}(\{x\})$, and the minimal requirement for a set x to have a rank is that $\varrho_{\mathrm{tcl}(\{x\})}$, for which in future we write $\varrho_{(x)}$, should exist.

1.7 REMARK $y = \mathrm{tcl}(\{x\})$ is equivalent in M_1 to the $\Delta_0^{\mathcal{P}}$ formula

$$\bigcup y \subseteq y \ \&\ x \in y \ \&\ \forall v :\subseteq y[(\bigcup v \subseteq v \ \&\ x \in v) \Longrightarrow v = y].$$

The following is proved in [M] as proposition 1·24.

1.8 LEMMA (M_0) $x \times y \in V$.

Henceforth we reason in $\mathsf{M}_1 + \mathsf{Repcoll}$.

1.9 PROPOSITION *Each transitive set u carries a rank function ϱ_u.*

Proof : Suppose there is a transitive set u with no rank function with domain u. For each $x \in u$, define $F(x)$ to be the ordinal $\varrho_{(x)}(x)$ assigned to x if there is a rank function $\varrho_{(x)}$ defined on $\mathrm{tcl}(\{x\})$, and to be 0 if there is no such function. Apply Repcoll to obtain a w containing all values $F(x)$ for $x \in u$; $w \cap ON$ is a set by Δ_0 separation; take its union and add 1 if necessary to obtain an ordinal κ such that for $x \in u$, $\varrho_{(x)}(x)$, if defined, is less than κ.

Now form the class $A = u \cap \{z \mid \exists f :\in \mathcal{P}(\kappa \times u)\ f = \varrho_{(z)}\}$ of members of u which have a rank function into κ; that will be a set, as it equals $u \cap \{z \mid \exists v :\in \mathcal{P}(u)\ \exists f :\in \mathcal{P}(\kappa \times u)\ f = \varrho_v \ \&\ z \in v\}$, which is the intersection of a set and a class that is Δ_0 in the parameters $\mathcal{P}(u)$ and $\mathcal{P}(\kappa \times u)$.

A is transitive since $y \in z \Longrightarrow \mathrm{tcl}(\{y\}) \subseteq \mathrm{tcl}(\{z\})$. Let $\bar{x} = A$ if $A = u$, and otherwise, using Foundation, let $\bar{x}$ be a member of $u \smallsetminus A$ with $\bar{x} \subseteq A$. By Repcoll the class $R =_{\mathrm{df}} \{\varrho_{(x)} \mid x \in \bar{x}\}$ is included in some set w; by $\Delta_0^{\mathcal{P}}$ separation that will be a set, since

$$R = w \cap \{f \mid \exists x :\in \bar{x}\ \exists y :\in \mathcal{P}(u)\ y = \mathrm{tcl}(\{x\}) \ \&\ f = \varrho_y\}.$$

The set $\bigcup R$ will be a rank function with domain $\mathrm{tcl}(\bar{x})$: in the case $\bar{x} = u$, that gives a rank function on u and therefore a contradiction.

In the case $\bar{x} \in u \smallsetminus A$, use $\bigcup R$ to form $\{\zeta \leqslant \kappa \mid \forall y :\in \bar{x}\ \varrho_{(y)}(y) < \zeta\}$; let its infimum be η. Now we have a rank function on $\mathrm{tcl}(\{\bar{x}\})$ after all, with $\varrho_{(\bar{x})}(\bar{x}) = \eta$, and again a contradiction. ⊣ (1.9)

1.10 COROLLARY *There is a Σ_1 function ϱ with domain the universe such that for each set x,*

$$\varrho(x) = \bigcup \{\varrho(y) + 1 \mid y \in x\}.$$

Our next task is to prove that, for each ordinal η, the class of sets of rank less than η is a set.

1.11 DEFINITION Call f an *attempt* if f is a function with domain some ordinal η, and for ordinals less than η, $f(0) = \varnothing$, $f(\xi + 1) = \mathcal{P}(f(\xi))$ and if $0 < \lambda = \bigcup \lambda < \eta$, $f(\lambda) = \bigcup_{\nu < \lambda} f(\nu)$.

1.12 Remark "f is an attempt" is a $\Delta_0^{\mathcal{P}}$ formula.

1.13 Lemma *There is at most one attempt with a given domain.*

Proof : Let f and g be attempts with domain η. Consider $\eta \cap \{\xi \mid f(\xi) \neq g(\xi)\}$, a set by Δ_0 separation, and therefore, if non-empty, possessed of a minimal element $\bar\zeta$. But then $f \restriction \bar\zeta = g \restriction \bar\zeta$, and so by cases on $\bar\zeta$ one shows that $f(\bar\zeta) = g(\bar\zeta)$, a contradiction. ⊣ (1.13)

1.14 Remark If f is an attempt with domain η then for each $\zeta < \eta$, $f \restriction \zeta$ will be an attempt with domain ζ.

1.15 Proposition *For each ordinal ζ there is an attempt with domain ζ.*

Proof : Suppose there is a ζ for which there is no attempt with domain ζ. Our problem is to find a minimal such ζ.

Define, for the ordinals ν less than ζ, $G(\nu)$ to be the unique attempt with domain ν, if it exists, and if no such attempt exists, set $G(\nu) = 0$. By Repcoll, there is some set w, which we may take to be transitive, such that every f that is an attempt is in w.

We may form $w \cap \{f \mid f \text{ is an attempt}\}$, which is a set by $\Delta_0^{\mathcal{P}}$ separation; its union, also a set, will be the maximal attempt h, and we may find the least ordinal $\bar\zeta \leqslant \zeta$ not in the domain of that union. But then by cases on $\bar\zeta$ we may extend the definition of h to an attempt defined for argument $\bar\zeta$, a contradiction. ⊣ (1.15)

1.16 Corollary *For each ζ, the class $V_\zeta =_{\mathrm{df}} \{x \mid \varrho(x) < \zeta\}$ is a set, and the sequence $\langle V_\xi \mid \xi < \zeta\rangle$ is a set.*

Proof : one shows that for f an attempt and ξ in its domain, $f(\xi) = V_\xi$. ⊣ (1.16)

1.17 Proposition *For any ordinal η_0 and set a, the class $\eta_0 \cap \{\eta \mid \exists x {:\in} a\ \varrho(x) = \eta\}$ is a set.*

Proof : let $a \subseteq u$, transitive, and consider $\varrho_u[a] \cap \eta_0$. ⊣ (1.17)

1.18 Remark In the next section we shall establish an induction on various classes of formulæ. To preserve the rhythm of the induction it will be convenient to use the alternative names Π_0 and Σ_0 for the class of Δ_0 formulæ.

2. Principles of logic.

Fix a concrete $\mathfrak{k}$. We write $\Theta(\vec{y})$ for a typical $\Sigma_{\mathfrak{k}}$ formula, which we take in the form

$$\exists x_{\mathfrak{k}} \forall x_{\mathfrak{k}-1} \exists x_{\mathfrak{k}-2} \dots \mathcal{Q}_2 x_2 \mathcal{Q}_1 x_1 \Phi_0(x_1, \dots, x_{\mathfrak{k}}, \vec{y})$$

where Φ_0 is a Δ_0 formula and the quantifiers $\mathcal{Q}_2$ and $\mathcal{Q}_1$ are $\exists\forall$ or $\forall\exists$ according to the parity of $\mathfrak{k}$. The reverse numbering of variables is deliberate.

2.0 DEFINITION Denote by $\mathsf{Logik}(\Sigma_{\mathfrak{k}}(\vec{y}))$ the principle which for each $\Sigma_{\mathfrak{k}}$ formula $\overrightarrow{\mathcal{Q}x}\,\Phi_0(\vec{x},\vec{y})$ as above asserts that starting from any ordinal η there is a sequence $\eta < \xi_{\mathfrak{k}} < \xi_{\mathfrak{k}-1} < \xi_{\mathfrak{k}-2} < \ldots < \xi_2 < \xi_1$ such that

$$\overrightarrow{\forall y} \in V_\eta \Big[\exists x_{\mathfrak{k}} \forall x_{\mathfrak{k}-1} \exists x_{\mathfrak{k}-2} \ldots \mathcal{Q}_2 x_2 \mathcal{Q}_1 x_1 \Phi_0(x_1,\ldots,x_{\mathfrak{k}},\vec{y}) \iff \exists x_{\mathfrak{k}} :\in V_{\xi_{\mathfrak{k}}}\ \forall x_{\mathfrak{k}-1} :\in V_{\xi_{\mathfrak{k}-1}}\ \exists x_{\mathfrak{k}-2} :\in V_{\xi_{\mathfrak{k}-2}}\ \ldots \ldots \mathcal{Q}_2 x_2 :\in V_{\xi_2}\ \mathcal{Q}_1 x_1 :\in V_{\xi_1}\ \Phi_0(x_1,\ldots,x_{\mathfrak{k}},\vec{y})\Big].$$

2.1 PROPOSITION $\mathsf{Logik}(\Sigma_0(\vec{y}))$

Proof : vacuous. ⊣ (2.1)

2.2 DEFINITION Denote by $\mathsf{Separation}(\Sigma_{\mathfrak{k}}(\hat{z},\vec{y}))$ the principle which for each $\Sigma_{\mathfrak{k}}$ formula $\Theta(z,\vec{y})$ asserts that for all sets $\vec{y}$ and a the class $a \cap \{z \mid \Theta(z,\vec{y})\}$ is a set.

2.3 PROPOSITION $\mathsf{Separation}(\Sigma_0(\hat{z},\vec{y}))$.

Proof : among the axioms of M_1.

2.4 PROPOSITION *For* $\mathfrak{k} \geqslant 1$, $\mathsf{Logik}(\Sigma_{\mathfrak{k}}(z,\vec{y}))$ *implies* $\mathsf{Separation}(\Sigma_{\mathfrak{k}}(\hat{z},\vec{y}))$*; more precisely, to each instance* $\mathfrak{A}$ *of* $\mathsf{Separation}(\Sigma_{\mathfrak{k}}(\hat{z},\vec{y}))$ *there is an instance* $\mathfrak{B}$ *of* $\mathsf{Logik}(\Sigma_{\mathfrak{k}}(z,\vec{y}))$ *such that* $\vdash_{\mathsf{M}_1} (\mathfrak{B} \Longrightarrow \mathfrak{A})$.

Proof : let $\Theta(z,\vec{y})$ be the formula

$$\exists x_{\mathfrak{k}} \forall x_{k-1} \exists x_{k-2} \ldots \mathcal{Q}_2 x_2 \mathcal{Q}_1 x_1 \Phi_0(x_1,\ldots,x_{\mathfrak{k}},z,\vec{y})$$

where Φ_0 is Δ_0. Given a, let $\eta = \varrho(a)$, and let $\eta < \xi_{\mathfrak{k}} < \xi_{\mathfrak{k}-1} < \xi_{\mathfrak{k}-2} < \ldots < \xi_2 < \xi_1$ be the sequence of ordinals promised by $\mathsf{Logik}(\Sigma_{\mathfrak{k}}(z,\vec{y}))$. Form the sets $a_i = V_{\xi_i}$ for $1 \leqslant i \leqslant \mathfrak{k}$. Then the class $a \cap \{z \mid \Theta\}$ equals

$$a \cap \{z \mid \exists x_{\mathfrak{k}} :\in a_{\mathfrak{k}}\ \forall x_{\mathfrak{k}-1} :\in a_{\mathfrak{k}-1}\ \exists x_{\mathfrak{k}-2} :\in a_{\mathfrak{k}-2}\ \ldots \ldots \mathcal{Q}_2 x_2 :\in a_2\ \mathcal{Q}_1 x_1 :\in a_1\ \Phi_0(x_1,\ldots,x_{\mathfrak{k}},z,\vec{y}).\}$$

which is a set by Δ_0 Separation.

2.5 DEFINITION Denote by $\mathsf{Minimisation}(\Pi_{\mathfrak{k}}(\hat{z},\vec{y}))$ the principle which for each $\Pi_{\mathfrak{k}}$ formula $\Psi(z,\vec{y})$ asserts that for any sets $\vec{y}$, if there is a z such that $\Psi(z,\vec{y})$ then there is an ordinal η such that there is a z of rank η such that $\Psi(z,\vec{y})$ but there is no such z of rank less than η.

That η is evidently unique.

2.6 PROPOSITION Minimisation $(\Pi_0(\hat{z}, \vec{y}))$.

Proof : easy from Δ_0 Separation, using the existence of $V_{\eta+1}$.

2.7 PROPOSITION $(\mathfrak{k} \geqslant 1)$ Separation$(\Sigma_{\mathfrak{k}}(\hat{z}, \vec{y}))$ *implies* Minimisation$(\Pi_{\mathfrak{k}}(\hat{z}, \vec{y}))$.

Proof : Fix $\vec{y}$ and let $\Psi(z, \vec{y})$ be $\Pi_{\mathfrak{k}}$. Suppose that z_0 and η_0 are such that $\Psi(z_0, \vec{y})$ and $\varrho(z_0) = \eta_0$. Since the negation of a $\Pi_{\mathfrak{k}}$ formula is logically equivalent to a $\Sigma_{\mathfrak{k}}$ one, we may use $\Sigma_{\mathfrak{k}}$ Separation to form the set $V_{\eta_0} \cap \{z \mid \Psi(z, \vec{y})\}$ and call it A. If A is empty, η_0 is the desired ordinal. Otherwise form $\eta_0 \cap \{\eta \mid \exists z :\in A\ \rho(z) = \eta\}$, which is a set by Proposition 1·17, and, using Foundation, take its least element, which will be the desired ordinal. ⊣ (2.7)

2.8 DEFINITION Strong Collection$(\Pi_{\mathfrak{k}}(\hat{z}, \vec{y}))$ is the principle, for a $\Pi_{\mathfrak{k}}$ formula $\Psi(z, \vec{y})$, that to each ordinal ζ there is a strictly larger ordinal ξ such that $\overrightarrow{\forall y} \in V_\zeta$, if there is a z such that Ψ then such a z will be found in V_ξ.

2.9 PROPOSITION Strong Collection$(\Pi_0(\hat{z}, \vec{y}))$.

Proof : Use Minimisation (Π_0) to produce minimal ranks of witnesses; then use Repcoll to collect those minimal ranks and thus to obtain a strict upper bound ξ for them; finally use the existence of V_ξ. ⊣ (2.9)

More generally,

2.10 PROPOSITION Minimisation$(\Pi_{\mathfrak{k}}(\hat{z}, \vec{y}))$ *implies* Strong Collection$(\Pi_{\mathfrak{k}}(\hat{z}, \vec{y}))$.

Proof : fix ζ. Let $\Psi(z, \vec{y})$ be $\Pi_{\mathfrak{k}}$. We define a function (of several variables) on V_ζ: $F(\vec{y})$ is the least η such that there is an z of rank η for which $\Psi(z, \vec{y})$, if there is such an z; otherwise $F(\vec{y}) = 0$.

By Repcoll, there is a bound ξ to such η's: we may take $\xi > \zeta$. Then

$$\overrightarrow{\forall y} :\in V_\zeta\ (\exists z \Psi \Longleftrightarrow \exists z :\in V_\xi\ \Psi). \qquad ⊣ (2.10)$$

2.11 PROPOSITION Logik$(\Sigma_1(\vec{y}))$.

Proof : Immediate from Strong Collection$(\Pi_0(\widehat{x_1}, \vec{y}))$. ⊣ (2.11)

2.12 PROPOSITION *Let* $\mathfrak{k} \geqslant 1$. *Suppose that both* Logik$(\Sigma_{\mathfrak{k}}(x_{\mathfrak{k}+1}, \vec{y}))$ *and* Strong Collection$(\Pi_{\mathfrak{k}}(\widehat{x_{\mathfrak{k}+1}}, \vec{y}))$ *hold. Then* Logik$(\Sigma_{\mathfrak{k}+1}(\vec{y}))$.

Proof : Let $\Phi(\vec{y})$ be $\exists x_{\mathfrak{k}+1} \Psi(x_{\mathfrak{k}+1}, \vec{y})$ where for some Δ_0 formula Φ_0, Ψ is the $\Pi_{\mathfrak{k}}$ formula

$$\forall x_{\mathfrak{k}} \exists x_{\mathfrak{k}-1} \forall x_{\mathfrak{k}-2} \ldots \mathcal{Q}_2 x_2 \mathcal{Q}_1 x_1 \Phi_0(x_1, \ldots, x_{\mathfrak{k}}, x_{\mathfrak{k}+1}, \vec{y})$$

where the quantifiers $\mathcal{Q}_2$ and $\mathcal{Q}_1$ are $\forall\exists$ if $\mathfrak{k}$ is even and $\exists\forall$ otherwise. For given η, Strong Collection$(\Pi_{\mathfrak{k}}(\widehat{x_{\mathfrak{k}+1}}, \vec{y}))$ yields a $\xi_{\mathfrak{k}+1} > \eta$ such that

$$\overrightarrow{\forall y} :\in V_\eta \left[\exists x_{\mathfrak{k}+1} \Psi \Longleftrightarrow \exists x_{\mathfrak{k}+1} :\in V_{\xi_{\mathfrak{k}+1}}\ \Psi\right].$$

Now starting from that ordinal $\xi_{\mathfrak{k}+1}$ apply $\mathsf{Logik}(\Sigma_{\mathfrak{k}}(x_{\mathfrak{k}+1}, \vec{y}))$ to the formula $\neg\Psi$ to obtain a strictly increasing sequence $\xi_{\mathfrak{k}} < \ldots < \xi_1$, with $\xi_{\mathfrak{k}+1} < \xi_{\mathfrak{k}}$, such that for all $x_{\mathfrak{k}+1}$ and $\vec{y}$ in $V_{\xi_{\mathfrak{k}+1}}$,

$$\begin{aligned}\Psi(x_{\mathfrak{k}+1}, \vec{y}) \Longleftrightarrow & \forall x_{\mathfrak{k}} :\in V_{\xi_{\mathfrak{k}}}\ \exists x_{\mathfrak{k}-1} :\in V_{\xi_{\mathfrak{k}-1}}\ \ldots \\ & \ldots \mathcal{Q}_2 x_2 :\in V_{\xi_2}\ \mathcal{Q}_1 x_1 :\in V_{\xi_1}\ \Phi_0(x_1, \ldots, x_{\mathfrak{k}}, x_{\mathfrak{k}+1}, \vec{y}).\end{aligned}$$

Then for all $\vec{y}$ in V_η,

$$\begin{aligned}\Phi(\vec{y}) & \Longleftrightarrow \exists x_{\mathfrak{k}+1} \Psi(x_{\mathfrak{k}+1}, \vec{y}) \\ & \Longleftrightarrow \exists x_{\mathfrak{k}+1} :\in V_{\xi_{\mathfrak{k}+1}}\ \Psi(x_{\mathfrak{k}+1}, \vec{y}) \\ & \Longleftrightarrow \exists x_{\mathfrak{k}+1} :\in V_{\xi_{\mathfrak{k}+1}}\ \forall x_{\mathfrak{k}} :\in V_{\xi_{\mathfrak{k}}}\ \ldots \\ & \qquad \ldots \mathcal{Q}_2 x_2 :\in V_{\xi_2}\ \mathcal{Q}_1 x_1 :\in V_{\xi_1}\ \Phi_0(x_1, \ldots, x_{\mathfrak{k}}, x_{\mathfrak{k}+1}, \vec{y}).\end{aligned}$$

⊣ (2.12)

3. PROOF OF THE MAIN THEOREM

$\mathsf{Logik}(\Sigma_0)$ is trivial; we are given $\mathsf{Separation}(\Sigma_0)$; we have established $\mathsf{Minimization}(\Pi_0)$, $\mathsf{Strong\ Collection}(\Pi_0)$ and $\mathsf{Logik}(\Sigma_1)$; and now for each $\mathfrak{k} = 1, 2, \ldots$, having obtained $\mathsf{Logik}(\Sigma_{\mathfrak{k}})$, we obtain successively $\mathsf{Separation}(\Sigma_{\mathfrak{k}})$, $\mathsf{Minimization}(\Pi_{\mathfrak{k}})$, $\mathsf{Strong\ Collection}(\Pi_{\mathfrak{k}})$, and $\mathsf{Logik}(\Sigma_{\mathfrak{k}+1})$. Thence we may derive all instances of the schemes of Replacement and Foundation.

We have proved

3.0 METATHEOREM *All instances of the schemes of Separation, Collection, Replacement and Foundation are derivable in the system* $\mathsf{M}_1 + \mathsf{Repcoll}$.

4. CLOSING REMARKS

Our proof has used TCo but not the Axiom of Infinity. If we replace TCo by the axiom of infinity in our base theory, we may, as we show below, derive TCo, and thus the rest of our argument will hold good, yielding this second result:

4.0 METATHEOREM *All instances of the schemes of Separation, Collection, Replacement and Foundation are derivable in the system* $\mathsf{M}_0 + \mathsf{Foundation} + \mathsf{Repcoll} + \omega \in V$.

Thus it remains only to establish the following

4.1 PROPOSITION TCo *is provable in* $\mathsf{M}_0 + \mathsf{Foundation} + \mathsf{Repcoll} + \omega \in V$.

Proof : Fix a set x. For $0 < n < \omega$, we define an *attempt of length* n to be a function f defined on n such that $f(0) = x$ and that $\forall m(m + 1 < n \Longrightarrow$

$f(m+1) = \bigcup f(m)$). Given two distinct attempts f and g with the same domain n, we may use Foundation for sets to find the least $m < n$ such that $f(m) \neq g(m)$, and thence obtain a contradiction showing $f = g$. Thus for each n there is at most one attempt of length n.

Hence we may, for each n, define $G(n)$ to be the unique attempt of length n, if it exists, and to be 0 otherwise. By Repcoll, there is a set v which contains all values of G.

Now the property of being an attempt is Δ_0 in the parameter x; hence we may apply Δ_0 Separation to v to form the set $\bar{v}$ of all attempts, and show that the union of the set of all attempts is a function, with domain $\bigcup\{n \mid \exists f :\in \bar{v} \;\; \mathrm{dom}\, f = n\}$, the supremum of the lengths of all attempts; which will be ω if we are lucky, and some $\bar{n} < \omega$ if we are not. But in the latter case, the given union will itself be an attempt and we may extend it to an attempt of length $\bar{n}+1$, a contradiction.

Thus the union of the set of attempts will be a function of domain ω, and the union of its values will be the desired transitive closure of x. ⊣ (4.1)

Without Infinity

It follows from the work of Hájek and Vopěnka [HV] and of K. Hauschild [H, Theorem II] that TCo is not provable in ZF without the axiom of infinity but with the set form of Foundation. The reason is slightly indirect: class forms of foundation may be derived from the set form if use of TCo is permitted: and that statement may be construed as applying both to set theories like ZF and to class theories like NBG. First Hájek and Vopěnka (for NBG) and then Hauschild (for ZF) have shown that if the axiom of infinity be dropped from either system, then a failure of class foundation may occur.

Hauschild's construction is roughly this: reason in a universe containing a set C_0 of distinct objects c_i (for $i \in \omega$) such that for each i, $c_i = \{c_{i+1}\}$; then, treating the elements of C_0 as Urelemente, form the sequence C_i where C_{i+1} is the set of all finite subsets of C_i, and consider its union $\bigcup_{i\in\omega} C_i$. That will be a model of ZF less the axiom of infinity, and in it, c_0 is a member of no transitive set, since any such would have to contain c_1, c_2, $\ldots$; but all members of the model are finite sets.

Let us look further at Hauschild's model. Something must go awry, for if each set has a rank, and each class $\{x \mid \varrho(x) < \zeta\}$ is a set, then TCo holds.

In this model, Repcoll is true, plus all the axioms of M_0, plus set foundation.

All instances of Separation are true, as are all instances of Strong Collection; so there must be a failure of Π_1 foundation, since TCo is provable using Δ_0 separation, Δ_0 collection, elementary set theory and Π_1 foundation; and indeed we know what failure that will be: the class of those sets that are not members of a transitive set can have no minimal element.

There is also a failure of Σ_1 foundation. The set C_0 is a class of the model, being definable as the set of all sets at the tail end of a finite $\in$-chain starting from c_0, and it has no $\in$-minimal element. As we have set foundation and full separation, we may conclude that rank is not definable in that model.

Our base theory above has been $\mathsf{M}_1 + \mathsf{Repcoll}$: and the principle TCo has played an important part in our proof. As we have not used the axiom of infinity, Hauschild's result shows that we must needs add TCo explicitly to our base theory for our proof to succeed.

In Hauschild's model, Logik and Minimization become false or, rather, meaningless, as one uses the notion of rank and the other the set-hood of the V_ζ's. Separation and Strong Collection make no mention of either, so it is a beguiling question whether part of our theorem might still hold without TCo.

Without Power Set

Finally, I am grateful to Robert Lubarsky for drawing my attention to the paper [Z] of Andrzej Zarach in which he constructs a model of TCo and Replacement, but without Power Set, in which Collection fails. In such a model, Repcoll will hold, it being trivially weaker than Replacement, and hence the essential character of our use of the Power Set Axiom is confirmed.

REFERENCES

[HV] P. Hájek and P. Vopěnka, Über die Gültigkeit des Fundierungsaxioms in speziellen Systemen der Mengentheorie, *Zeitschr. f. math Logik und Grundlagen d. Math.* **9** (1963) 235–241.

[H] K. Hauschild, Bemerkungen, das Fundierungsaxiom betreffend, *Zeitschr. f. math. Logik und Grundlagen d. Math.* **12** (1966) 51–56.

[M] A. R. D. Mathias, The Strength of Mac Lane Set Theory, *Annals of Pure and Applied Logic,* **110** (2001) 107–234.

[Z] A. Zarach, *Springer Lecture Notes in Logic* **6** (1996) 307–322.