CURRENTS ON FREE GROUPS

ILYA KAPOVICH

ABSTRACT. We study the properties of geodesic currents on free groups, particularly the "intersection form" that is similar to Bonahon's notion of the intersection number between geodesic currents on hyperbolic surfaces.

1. Introduction

A *geodesic current* on a word-hyperbolic group G is a positive G-invariant Borel measure on the space $\partial^2 G := \{(x, y) : x, y \in \partial G, x \neq y\}$. The study of geodesic currents on free groups is motivated by obtaining information about the geometry and dynamics of individual automorphisms as well as of groups of automorphisms of a free group. A similar programme has proved to be very successful for the case of surface groups and hyperbolic surfaces. There Bonahon's foundational work [3, 4] showed the relevance of currents to the study of the geometry of the Teichmuller space and the Thurston compactification of it, and to understanding the dynamical properties of the mapping class group and its individual elements. Other interesting and important results about geodesic currents in the hyperbolic surface case can be found in [6, 7, 11, 23, 22] and other sources.

We believe that in the free group case the study of currents is particularly promising, in part since they are naturally defined in the context of symbolic dynamics which is, in a sense, "native" to the free group case. Some examples of geometric information about free group automorphisms obtained by studying currents are contained in the previous work of the author [12] as well as in [15]. Two very important precursors and sources of inspiration for the present paper are the article of Bonahon [5], where currents on general word-hyperbolic groups are considered, and the 1995 doctoral dissertation of Reiner Martin from UCLA [21].

2000 *Mathematics Subject Classification.* Primary 20F36, Secondary 20E36, 57M05.
Key words and phrases. geodesic currents, free groups, random geodesics.
The author was supported by the NSF grant DMS#0404991 and by Centre de Recerca Matemàtica at Barcelona.

The main purpose of this paper is to collect together and clarify various background facts and ideas related to geodesic currents on free groups as well as to explain the relationships between them. We also aim to set up the basic machinery (and even notations) for future use and to clarify a number of typically confusing points (such as those related to left and right actions of $Out(F_n)$ on the outer space and on the space of currents. We should make it clear that many basic results here are not new and, in many instances are already present, either implicitly or explicitly, in Bonahon and Martin's work as well as in the authors article [12].

Although the present paper is broad in scope, its main underlying theme is to explain the significance and the geometric meaning of the "intersection form", which we believe to be a fundamentally important object.

A central point of Bonahon's work [3, 4] on geodesic currents on surfaces is that the notion of the geometric intersection number between (free homotopy classes of) closed curves on a closed hyperbolic surface S_g extends to a continuous symmetric bilinear map $i : Curr(S_g) \times Curr(S_g) \to \mathbb{R}$. A crucial feature of this construction is that if η_c is the current determined by a free homotopy class c of closed curves and if L_ρ is the Liouville currents corresponding to a hyperbolic structure ρ on S_g then $i(L_\rho, \eta_c) = \ell_\rho(c)$ where ℓ_ρ is the marked length spectrum corresponding to ρ. That is $i(L_\rho, \eta_c)$ is the ρ-length of the shortest with respect to ρ curve in the class c.

It turns out that in the context of a free group F of finite rank $k \geq 2$ there exists a natural "intersection form"

$$I : FLen(F) \times Curr(F) \to \mathbb{R}$$

where $FLen(F)$ is the space of all hyperbolic length functions on F corresponding to free and discrete isometric actions of F on $\mathbb{R}$-trees. The form I is continuous, linear with respect to the second argument and $\mathbb{R}$-homogeneous with respect to the first argument. Also, the form I is equivariant with respect to the left action of $Out(F)$ on $FLen(F)$ and $Curr(F)$. Moreover, as in the surface case, if η_g is the current determined by the conjugacy class $[g]$ of $g \in F$ and if $\ell \in FLen(F)$ is arbitrary then $I(\ell, \eta_g) = \ell(g)$. A normalized version of the intersection form I, as explained in Section 11 below, already appears in [12] where it serves as the main tool for computingthe conjugacy distortion spectrum of free groups automorphisms. The definition given in the present paper is due to Lustig and Hubert [19].

It turns out that some symmetries and dualities applicable to hyperbolic surfaces break down for the case of currents of free groups. Thus in Theorem 9.2 we prove that there does not exist a natural symmetric extension of I to a map $Curr(F) \times Curr(F) \to \mathbb{R}$. To do that we interpret the value $I(\ell, n_A)$, where $\ell \in FLen(F)$ is any and where n_A is the uniform current on

F corresponding to a free basis A, as the "generic stretching factor" $\lambda_A(\ell)$ of ℓ with respect to A. Here $\lambda_A(\ell)$ is approximated by the ℓ-distortion of a "random" A-geodesic. That is, for a long random cyclically reduced word w of length n over A we have $\ell(w)/n \approx \lambda_A(\ell)$. The obstruction to a symmetric extension of I is caused by the fact that there exists $\phi \in Out(F)$ such that $\lambda_A(\phi\ell_A) \neq \lambda_A(\phi^{-1}\ell_A)$ where ℓ_A is the length function on F corresponding to the action of F on its Cayley graph with respect to A.

There is one obvious and notable exception in the topics covered in that we do not discuss the Patterson-Sullivan-Bowen-Margulis embedding from the outer space into the space of geodesic currents on a free group (see [10] for an excellent discussion Patterson-Sullivan measures in the general word-hyperbolic group context). We think that this topic deserves an extensive treatment and we intend to address it separately in subsequent papers.

Because this paper is, in part, expository, some of the proofs are omitted or just sketched and there is an emphasis on aspects that are not addressed in detail in [12].

The author is very grateful to Vadim Kaimanovich, Martin Lustig, Tatiana Smirnova-Ngnibeda Jerome Los and Paul Schupp for the many helpful discussions. The author expresses special thanks to Centre de Recerca Matemàtica at Barcelona where most of this paper was written, and to the organizers of the special year in Geometric Group Theory at CRM, Enric Ventura and Jose Burillo, for the invitation to come there.

2. Basic definitions

Convention 2.1. For the remainder of the paper, unless specified otherwise, let F be a finitely generated free group of rank $k \geq 2$. We will denote by ∂F the hyperbolic boundary of F in the sense of the theory of word-hyperbolic groups. Since F is free, ∂F can also be viewed as the space of ends of F with the standard ends-space topology.

Thus ∂F is a topological space homeomorphic to the Cantor set. We will also denote

$$\partial^2 F := \{(\zeta, \xi) : \zeta, \xi \in \partial F \text{ and } \zeta \neq \xi\}.$$

Definition 2.2 (Geodesic Currents)**.** Let F be a free group of finite rank $k \geq 2$. A *geodesic current* on F is a positive Borel measure on $\partial^2 F$ that is F-invariant. We denote the space of all geodesic currents on F by $Curr(F)$.

The space $Curr(F)$ comes equipped with a weak topology: for $\nu_n, \nu \in Curr(F)$ we have $\lim_{n\to\infty} \nu_n = \nu$ iff for every two disjoint closed-open sets $S, S' \subseteq \partial F$ we have $\lim_{n\to\infty} \nu_n(S \times S') = \nu(S \times S')$.

Definition 2.3 (Projectivized Geodesic Currents)**.** For two nonzero geodesic currents $\nu_1, \nu_2 \in Curr(F)$ we say that ν_1 is equivalent to ν_2, denoted

$\nu_1 \sim \nu_2$, if there exists a nonzero scalar $r \in \mathbb{R}$ such that $\nu_2 = r\nu_1$. We denote

$$\mathbb{P}Curr(F) := \{\nu \in Curr(F) : \nu \neq 0\}/\sim$$

and call it the *space of projectivized geodesic currents on* F. Elements of $\mathbb{P}Curr(F)$ (that is, scalar equivalence classes of elements of $Curr(F)$) are called *projectivized geodesic currents*. The space $\mathbb{P}Curr(F)$ is endowed with the quotient topology. We will denote the $\sim$-equivalence class of a nonzero geodesic current ν by $[\nu]$.

We will see later on that, once a simplicial chart (defined below) on F is fixed, there is a natural embedding $\mathbb{P}Curr(F) \to Curr(F)$ that provides a section to the natural quotient map $Curr(F) - \{0\} \to \mathbb{P}Curr(F)$.

Remark 2.4 (A note on the symmetrization)**.** Denote by $\sigma : \partial^2 F \to \partial^2 F$ the *flip* map $\sigma : (\zeta, \xi) \mapsto (\xi, \zeta)$ for $(\zeta, \xi) \in \partial^2 F$.

Let $Curr_s(F)$ be the set of all $\nu \in Curr(F)$ such that ν is σ-invariant. It is clear that $Curr_s(F)$ is a closed linear subspace of $Curr(F)$. The image of $Curr_s(F)$ in $\mathbb{P}Curr(F)$ is denoted by $\mathbb{P}Curr_s(F)$.

Frequently the requirement for ν to be flip-invariant is included in the definition of a (projectivized) geodesic current. We will not do that here since most arguments and statements (at least those discussed in this paper) work in exactly the same way in both contexts and for the reasons of savings space and simplifying notations we prefer not to impose an extra condition at the definition level.

We should also add that there is a natural retraction $r : Curr(F) \to Curr_s(F)$ defined as follows. If $\nu \in Curr(F)$ and $S \subseteq \partial^2 F$, we have $(r(\nu))(S) = \frac{1}{2}(\nu(S) + \nu(\sigma S))$.

An important basic fact about geodesic currents is:

Proposition 2.5. *The space* $\mathbb{P}Curr(F)$ *is compact.*

Convention 2.6. If γ is an edge-path or a circuit in some graph, we will denote by $|\gamma|$ the edge-length of γ. Similarly, if w is a word in some alphabet or a *cyclic word* in some alphabet (to be defined later), we denote by $|w|$ the length of w, that is, the number of letters in w. If Δ is a graph, we will denote by $\mathcal{P}(\Delta)$ the set of all edge-paths of finite positive edge-length in Δ.

Definition 2.7 (Simplicial charts)**.** Let Γ be a finite connected graph such that $\pi_1(\Gamma) \cong F$. Let $\alpha : F \to \pi_1(\Gamma, p)$ be an isomorphism, where p is a vertex of Γ. We will call such α a *simplicial chart* for F.

Convention 2.8. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart.

Then $X := \widetilde{\Gamma}$ is a topological tree. Denote the covering map from X to Γ by $q : X \to \Gamma$. For $\gamma \in \mathcal{P}(X)$ we will call the reduced edge-path $v = q(\gamma)$ in Γ the *label* of γ.

Let ∂X denote the space of ends of X with the natural ends-space topology. Then we obtain a canonical α-equivariant homeomorphism $\hat{\alpha} : \partial F \to \partial X$.

This homeomorphism can be thought of in the following way. Suppose we endow Γ with the structure of a metric graph, that is, we assign each edge of Γ a positive length. This turns X into an $\mathbb{R}$-tree with a discrete isometric action of $\pi_1(\Gamma, p)$. Moreover, X is quasi-isometric to F and, if F is equipped with a word metric and p' is a lift of p to $X = \widetilde{G}$ then the orbit map $\tilde{\alpha} : F \to X$, $\tilde{\alpha} : f \to \alpha(f)p'$, is a quasi-isometry. This quasi-isometry extends to a homeomorphism $\partial F \to \partial X$ that is equal to $\hat{\alpha}$.

If α is fixed, we will usually suppress explicit mention of $\hat{\alpha}$ and also of the map α itself when talking about the action of F on X and on ∂F arising from this situation. Thus we also have an identification $\hat{\alpha} : \partial^2 F \to \partial^2 X$. A crucial feature of this construction is that $\hat{\alpha}$ does not depend on the choice of a metric graph structure on Γ.

Convention 2.9. Suppose Γ, α, p and X are as in Convention 2.8.

Similarly to the case of F, we denote by $\partial^2 X$ the set of all pairs (ζ_1, ζ_2) such that $\zeta_1, \zeta_2 \in \partial X$ and $\zeta_1 \neq \zeta_2$. For $(\zeta_1, \zeta_2) \in \partial^2 X$ we denote by $[\zeta_1, \zeta_2]$ the simplicial (non-parameterized) geodesic from ζ_1 to ζ_2 in X. Thus $[\zeta_1, \zeta_2]$ is a subgraph of X isomorphic to the simplicial line, together with a choice of direction on that line.

Definition 2.10. For every (oriented) reduced edge-path γ in X of positive edge-length denote

$$\begin{aligned} Cyl_\alpha(\gamma) := \{(x, y) \in \partial^2 F : \gamma \subseteq [\hat{\alpha}(x), \hat{\alpha}(y)] \text{ in } X \\ \text{and the orientations on } \gamma \text{ and on } [\hat{\alpha}(x), \hat{\alpha}(y)] \text{ agree}\} \end{aligned}$$

Note that by definition $Cyl_\alpha(\gamma)$ is a subset of $\partial^2 F$ (rather than of $\partial^2 X$). This distinction becomes important when α and Γ are not fixed but come from different points in the outer space. However, when α is fixed, we will often suppress the subscript X and denote $Cyl(\gamma) = Cyl_\alpha(\gamma)$.

The collection of all sets $Cyl(\gamma)$, where γ varies over $\mathcal{P}(X)$, gives a basis of closed-open sets for $\partial^2 F$. Hence it is easy to see that:

Lemma 2.11. *For $\nu_n, \nu \in Curr(F)$ $\lim_{n\to\infty} \nu_n = \nu$ iff $\lim_{n\to\infty} \nu_n(Cyl(\gamma)) = \nu(Cyl(\gamma))$ for every reduced edge-path γ in X of positive edge-length. Moreover, for $\nu, \nu' \in Curr(F)$ we have $\nu = \nu'$ iff $\nu(Cyl(\gamma)) = \nu'(Cyl(\gamma))$ for every $\gamma \in \mathcal{P}(X)$.*

Note that for any $f \in F$ and $\gamma \in \mathcal{P}(X)$ we have $fCyl(\gamma) = Cyl(f\gamma)$. Since geodesic currents are, by definition, F-invariant, for a geodesic current

ν and for $\gamma \in \mathcal{P}(X)$ the value $\nu(Cyl(\gamma))$ only depends on the label $q(\gamma)$ of γ.

Definition 2.12 (Number of occurrences of a path in a current)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart and let $X = \widetilde{\Gamma}$. For a path $v \in \mathcal{P}(\Gamma)$ and for $\nu \in Curr(F)$ we denote $\langle v, \nu \rangle_\alpha := \nu(Cyl(\gamma))$ where γ is any lift of v to X. We call $\langle v, \nu \rangle_\alpha$ the *number of occurrences of v in ν*.

In view of Lemma 2.11 one can view a simplicial chart $\alpha : F \to \pi_1(\Gamma, p)$ as providing a "coordinate system" on $Curr(F)$. A current $\nu \in Curr(F)$ is uniquely defined by its collection of "coordinates" $(\langle v, \nu \rangle_\alpha)_{v \in \mathcal{P}(\Gamma)}$. Each path $v \in \mathcal{P}(\Gamma)$ can be thought of as defining the "coordinate function" $\langle v, \cdot \rangle_\alpha$ on $Curr(F)$.

As usual we will often omit the subscript α if the chart α is fixed.

The following lemma is a well-known fact about word-hyperbolic groups.

Lemma 2.13. *Let G be a group acting properly discontinuously and co-compactly by isometries on a geodesic hyperbolic metric space X (hence G is word-hyperbolic). Let ϕ be an automorphism of G. Then for any $g \in G$ and $\xi \in \partial X$ we have $\phi(g)\phi(\xi) = \phi(g\xi)$*

Lemma 2.13 is a crucial ingredient in defining the action of $Out(F)$ on $Curr(F)$.

Lemma 2.14. *Let ϕ be an automorphism of F and let $\nu \in Curr(F)$. Define a measure $\phi\nu$ on $\partial^2 F$ by setting $\phi\nu(S) := \nu(\phi^{-1}(S))$ for a Borel subset $S \subseteq \partial^2 F$. Then $\phi\nu$ is a geodesic current on F.*

Proof. Let $S \subseteq \partial^2 F$ and $f \in F$. We need to check that $\phi\nu(fS) = \phi\nu(S)$.

By definition $\phi\nu(fS) = \nu(\phi^{-1}(fS))$. By Lemma 2.13 $\phi^{-1}(fS) = \phi^{-1}(f)\phi^{-1}(S)$. Since ν is F-invariant, we have

$$\nu(\phi^{-1}(fS)) = \nu(\phi^{-1}(f)\phi^{-1}(S)) = \nu(\phi^{-1}(S)) = \phi\nu(S).$$

Thus $\phi\nu(fS) = \phi\nu(S)$, as required. □

Proposition 2.15. *The map $(\phi, \nu) \mapsto \phi\nu$, where $\phi \in Aut(F), \nu \in Curr(F)$, defines a left action of $Aut(F)$ on $Curr(F)$ by continuous linear transformations. Moreover, $Inn(F)$ is contained in the kernel of this action, which, therefore, factors to the action of $Out(F)$ on $Curr(F)$.*

Proof. It is clear from the definition that for a fixed $\phi \in Aut(F)$ the map $\nu \mapsto \phi\nu$ is linear. Let us check the continuity of this map. Suppose that $\nu_n, \nu \in Curr(F)$ and $\lim_{n\to\infty} \nu_n = \nu$. We need to show that $\lim_{n\to\infty} \phi\nu_n = \phi\nu$. Consider an arbitrary closed-open set $S \subseteq \partial^2 F$. Since ϕ defines a homeomorphism of $\partial^2 F$, the set $\phi^{-1}(S)$ is also closed-open. Hence $\lim_{n\to\infty} \nu_n = \nu$

implies that $\lim_{n\to\infty} \nu_n(\phi^{-1}(S)) = \nu(\phi^{-1}(S))$, that is $\lim_{n\to\infty} \phi\nu_n(S) = \phi\nu(S)$. Since S was an arbitrary closed-open set, this implies that $\lim_{n\to\infty} \phi\nu_n = \phi\nu$.

Let us now check that $(\phi,\nu) \mapsto \phi\nu$ gives a left action of $Aut(F)$ on $Curr(F)$. Suppose $\phi,\psi \in Aut(F)$ and $\nu \in Curr(F)$. We need to check that $(\phi\psi)(\nu) = \phi(\psi\nu)$. Take an arbitrary closed-open $S \subseteq \partial^2 F$. We have $[(\phi\psi)\nu](S) = \nu((\phi\psi)^{-1}(S)) = \nu(\psi^{-1}\phi^{-1}(S))$. On the other hand $[\phi(\psi\nu)](S) = (\psi\nu)(\phi^{-1}(S)) = \nu(\psi^{-1}\phi^{-1}(S))$, as required.

Finally, observe that inner automorphisms act trivially on $Curr(F)$. Let $f \in F$ and consider an automorphism τ_f of F defines as $\tau_f(g) = fgf^{-1}$ for $g \in F$. Note that for every point $\xi \in \partial F$ we have $\tau_f(\xi) = f\xi$ and $\tau_f^{-1}(\xi) = f^{-1}\xi$. Hence for any closed-open $S \subseteq \partial^2 F$ we have $\tau_f^{-1}(S) = f^{-1}(S)$. Thus for any geodesic current ν and any Borel set S as above

$$(\tau_f\nu)(S) = \nu(\tau_f^{-1}S) = \nu(f^{-1}S) = \nu(S)$$

where the last equality holds by F-invariance of ν. Thus $\tau_f\nu = \nu$ and we see that inner automorphisms of F act trivially on $Curr(F)$, as required. □

Remark 2.16. Suppose that $\alpha,\beta : F \to \pi_1(\Gamma,p)$ are simplicial charts such that $\alpha^{-1}\beta$ is an inner automorphism of F and let $X = \widetilde{\Gamma}$. Then the maps $\hat{\alpha},\hat{\beta} : \partial^2 F \to \partial^2 X$ differ by a translation by an element of F. That is, there is $g \in F$ such that for every $(\zeta,\xi) \in \partial^2 F$ we have $(\hat{\alpha}^{-1}\hat{\beta})(\zeta,\xi) = (g\zeta, g\xi)$. Thus we have $Cyl_\alpha(\gamma) = gCyl_\beta(\gamma)$ for every $\gamma \in \mathcal{P}(X)$. Hence for every $\nu \in Curr(F)$ and every path $v \in \mathcal{P}(\Gamma)$ we have $\langle v;\nu\rangle_\alpha = \nu(Cyl_\alpha(v)) = \nu(Cyl_\beta(v)) = \langle v;\nu\rangle_\beta$.

3. Projectivization and related questions

Convention 3.1. We will denote by $FLen(F)$ the space of all hyperbolic length functions $\ell : F \to \mathbb{R}$ corresponding to free and discrete actions of F on $\mathbb{R}$-trees. We will denote by $\mathbb{P}FLen(F)$ or by $CV(F)$ the space of projective equivalence classes of nonzero elements of $FLen(F)$. Here two functions in $FLen(F)$ are equivalent if they are scalar multiples of each other. The space $FLen(F)$ comes equipped with the weak topology of pointwise convergence on finite subsets of F. The space $CV(F)$ inherits the quotient topology. We will denote the projective equivalence class of $\ell \in FLen(F)$ by $[\ell]$.

It is well-known that $CV(F)$ is precisely the *Culler-Vogtmann outer space* of F, as defined in [9], and we shall exploit both points of view here. Thus an element $\ell' \in CV(F)$ can also be represented as a simplicial chart $\alpha : F \to \pi_1(\Gamma,p)$ where Γ has no degree-one or degree-two vertices and is equipped with the structure of a metric graph so that the sum of the length of unoriented edges of Γ is equal to 1. Then $X = \widetilde{\Gamma}$ is an $\mathbb{R}$-tree with a free and discrete isometric action of F on X via α. and the Let ℓ be the hyperbolic length function for this action of F on X. Then $[\ell] = \ell'$.

If A is a free basis of F, we denote by $\ell_A \in Len(F)$ the hyperbolic length function on F corresponding to the action of F on its Cayley graph with respect to A. Thus for $g \in F$ $\ell_A(g)$ is the cyclically reduced length of g with respect to A. We will denote the freely reduced length of g with respect to A by $|g|_A$.

Convention 3.2. Suppose that F acts minimally, freely and discretely on $\mathbb{R}$-tree X and that $\ell : F \to \mathbb{R}$ is a hyperbolic length function associated to this action.

Then $\Gamma = X/F$ is a finite metric graph. If e is an edge of X or of Γ, we will still denote the length of e by $\ell(e)$.

Remark 3.3 (Left and Right actions on the outer space)**.** The group $Out(F)$ acts on both $FLen(F)$ and $CV(F)$ by homeomorphisms. The traditional action is the *right* action that, at the level of length functions, is given as follows.

Let $\ell \in FLen(F)$ and let $\phi \in Aut(F)$ be an automorphism representing its outer automorphism class $[\phi] \in Out(F)$. Then $\ell[\phi] := (\ell \circ \phi) : F \to \mathbb{R}$.

However, the natural action of $Out(F)$ on $Curr(F)$ and $\mathbb{P}Curr(F)$, which we described above, is a *left* action. Hence, for the equivariance and embeddability purposes, in this paper we consider the *left* action of $Out(F)$ on $FLen(F)$ and $CV(F)$. In the above notations it is defined as follows: $[\phi]\ell := (\ell \circ \phi^{-1}) : F \to \mathbb{R}$. Thus $[\phi]\ell(g) = \ell(\phi^{-1}g)$ for $g \in F$. It is easy to see that this indeed defines left actions of $Out(F)$ on $FLen(F)$ and $CV(F)$.

The left action is "natural" in the following sense. There is a natural left action of $Aut(F)$ on elements of F and on free bases of F. Let $\phi \in Aut(F)$ and let A be a free basis of F. Let $\ell_A : F \to \mathbb{R}$ be the hyperbolic length function corresponding to the left action of F on its Cayley graph with respect to A. Thus $\ell_A(g)$ is the cyclically reduced length of g over A. Then it is clear that $\ell_{\phi(A)}(g) = l_A(\phi^{-1}g)$. Thus under our convention $\phi l_A = l_{\phi(A)}$.

Definition 3.4 (Normalization)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and $X = \widetilde{\Gamma}$. Let $\nu \in Curr(F)$ be a geodesic current. Denote by $E\Gamma$ the set of all oriented edges of Γ. Put

$$\omega_\alpha(\nu) := \sum_{e \in E\Gamma} \langle e, \nu \rangle_\alpha.$$

We call $\omega_\alpha(\nu)$ the *weight of* ν *with respect to* α. For a nonzero $\nu \in Curr(F)$ denote $\nu_\alpha := \nu/\omega_\alpha(\nu)$. Thus $[\nu] = [\nu_\alpha]$ and ν_α is the unique scalar multiple of ν that has α-weight 1. We call ν_α the *α-normalized representative of* ν

and, in general, we will say that a current is *α-normalized* if it has α-weight 1.

The following lemma is an easy exercise. It gives an explicit criterion for the convergence of projectivized currents.

Lemma 3.5. *Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F.*

Then for any nonzero $\nu, \nu_n \in Curr(F)$ (where $n = 1, 2, \dots$) we have: $\lim_{n\to\infty}[\nu_n] = [\nu]$ in $\mathbb{P}Curr(F)$ if and only if $\lim_{n\to\infty}(\nu_n)_\alpha = \nu_\alpha$ in $Curr(F)$.

The map $i_\alpha : \mathbb{P}Curr(F) \to Curr(F)$, $[\nu] \mapsto \nu_\alpha$, is an $Out(F)$-equivariant topological embedding of $\mathbb{P}Curr(F)$ in $Curr(F)$.

Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and $X = \widetilde{\Gamma}$. Let $g \in F$ be a nontrivial element and let $c = c(g)$ be the cyclic path in Γ representing $[g]$. Note that

$$|c| = \sum_{e\in E\Gamma} \langle e, c\rangle = \sum_{e\in E\Gamma} \langle e, [g]\rangle = \sum_{e\in E\Gamma} \langle e, \eta_g\rangle = \omega_\alpha(\eta_g)$$

Thus for the α-normalized rational current $(\eta_g)_\alpha$ we have $(\eta_g)_\alpha = \frac{\eta_g}{|c|}$. Hence for every path $v \in \mathcal{P}(\Gamma)$ we have

$$\langle v, (\eta_g)_\alpha\rangle = \langle v, \eta_g\rangle/\omega_\alpha(\eta_g) = \langle v, (\eta_g)\rangle/|c(g)|.$$

So $\langle v, (\eta_g)_\alpha\rangle$ is equal to the number of occurrences of v in $c(g)$ divided by the length of $c(g)$. This motivates the following:

Definition 3.6 (Frequencies)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F. For an arbitrary nonzero current $\nu \in Curr(F)$ and for any path $v \in \mathcal{P}(\Gamma)$ we call $\langle v, (\nu)_\alpha\rangle = \langle v, \nu\rangle/\omega_\alpha(\nu)$ the *frequency* of v in ν.

Similarly to the situation in $Curr(F)$, once a simplicial chart α is fixed, it can be thought of as providing a "coordinate system" on $\mathbb{P}Curr(F)$. Every projective class $[\nu] \in \mathbb{P}Curr(F)$ is uniquely determined by its "frequency coordinates" $(\langle v, (\nu)_\alpha\rangle)_{v\in\mathcal{P}(\Gamma)}$. This point of view is explored in detail in [12].

4. Other models

There are several other spaces that are naturally homeomorphic to $Curr(F)$ and $\mathbb{P}Curr(F)$.

We will briefly discuss them here omitting most of the details.

Definition 4.1 (Semi-infinite sequences and one-sided shift)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and let $X = \widetilde{\Gamma}$. Let $\Omega(\Gamma)$ be the set of all semi-infinite reduced edge-paths in Γ. For each vertex $a \in V\Gamma$ denote by $\Omega_a(\Gamma)$ the set of all $\gamma \in \Omega(\Gamma)$ that start at a. Thus $\Omega(\Gamma) = \sqcup_{a\in V\Gamma}\Omega_a(\Gamma)$.

Each $\Omega_a(\Gamma)$ is identified in the obvious way with ∂X and topologized accordingly, making it into a Cantor set. We give $\Omega_a(\Gamma)$ the topology of the disjoint union of several Cantor sets.

Let $T_\Gamma : \Omega(\Gamma) \to \Omega(\Gamma)$ be the *shift map*, that erases the first edge of each $\gamma \in \Omega(\Gamma)$. Then T_Γ is easily seen to be continuous.

Let $\mathcal{S}(\Gamma)$ be the space of all positive Borel measures μ on $\Omega(\Gamma)$ that are T_Γ-invariant, that is, have the property that for every Borel set $A \subseteq \Omega(\Gamma)$ we have $\mu(A) = \mu(T_\Gamma^{-1}A)$. Let $\mathbb{P}S(\Gamma)$ be the set of all $\mu \in S(\Gamma)$ such that $\mu(\Omega(\Gamma)) = 1$, that is, μ is a probability measure.

Definition 4.2 (Bi-infinite sequences and two-sided shift)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and let $X = \widetilde{\Gamma}$. Denote by $\Sigma(\Gamma)$ the set of all maps $\varsigma : \mathbb{Z} \to E\Gamma$ such that for every $i \in \mathbb{Z}$ we have $\varsigma(i)\varsigma(i+1) \in \mathcal{P}(\Gamma)$, that is, $\varsigma(i)\varsigma(i+1)$ is a reduced edge-path in Γ. We give $\Sigma(\Gamma)$ the natural weak topology of pointwise convergence on all finite subintervals of $\mathbb{Z}$ which makes $\Sigma(\Gamma)$ compact. The space $\Sigma(\Gamma)$ comes equipped with a natural shift action of $\mathbb{Z}$: for each $n \in \mathbb{Z}$ and $\varsigma \in \Sigma(\Gamma)$ we have $(\tau_n\varsigma)(i) := \varsigma(i+n)$. Then τ_n is a homeomorphism of $\Sigma(\Gamma)$ and $\tau_n\tau_m = \tau_{n+m}$ for every $n, m \in \mathbb{Z}$.

We denote by $\mathcal{T}(\Gamma)$ the space of all positive Borel measures on $\Sigma(\Gamma)$ that are invariant with respect to this shift action of $\mathbb{Z}$.

If we choose a lift $\tilde{V}$ of the vertex set $V\Gamma$ to X, we can also think of $\Sigma(\Gamma)$ as the set of all bi-infinite geodesic paths in X that at time 0 pass through one of the elements of $\tilde{V}$

Definition 4.3 (Geodesic flow)**.** Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and let $X = \widetilde{\Gamma}$. We now give both Γ and X simplicial metrics with every edge having unit length, so that X becomes an $\mathbb{R}$-tree and Γ becomes a metric graph.

Define the *geodesic flow space* $\mathcal{G}(\Gamma)$ as the set of all isometric embeddings $\gamma : \mathbb{R} \to X$. The set $\mathcal{G}(\Gamma)$ is endowed with the compact-open topology, which in this case coincides with the weak (pointwise convergence) topology.

There is an obvious F-action and $\mathbb{R}$-action on $\mathcal{G}(\Gamma)$ by homeomorphisms defined as follows. For $g \in F$ and $\gamma : \mathbb{R} \to X$ in $\mathcal{G}(\Gamma)$ put $g\gamma := g \circ \gamma$, so that for each $r \in \mathbb{R}$ we have $(g\gamma)(r) = g\gamma(r)$. Similarly, if $t \in \mathbb{R}$ then $\varrho_t : \mathcal{G}(\Gamma) \to \mathcal{G}(\Gamma)$ is defined as $(\varrho_t\gamma)(r) = \gamma(r + t)$. Then $\varrho_0 = Id$ and $\varrho_t \circ \varrho_s = \varrho_{t+s}$ for any $t, s \in \mathbb{R}$.

Note that the quotient by the shift action $\mathcal{G}(\Gamma)/\mathbb{R}$ is equal to $\partial^2 X$.

Definition 4.4 (Bi-invariant measures on $\mathcal{G}(\Gamma)$)**.** We denote by $BI(\Gamma)$ the space of all positive Borel measures on $\mathcal{G}(\Gamma)$ that are both $\mathbb{R}$- and F-invariant. This space comes equipped with the weak topology.

We denote by $\mathbb{P}\mathcal{BI}(\Gamma)$ the space of projective equivalence classes of nonzero measures from $\mathcal{BI}(\Gamma)$, equipped with the quotient topology.

Proposition 4.5. *Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and let $X = \widetilde{\Gamma}$. Then*

(a) The space $Curr(F)$ is naturally homeomorphic to $\mathcal{S}(\Gamma)$, to $\mathcal{BI}(\Gamma)$ and to $\mathcal{T}(\Gamma)$.

(b) The space $\mathbb{P}Curr(F)$ is naturally homeomorphic to $\mathbb{P}\mathcal{S}(\Gamma)$, to $\mathbb{P}\mathcal{BI}(\Gamma)$ and to $\mathbb{P}\mathcal{T}(\Gamma)$.

Proof. We will discuss briefly only the proof of (a) since part (b) is completely analogous.

For a path $v \in \mathcal{P}(\Gamma)$ let $Cyl(v)$ denote the set of all paths $\gamma \in \Omega(\Gamma)$ that start with v. Then $Cyl(v)$ is an open-closed subset of $\Omega(\Gamma)$ and the sets $Cyl(v)$, where v varies over all $\mathcal{P}(\Gamma)$ generates the Borel σ-algebra for $\Omega(\Gamma)$.

Suppose now that $\mu \in S(\Gamma)$ is a shift-invariant measure on $\Omega(\Delta)$. We define a measure $\hat{\mu}$ on $\partial^2 X$ as follows. For each path $\beta \in \mathcal{P}(X)$ put $\hat{\mu}(Cyl(\beta)) := \mu(Cyl(v))$ where v is the label of β. It is not hard to see that $\hat{\mu}$ is a measure on $\partial^2 X$ that, by construction, is F-invariant. The map $\mu \mapsto \hat{\mu}$ provides a homeomorphism $S(\Gamma) \to Curr(F)$ that factors to a homeomorphism between the projectivized versions of these spaces.

Consider now the geodesic flow space $\mathcal{G}(\Gamma)$. Choose $t \in \mathbb{R}$ and a path $\beta \in \mathcal{P}(X)$. Define $Cyl(\beta, t)$ to be the set of all geodesics $\gamma : \mathbb{R} \to X$ such that $\gamma([t, t+|\beta|]) = \beta$ and γ maps $[t, t+|\beta|]$ to β preserving the orientation. Then $Cyl(\beta, t) \subset \mathcal{G}(\Gamma)$ is closed-open and compact. The sets $Cyl(\beta, t)$, where $t \in \mathbb{R}, \beta \in \mathcal{P}(X)$, generate the Borel σ-algebra for $\mathcal{G}(\Gamma)$.

Suppose now that $\nu \in Curr(F)$ is a geodesic current. We define a measure $\tilde{\nu}$ on $\mathcal{G}(\Gamma)$ as follows. For every cylinder $Cyl(\beta, t)$ put $\tilde{\nu}(Cyl(\beta, t)) := \nu(Cyl(\beta))$. Again, it is easy to see that $\tilde{\nu}$ is a measure which is both F- and $\mathbb{R}$-invariant, so that $\tilde{\nu} \in B(\Gamma)$. Also, the map $\nu \mapsto \tilde{\nu}$ is a homeomorphism between $Curr(F)$ and $B(\Gamma)$ that factors through to a homeomorphism between their projectivizations.

Finally, let us discuss the identification between $Curr(F)$ and $\mathcal{T}(\Gamma)$. In the context of $\Sigma(\Gamma)$ the cylinder sets look as follows. Let $v \in \mathcal{P}(\gamma)$ be a path with $|v| = n$ and let $t \in \mathbb{Z}$. Then $Cyl_t(v)$ is defined as the set of all $\varsigma \in Sigma(\Gamma)$ such that $\varsigma(i)\varsigma(i+1)\dots\varsigma(i+n-1) = v$. Again, the sets $Cyl_t(v)$, where $v \in \mathcal{P}(\Gamma)$ and $t \in \mathbb{Z}$, generate the Borel σ-algebra for $\Sigma(\Gamma)$.

Suppose $\nu \in Curr(F)$. We associate to ν a Borel measure μ on $\Sigma(\Gamma)$ as follows. For any cylinder $Cyl_t(v)$ set $\mu(Cyl_t(v)) := \langle v, \nu \rangle = \nu(Cyl(\gamma))$ where γ is any lift of v to X. It is not hard to see that μ is shift-invariant, so that $\mu \in \mathcal{T}(\Gamma)$. This determines a map from $Curr(F)$ to $\mathcal{T}(\Gamma)$ that is easily seen to be bijective and continuous. □

The identification between $\mathbb{P}Curr(F)$ and the space $\mathbb{P}\mathcal{S}(\Gamma)$ is particularly useful, for example, for proving the density of rational currents in $Curr(F)$

and in $\mathbb{P}Curr(F)$. Indeed, it is not hard to see that the elements of $\mathbb{P}\mathcal{T}(\Gamma)$ corresponding to rational currents are precisely the shift-invariant probability measures supported on the T_Γ-periodic orbits in $\Omega(\Gamma)$. This connection is explored in more detail in [12].

5. Currents determined by conjugacy classes and the intersection form

Let $g \in F$ be a nontrivial element. It canonically defines a pair of distinct points $g^\infty, g^{-\infty} \in \partial F$, where $g^\infty = \lim_{n\to\infty} g^n$ and $g^{-\infty} = \lim_{n\to\infty} g^{-n}$. Note that $(fgf^{-1})^\infty = fg^\infty$ and $(fgf^{-1})^{-\infty} = fg^{-\infty}$ for every $f \in F$.

Definition 5.1 (Rational currents)**.** Let $g \in F$ be a nontrivial element that is not a proper power. Define a Borel measure η_g on $\partial^2 F$ as follows. For a closed-open subset $S \subseteq \partial^2 F$ let $\eta_g(AS)$ be the number of those F-translates of $(g^\infty, g^{-\infty})$ that belong to S. Obviously, η_g is F-invariant, so that $\eta_g \in Curr(F)$.

In view of the remark above $\eta_g(S)$ is equal to the number of points of the form $f(g^\infty, g^{-\infty})f^{-1}$, where $f \in F$, that belong to S. Thus η_g only depends on the conjugacy class of g in F.

Suppose now that g is an arbitrary nontrivial element of F. Then we can uniquely represent g as $g = h^t$ where $t \geq 1$ is an integer and $h \in F$ is not a proper power. We define $\eta_g := t\eta_h$. Again, we see that η_g only depends on the conjugacy class of g.

Finally, for any nontrivial conjugacy class $[g]$ in F define $\eta_{[g]} := \eta_g$, where $g \in [g]$ is an arbitrary element.

We shall refer to multiplies of currents of the form $\eta_{[g]}$, where $[g]$ is a nontrivial conjugacy class in F, as *rational* currents.

For any nontrivial $g \in F$ we denote the projective class $[\eta_g]$ of η_g in $\mathbb{P}Curr(F)$ by μ_g or by $\mu_{[g]}$. Note that by definition if $n \geq 1$ then $\mu_g = \mu_{g^n}$.

Suppose now that ∂F is identified with $\partial X = \partial\widetilde{\Gamma}$, as in the previous section. Let w be a nontrivial root-free conjugacy class in F. Then w is represented by a unique cyclically reduced closed circuit w' in Γ. Choose a particular cyclically reduced path γ in Γ representing w' (which can be done by choosing a vertex on w').

Then it is easy to see the for every path $\gamma \in \mathcal{P}(X)$ $\eta_g(Cyl(\gamma))$ is equal to the number of those $(x, y) \in \partial^2 X$ such that the geodesic $[x, y]$ contains γ (with the agreement of orientations) and such that $[x, y]$ is labelled by a bi-infinite power of γ:

$$\dots\gamma\gamma\gamma\dots$$

Definition 5.2 (Cyclic paths and cyclic words)**.** A *cyclic path* or *circuit* in Γ is an immersion graph-map $c : \mathbb{S} \to \Gamma$ from a simplicially subdivided oriented circle $\mathbb{S}$ to Γ. Let u be an edge-path in Γ. An *occurrence of* u *in* c is a vertex of $\mathbb{S}$ such that, going from this vertex in the positive direction along $\mathbb{S}$, there exists an edge-path in $\mathbb{S}$ (not necessarily simple and not necessarily closed) which is labelled by u, that is, which is mapped to u by c. We denote by $\langle u, c\rangle$ the number of occurrences of u in c.

If A is a free basis of F and Γ is a bouquet of edges labelled by the elements of A, then a cyclic path in Γ can also be thought of as a *cyclic word* over A. A *cyclic word* is an equivalence class of cyclically reduced words, where two cyclically reduced words are equivalent if they are cyclic permutations of each other.

Notation 5.3. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and let $X = \widetilde{\Gamma}$. It is easy to see that every nontrivial conjugacy class $[g]$ in F is uniquely represented by a reduced circuit in Γ which in turn is uniquely represented by a cyclic path $c = c_\alpha(g)$ in Γ.

The following is an immediate corollary of the definitions:

Lemma 5.4. *For every* $\gamma \in \mathcal{P}(X)$ *with label* $v = q(\gamma)$ *and for every nontrivial conjugacy class* $[g]$ *in* F *with* $c = c(g)$ *we have*

$$\eta_{[g]}(Cyl(\gamma)) = \langle v, c\rangle,$$

that is,

$$\langle v, \eta_{[g]}\rangle = \langle v, c\rangle$$

A similar statement holds at the level of frequencies. For example, suppose that (α, Γ) is the bouquet of circles corresponding to a free basis A of F, and that w is a cyclic word over A representing g. If v is a freely reduced word over A then the frequency of v in η_g is equal to the number of occurrences of v in w divided by the length of w, which is the frequency of v in w.

Proposition 5.5. *Let* $g \in F$ *be a nontrivial element and let* $\phi \in Aut(F)$*. Then* $\phi\eta_g = \eta_{\phi(g)}$*.*

Proof. Let $S \subseteq \partial^2 F$ be a closed-open set and let $S' = \phi^{-1}(S)$. Recall that for $h \neq 1$ the measure η_h counts the number of points that are F-translates (or F-conjugates) of the pair $(h^{-\infty}, h^{\infty})$ in a set.

Thus $\phi\eta_g(S) := \eta_g(\phi^{-1}(S))$ is the number of F-translates of $(g^{-\infty}, g^{\infty})$ in $S' = \phi^{-1}(S)$. Similarly, $\eta_{\phi(g)}(S)$ is the number of F-translates of $(\phi(g)^{-\infty}, \phi(g)^{\infty})$ in S. Lemma 2.13 implies that ϕ maps bijectively F-translates of $(g^{-\infty}, g^{\infty})$ in $\phi^{-1}(S)$ to F-translates of $(\phi(g)^{-\infty}, \phi(g)^{\infty})$ in S. Hence $\phi\eta_g(S) = \eta_{\phi(g)}(S)$ and, since S was arbitrary, $\phi\eta_g = \eta_{\phi(g)}$. □

Notation 5.6. Let $C = C(F)$ denote the set of all nontrivial conjugacy classes in F and let $C_0 = C_0(F)$ the the set of all elements of C that are not proper powers. Denote by $r : C \to Curr(F)$ the map $r : [g] \mapsto \eta_{[g]}$ and denote by $\hat{r} : C \to \mathbb{P}Curr(F)$ the map $\hat{r} : [g] \mapsto \mu_{[g]}$.

We summarize some of the properties of r and $\hat{r}$ in the following proposition (see [12] for detailed arguments):

Proposition 5.7. *We have:*

(1) *The set $\hat{r}(C) = \hat{r}(C_0)$ is dense in $\mathbb{P}Curr(F)$ and the set of all scalar multiples of elements of $r(C)$ is dense in $Curr(F)$.*
(2) *The map $r : C \to Curr(F)$ is injective and the map $\hat{r}|_{C_0} : C_0 \to \mathbb{P}Curr(F)$ is injective; consequently, the actions of $Out(F)$ on $Curr(F)$ and $\mathbb{P}Curr(F)$ are effective.*

We can now define a natural "intersection form" $I: FLen(F) \times Curr(F) \to \mathbb{R}$ that will be $Out(F)$-equivariant and linear in the second coordinate. Moreover, it will be "natural" in the sense that $I(\eta_g, l) = \ell(g)$ for every $g \in F$.

Definition 5.8 (Intersection form)**.** Let $\ell \in FLen(F)$ be a length function and let $\nu \in Curr(F)$ is a geodesic current. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart and let Γ be given a metric graph structure so that ℓ is the hyperbolic length function corresponding to the action of F on the $\mathbb{R}$-tree $X = \widetilde{\Gamma}$ via α, where the metric on X correspnds to the metric graph structure on Γ. We define

$$I(\ell, \nu) := \sum_{e \in E\Gamma} \ell(e) \langle e, \nu \rangle_\alpha,$$

where $E\Gamma$ is the set of oriented edges of Γ.

Proposition 5.9. *The map $I : FLen(F) \times Curr(F) \to \mathbb{R}$ is continuous, linear with respect to the second argument and $Out(F)$-equivariant with respect to the left diagonal action of $Out(F)$ on $FLen(F) \times Curr(F)$. Also, I is homogeneous with respect to the first argument, that is $I(r\ell, \nu) = rI(\ell, \nu)$ for every $r \geq 0$ and every $\ell \in FLen(F), \nu \in Curr(F)$.*

Moreover, for every $g \in F$, and $\ell \in FLen(F)$ we have $I(\ell, \eta_g) = \ell(g)$.

Proof. It is clear from the definition that $I : FLen(F) \times Curr(F) \to \mathbb{R}$ is linear with respect to the second argument, homogeneous with respect to the first argument, and, moreover, for any $\ell \in FLen(F)$ the function $I(\ell, -)$ is continuous on $Curr(F)$.

Suppose that $\ell \in FLen(F)$ is a length function and that $\nu \in Curr(F)$ is a geodesic current. The length function ℓ defines a free and discrete isometric action of F on an $\mathbb{R}$-tree X such that $\Gamma = X/F$ is a finite metric graph and that $X = \widetilde{\Gamma}$. Let E be the collection of oriented edges of Γ. Let $g \in F$ be a nontrivial element. Let $c = c(g)$ be the reduced cyclic path in Γ representing $[g]$. Then $\ell(g)$ is equal to the sum of the length of the edges of c, that is $\ell(g) = \sum_{e \in E} \ell(e) n_c(e)$. By Lemma 5.4 we have $n_c(e) = n_{\eta_g}(e)$. Thus

$$\ell(g) = \sum_{e \in E} \ell(e) n_c(e) = \sum_{e \in E} \ell(e) n_{\eta_g}(e) = I(\ell, \eta_g),$$

as claimed.

Assuming the global continuity of I, let us check its $Out(F)$-equivariance. Suppose $\ell \in FLen(F)$ and $g \in F$ and let $\phi \in Aut(F)$. Recall that $\phi \eta_g = \eta_{\phi(g)}$. Also, by definition of the left action of $Out(F)$ on $FLen(F)$ we have $\phi l = \ell \circ \phi^{-1}$. Thus we have

$$I(\phi\ell, \phi\eta_g) = I(\phi\ell, \eta_{\phi(g)}) = (\phi\ell)(\phi(g)) = (\ell \circ \phi^{-1})(\phi(g)) = \ell(g) = I(\ell, \eta_g).$$

Since the scalar multiples of rational currents are dense in $Curr(F)$, the continuity of I implies that $I(\phi l, \phi\nu) = I(\ell, \nu)$ for every $\ell \in FLen(F)$ and every $\nu \in Curr(F)$.

It remains to show that I is continuous. We will give a sketch of the argument here and leave some of the details to the reader. Let $\ell \in CV, \nu \in Curr(F)$. We need to establish that I is continuous at (ℓ, ν). Because $FLen(F) \times Curr(F)$ is metrizable and locally compact, it suffices to prove that for every two sequences $\ell_n \in FLen(F)$ and $\nu_n \in Curr(F)$ with $\lim_{n\to\infty} \ell_n = l$ and $\lim_{n\to\infty} \nu_n = \nu$ we have $\lim_{n\to\infty} I(\ell_n, \nu_n) = I(\ell, \nu)$.

The length function ℓ defines a minimal action of F on an $\mathbb{R}$-tree X with a finite quotient graph $\Gamma = X/F$. Moreover, this action determines an isomorphism $\alpha : F \to \pi_1(G, p')$ where p' is a vertex of Γ.

Suppose first that Γ has maximal possible number of edges among all finite connected graphs with no degree-one and degree-two vertices whose fundamental group is isomorphic to F. Then for all length-functions ℓ' sufficiently close to ℓ we have $\Gamma' = X'/F = X/F = \Gamma$ and $\alpha = \alpha'$ where X' is the tree corresponding to ℓ' and where $\alpha' : F \to \pi_1(G, p')$ is defined similarly to α. Then the continuity of I at (ℓ, ν) follows directly from the definition of I.

Suppose now that the number of edges of Γ is not the maximal possible. Then it suffices to consider the situation when the sequence ℓ_n approximating ℓ has the following form. There is a finite graph Δ homotopy equivalent to Γ and such that Γ is obtained from Δ by contracting to points a certain (possibly empty) collection of edges E' of Δ. We will denote this contraction by $\kappa : \Delta \to \Gamma$. There is an isomorphism $\beta : F \to \pi_1(\Delta, p'')$, where p'' is a vertex of Δ such that $\kappa(p'') = p'$ and such that β factors through $\kappa_\#$ to α, that is $\alpha = \kappa_\# \circ \beta$. Each ℓ_n corresponds to making Δ into a finite metric graph. Moreover, the original length function ℓ corresponds to a semi-metric structure on Δ, where every edge of Δ is assigned a certain nonnegative length, with edges of E' being given zero length and the edges in $E\Delta - E'$ assigned the same length as their images in Γ. Then again every conjugacy class $[g]$ in F is represented, via β, by a unique reduced cyclic path and $\ell(g)$ is the length of that path. Note that $Y = \widetilde{\Delta}$ is no longer equivariantly homeomorphic to $X = \widetilde{\Gamma}$. However, X is obtained from Y by performing an equivariant collection of edge-contractions of the lifts of the edges of E'. Moreover, for every $e \in E'$ we have $\ell_n(e) \to 0$ as $n \to \infty$. Note that since ν_n converges to ν, we have $\lim_{n\to\infty} \nu_n(Cyl_\beta(e)) = \nu(Cyl_\beta(e))$ for every edge e of Δ. In particular, for every such e the sequence $\nu_n(Cyl_\beta(e))$ is bounded. In case $e \in E'$ this implies that $\lim_{n\to\infty} \ell_n(e)\nu_n(Cyl_\beta(e)) = 0$.

The crucial point is that for every edge e of Γ (which we still denote by e when thought of as an edge of Δ) we have $Cyl_\alpha(e) = Cyl_\beta(e) \subseteq \partial^2 F$. Consequently, for every edge e of Γ and for any current $\mu \in Curr(F)$ we have $\langle e, \mu\rangle_\beta = \langle e, \mu\rangle_\alpha$.

Hence

$$\lim_{n\to\infty} I(\ell_n, \nu_n) = \lim_{n\to\infty} \sum_{e\in E\Delta} \ell_n(e)\langle e, \nu_n\rangle_\beta =$$

$$\lim_{n\to\infty} [\sum_{e\in E'} \ell_n(e)\langle e, \nu_n\rangle_\beta + \sum_{e\in EG} \ell_n(e)\langle e, \nu_n\rangle_\beta] =$$

$$0 + \lim_{n\to\infty} \sum_{e\in EG} \ell_n(e)\langle e, \nu_n\rangle_\beta =$$

$$\lim_{n\to\infty} \sum_{e\in EG} \ell(e)\langle e, \nu\rangle_\alpha = I(\ell, \nu)$$

as required. □

Remark 5.10 (Weight and length)**.** The notion of weight is also natural in the following sense. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and $X = \widetilde{\Gamma}$. Endow X and Γ with simplicial metrics and let $\ell_\alpha \in FLen(F)$ be the corresponding length function on F. In view of the definition of the intersection form $I(-,-)$ we see that for any nonzero current ν we have $I(\ell_\alpha, \nu_\alpha) = 1$ and that ν_α is the only representative of $[\nu]$ with this property.

6. Local formulas

It is a natural and important question to understand what the transition functions between coordinate systems on $Curr(F)$ corresponding to two different simplicial charts look like.

Proposition 6.1. *Let* $\alpha : F \to \pi_1(\Gamma, p)$ *and* $\beta : F \to \pi_1(\Upsilon, s)$ *be two simplicial charts for* F*. Let* $X = \widetilde{\Gamma}$ *and* $Y = \widetilde{\Upsilon}$ *be the corresponding topological trees. There exists a constant* $K = K(\alpha, \beta) > 0$ *with the following property.*

For any path $v \in \mathcal{P}(\Upsilon)$ *there exist integers* $C(u, v) = C(u, v, \alpha, \beta) \geq 0$ *such that for every nontrivial* $g \in F$ *we have:*

$$(\ddagger) \qquad \langle v, c_\beta(g)\rangle_\beta = \sum_{u \in \mathcal{P}(\Gamma), |u| \leq K|v|} C(u, v)\langle u, c_\alpha(g)\rangle_\alpha.$$

Therefore, since rational currents are dense in $Curr(F)$*, for every* $\nu \in Curr(F)$ *we have:*

$$\langle v, \nu\rangle_\beta = \sum_{u \in \mathcal{P}(\Gamma), |u| \leq K|v|} C(u, v)\langle u, \nu\rangle_\alpha.$$

Proof. Since rational currents are dense in $Curr(F)$, by continuity it suffices to establish (‡).

This statement is proved in [12] for the case where both Γ and Υ are bouquets of k loop-edges. We will refer to this as the bouquet-bouquet case. Note that this is the crucial case where the heart of the argument lies. The idea there is that the bouquet-bouquet case corresponds to considering a single automorphism ϕ of F with respect to a fixed free basis A of F. The statement of the proposition then says that for a fixed freely reduced word v over A and for any reduced cyclic word w over A the number of occurrences of v in $\phi(w)$ is an integer linear combination of the numbers of occurrences in w of words u from a finite collection, where this collection depends only on ϕ and v but not on w. This, in turn, can be easily established by induction on the word-length of an automorphism ϕ, once the statement has been directly verified for the Nielsen automorphisms which generate $Aut(F)$.

Since the bouquet-bouquet case is already covered, by a composition argument it suffices to prove statement for the case where Γ is a bouquet of k loop-edges and Υ is arbitrary and for the case where Υ is a bouquet of k loop-edges and Γ is arbitrary.

Consider first the case where Γ is a bouquet of k loop edges and Υ is arbitrary. This case is in fact considered in [12], but we will repeat the argument.

Choose a maximal tree T in Υ. Choose an orientation $E\Upsilon = E^+ \sqcup E^-$ on Υ. This defines a *geometric basis* A_T of F as follows. For each edge $e \in E^+ - ET$ put $\gamma_e = [s, o(e)]_T e[t(e), s] \in \pi_1(\Upsilon, s)$. Note that γ_e is a reduced edge-path from s to s in Υ. Put $a_e = \beta^{-1}(\gamma_e)$ and put $A_T := \{a_e | e \in E^+ - ET\}$. Thus indeed A_T is a free basis of F.

By the bouquet-bouquet case we may assume that (Γ, α) is the the bouquet of edges corresponding to the free basis A_T of F.

There is an explicit way of rewriting a reduced cyclic word w over A_T into a reduced cyclic path $c_\Upsilon w$ as follows. Replace each $a_e^{\pm 1}$ in w by $e^{\pm 1}$ and then between each pair of the sort $e_1^\varepsilon e_2^\delta$, where $\varepsilon, \delta \in \{\pm 1\}$, insert $[t(e_1^\varepsilon), o(e_2^\delta)]$. The result is precisely the cyclic path $c_\Upsilon w$. It is now clear that every occurrence of a fixed path v in $c_\Upsilon w$ must come from an occurrence of one of a finite collection of paths u in Γ, where this collection depends only on v but not on w.

Note also that there is also an obvious way of rewriting a cyclic path c in Υ into a cyclic word w over A_T. Namely, we delete all the edges of ET from c and replace every $e^{\pm 1}$ by $a_e^{\pm 1}$. This shows that the number of occurrences of a reduced word z in w is equal to the number of occurrences of the path z' in c, where, again, z' is obtained from z by replacing each $a_e^{\pm 1}$ in w by $e^{\pm 1}$ and then inserting between each pair of the sort $e_1^\varepsilon e_2^\delta$ the path $[t(e_1^\varepsilon), o(e_2^\delta)]$.

This shows, in addition, that the statement of the proposition also holds when Γ and Υ as above exchange places. By the bouquet-bouquet case this implies that the statement holds when Γ is arbitrary and when Υ is a bouquet of edges.

This completes the proof of (‡) and of the proposition. □

Theorem 6.2. *Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F and let $\ell \in FLen(F)$ be a length function. Let $X = \widetilde{\Gamma}$.*

Then there exist an integer $K > 0$ and some constants $d(u) = d(u, \alpha, l) \geq 0$, where u varies over all reduced paths in Γ of length at most K, with the following property.

For any $\nu \in Curr(F)$ we have

$$I(\ell, \nu) = \sum_{|u| \leq K, u \in \mathcal{P}(\Gamma)} d(u) \langle u, \nu \rangle_\alpha.$$

In particular, for every nontrivial element $g \in F$

$$I(\ell, \eta_g) = \ell(g) = \sum_{|u| \leq K, u \in \mathcal{P}(\Gamma)} d(u) \langle u, [g] \rangle_\alpha.$$

Proof. Since the intersection form I is continuous and rational currents are dense in $Curr(F)$, it suffices to establish the statement of the theorem for rational currents.

The length function ℓ corresponds to a representation $\beta : F \to \pi_1(\Upsilon, s)$, where Υ is a metric graph and where ℓ is the hyperbolic length function associated to the action of F (via β) on the $\mathbb{R}$-tree $Y = \widetilde{\Upsilon}$. We will still denote the length of an edge e in Υ or in Y by $\ell(e)$.

Let $g \in F$ be an arbitrary nontrivial element. Let $c = c(g)$ be the reduced cyclic path in Υ representing $[g]$. Then

$$\ell(g) = \sum_{e \in E\Upsilon} \ell(e)\langle e, c(g)\rangle.$$

By Proposition 6.1 we have

$$I(\ell, \eta_g) = \ell(g) = \sum_{e \in E\Upsilon} \ell(e)\langle e, c(g)\rangle = \sum_{e \in E\Upsilon} \ell(e) \sum_{|u| \le K, u \in \mathcal{P}(\Gamma)} c(u, e)\langle u, [g]\rangle_\alpha =$$

$$= \sum_{|u| \le K, u \in \mathcal{P}(\Gamma)} \Big(\sum_{e \in E\Upsilon} \ell(e) c(u, e)\Big)\langle u, [g]\rangle_\alpha.$$

Thus the statement of the theorem holds with $d(u) = \sum_{e \in E\Upsilon} \ell(e)c(u, e)$. □

7. Uniform measures and uniform currents

Definition 7.1 (Uniform current)**.** Let A be a free basis of F and let Γ be a bouquet of k edges labelled by the generators of F and let $\alpha : F \to \pi_1(\Gamma)$ be the corresponding simplicial chart. Thus $X = \widetilde{\Gamma}$ is the Cayley graph of F with respect to A.

We define a current n_A on F by setting $n_A(Cyl(\gamma)) := \frac{1}{2k(2k-1)^{|\gamma|-1}}$ for any path $\gamma \in \mathcal{P}(X)$. It is easy to see that $n_A \in Curr(F)$ which has weight 1 with respect to α.

We refer to n_A as the *uniform current on F corresponding to A*.

Similarly, we define a measure m_A on ∂F (identified with the set of semi-infinite freely reduced words over A) as follows. For any freely reduced word v over A we have $m_A(Cyl(v)) := \frac{1}{2k(2k-1)^{|v|-1}}$. Then m_A is easily seen to be a Borel probability measure on ∂F that is invariant with respect to the shift map $T_A : \partial F \to \partial F$ that erases the first letter of each geodesic ray. We call m_A the *uniform measure on ∂F corresponding to A*.

Convention 7.2. For a point $\zeta \in \partial F$ and an integer $n \ge 0$ we denote by $\zeta_A(n)$ the element of F corresponding to the initial segment of ζ of length n, when ζ is expressed as a geodesic ray over A.

The following is an easy corollary of the law of large numbers applied to the finite state markov process generating freely reduced words over A:

Proposition 7.3. *Let A be a free basis of F. Then for a m_A-a.e. point $\zeta \in \partial F$ we have*

$$\lim_{n \to \infty} \frac{\eta_{\zeta_A(n)}}{n} = n_A$$

in $Curr(F)$ and

$$\lim_{n \to \infty} [\eta_{\zeta_A(n)}] = [n_A]$$

in $\mathbb{P}Curr(F)$.

Informally, the above statement says that in a long random freely reduced word $\zeta_A(n)$ the frequency of every fixed freely reduced word v approaches its equilibrium value $\frac{1}{2k(2k-1)^{|v|-1}}$ and that the word $\zeta_A(n)$ is "almost" cyclically reduced.

Lemma 7.4. *Let A be a free basis of F and let $\phi \in Aut(F)$. Then $\phi n_A = n_{\phi(A)}$ in $Curr(F)$.*

Proof. Let $B = \phi(A)$ and let $v = v(B)$ be a freely reduced word over B of length $t > 0$. Let $u = v(A)$, so that $\phi(u) = v$.

Let X_A and X_B be the Cayley graphs of F with respect to A and B respectively. Then at the level of subsets of ∂F we have $\phi(Cyl_A(u)) = Cyl_B(v)$. Similarly, at the level of subsets of $\partial^2 F$, if γ is a path in X_B labelled $v = v(B)$ then $\phi^{-1}(Cyl_B(\gamma)) = Cyl_A(\beta)$ where β is a path in X_A labelled by $u = v(A)$.

Hence

$$(\phi n_A)(Cyl_B(\gamma)) = n_A(\phi^{-1}(Cyl_B(\gamma))) = n_A(Cyl_A(\beta)) = \frac{1}{2k(2k-1)^{t-1}}.$$

This implies that $\phi n_A = n_{\phi(A)}$, as claimed. □

The following fact is a basic consequence of the Subadditive Ergodic Theorem and of the non-amenability of F. It is essentially a restatement of the results of Kapovich, Kaimanovich and Schupp in [15] where we refer the reader for a more detailed discussion about generic stretching factors.

Lemma-Definition 7.5 (Generic stretching factors)**.** Let A be a free basis of F, let $\ell \in Len(F)$ and let X be an $\mathbb{R}$-tree realizing the length function ℓ. Then there exists a unique number $\lambda_A(\ell) > 0$ with the following property.

For m_A-a.e point $\zeta \in \partial F$ we have:

(1) $\displaystyle\lim_{n\to\infty} \frac{\ell(\zeta_A(n))}{n} = \lim_{n\to\infty} \frac{\ell(\zeta_A(n))}{l_A(\zeta_A(n))} = \lambda_A(\ell)$;

(2) $\displaystyle\lim_{n\to\infty} \frac{d_X(p, \zeta_A(n)p)}{n} = \lambda_A(\ell)$ where $p \in X$ is any point.

The number $\lambda_A(\ell)$ is called the *generic stretching factor of ℓ with respect to A.*

Informally, for a long random freely reduced word w over A we have $\ell(w)/||w||_A \approx \lambda_A(\ell)$.

Definition 7.6. Let F be a free group with a free basis A and let $\phi \in Out(F)$. We call $\lambda_A(\ell_A\phi) = \lambda_A(\phi^{-1}l_A)$ the *generic stretching factor of ϕ with respect to A* and denote it by $\lambda_A(\phi)$.

If $\varphi \in Aut(F)$ is an automorphism representing $\phi \in Out(F)$, we set $\lambda_A(\varphi) := \lambda_A(\phi)$.

Thus if $\varphi \in Aut(F)$ represents ϕ then for a long random freely reduced word w over A we have $\frac{l_A(\varphi(w))}{l_A(w)} \approx \lambda_A(\phi)$.

8. Computation of generic stretching factors

Let $F = F(A)$ be a free group of finite rank $k \geq 2$ with a free basis $A = a, b, \dots$. Our goal in this section is to produce an automorphism ϕ of F such that $\lambda_A(\phi) \neq \lambda_A(\phi^{-1})$.

Consider the automorphisms $\tau, \sigma \in Aut(F)$ defined as follows. We have $\tau(b) = ba$ and $\tau(x) = x$ for each $x \in A - \{b\}$. We have $\sigma(a) = ab$ and $\sigma(x) = x$ for each $x \in A - \{a\}$. Finally put $\phi = \sigma\tau^2$. We claim that $\lambda_A(\phi) \neq \lambda_A(\phi^{-1})$.

We shall need the following series of lemmas for explicit computations of the generic stretching factors of ϕ and ϕ^{-1}. For a cyclic word w over A and for a freely reduced word v we will denote $n(v, w) := \langle v, w\rangle_A + \langle v^{-1}, w\rangle_A$.

Lemma 8.1. *Let w be any nontrivial cyclic word over A. Then:*

(1) $n(x, \tau(w)) = n(x, w)$ *for any* $x \in A$, $x \neq a$.
(2) $n(ab^{-1}, \tau(w)) = n(ba^{-1}, \tau(w)) = n(ba^{-1}b^{-1}, w) + n(ba^{-2}, w)$.
(3) $|\tau(w)| = |w| + n(b, w) - 2n(ba^{-1}, w) = |w| + n(a, \tau(w)) - n(a, w)$ *and therefore* $n(a, \tau(w)) = n(a, w) + n(b, w) - 2n(ba^{-1}, w)$.
(4) $n(ba^{-1}b^{-1}, \tau(w)) = n(ba^{-1}b^{-1}, w)$.
(5) $n(ba^{-2}, \tau(w)) = n(ba^{-3}, w) + n(ba^{-2}b^{-1}, w)$.

Proof. Note that $\tau(b^{-1}) = a^{-1}b^{-1}$. The lemma follows easily from the fact that the only cancellations in $\tau(w)$ after the letter-wise application of τ are of the form aa^{-1} and they come from the occurrences of ba^{-1} and ab^{-1} in w. In particular, no letter different from a^{pm1} is cancelled. After these cancellations of aa^{-1} are performed, the result is the cyclically reduced form of $\tau(w)$. □

The following two lemmas are essentially self-explanatory and we omit the details:

Lemma 8.2. *Let w be any nontrivial cyclic word over A. Then:*

(1) $n(x, \tau^2(w)) = n(x, w)$ *for any* $x \in A$, $x \neq a$.
(2) $n(ab^{-1}, \tau^2(w)) = n(ba^{-1}, \tau^2(w)) = n(ba^{-1}b^{-1}, \tau(w)) + n(ba^{-2}, \tau(w)) = n(ba^{-1}b^{-1}, w) + n(ba^{-3}, w) + n(ba^{-2}b^{-1}, w)$.
(3) $n(a, \tau^2(w)) = n(a, \tau(w)) + n(b, \tau(w)) - 2n(ba^{-1}, \tau(w)) = n(a, w) + n(b, w) - 2n(ba^{-1}, w) + n(b, w) - 2[n(ba^{-1}b^{-1}, w) + n(ba^{-2}, w)] = n(a, w) + 2n(b, w) - 2n(ba^{-1}, w) - 2[n(ba^{-1}b^{-1}, w) + n(ba^{-2}, w)]$.

(4) $|\tau^2(w)| = |\tau(w)| + n(b,\tau(w)) - 2n(ba^{-1},\tau(w)) = |w| + n(b,w) - 2n(ba^{-1},w) + n(b,w) - 2[n(ba^{-1}b^{-1},w) + n(ba^{-2},w)] = |w| + 2n(b,w) - 2n(ba^{-1},w) - 2[n(ba^{-1}b^{-1},w) + n(ba^{-2},w)]$.

Lemma 8.3. *Let w be any nontrivial cyclic word over A. Then:*

(1) $|\sigma(w)| = |w| + n(a,w) - 2n(ab^{-1},w)$
(2) $|\sigma\tau^2 w| = |\tau^2 w| + n(a,\tau^2 w) - 2n(ab^{-1},\tau^2 w) = |w| + 2n(b,w) - 2n(ba^{-1},w) - 2[n(ba^{-1}b^{-1},w) + n(ba^{-2},w)] + n(a,w) + 2n(b,w) - 2n(ba^{-1},w) - 2[n(ba^{-1}b^{-1},w) + n(ba^{-2},w)] - 2[n(ba^{-1}b^{-1},w) + n(ba^{-3},w) + n(ba^{-2}b^{-1},w)] = |w| + 4n(b,w) + n(a,w) - 4n(ba^{-1},w) - 6n(ba^{-1}b^{-1},w) - 4n(ba^{-2},w) - 2n(ba^{-3},w) - 2n(ba^{-2}b^{-1},w)$.

Proposition 8.4. *We have $\lambda_A(\phi) = 1 + \frac{5}{k} - \frac{4}{k(2k-1)} - \frac{10}{k(2k-1)^2} - \frac{4}{k(2k-1)^3}$.*

Proof. If w is a long random cyclic word, then the frequency of a fixed reduced word v of length t tends to the uniform frequency $\frac{1}{2k(2k-1)^{t-1}}$. Therefore $n(v,w)/|w|$ tends to $r(t) := \frac{1}{k(2k-1)^{t-1}}$.

Hence

$$\lambda_A(\phi) = 1 + 5r(1) - 4r(2) - 10r(3) - 4r(4) =$$
$$1 + \frac{5}{k} - \frac{4}{k(2k-1)} - \frac{10}{k(2k-1)^2} - \frac{4}{k(2k-1)^3}.$$

□

We now need to perform similar computations for $\phi^{-1} = \tau^{-2}\sigma^{-1}$. Note that $\tau^{-1}(b) = ba^{-1}$, $\sigma^{-1}(a) = ab^{-1}$ and that τ^{-1} and σ^{-1} fix all other letters.

The following two lemmas are again, easy corollaries of the definitions:

Lemma 8.5. *Let w be any nontrivial cyclic word over A. Then:*

(1) $|\tau^{-1}(w)| = |w| + n(b,w) - 2n(ba,w) = |w| + n(a,\tau^{-1}(w)) - n(a,w)$ *and hence* $n(a,\tau^{-1}(w)) = n(a,w) + n(b,w) - 2n(ba,w)$.
(2) $n(ba,\tau^{-1}w) = n(ba^2,w)$
(3) $|\tau^{-2}(w)| = |\tau^{-1}w| + n(b,\tau^{-1}w) - 2n(ba,\tau^{-1}w) = |w| + n(b,w) - 2n(ba,w) + n(b,w) - 2n(ba^2,w) = |w| + 2n(b,w) - 2n(ba,w) - 2n(ba^2,w)$.

Lemma 8.6. *Let w be any nontrivial cyclic word over A. Then:*

(1) $|\sigma^{-1}(w)| = |w| + n(a,w) - 2n(ab,w) = |w| + n(b,\sigma^{-1}(w)) - n(b,w)$ *and hence* $n(b,\sigma^{-1}w) = n(a,w) + n(b,w) - 2n(ab,w)$*;*
(2) $n(ba,\sigma^{-1}w) = n(ba,w) - n(aba,w)$*;*
(3) $n(ba^2,\sigma^{-1}w) = n(w,baba) - n(ababa,w)$*;*

(4) $|\tau^{-2}\sigma^{-1}w| = |w| + 2n(b,\sigma^{-1}w) - 2n(ba,\sigma^{-1}w) - 2n(ba^2,\sigma^{-1}w) = |w| + 2[n(a,w) + n(b,w) - 2n(ab,w)] - 2n(ba,w) + 2n(aba,w) - 2n(w,baba) + 2n(ababa,w)$.

Proposition 8.7. *We have*

$$\lambda_A(\phi^{-1}) = 1 + \frac{4}{k} - \frac{6}{k(2k-1)} + \frac{2}{k(2k-1)^2} - \frac{2}{k(2k-1)^3} + \frac{2}{k(2k-1)^4}.$$

Proof. We will use the same notations as in the proof of Proposition 8.4. Then it follows from the previous lemma that

$$\lambda_A(\phi) = 1 + 4r(1) - 6r(2) + 2r(3) - 2r(4) + 2r(5) =$$
$$1 + \frac{4}{k} - \frac{6}{k(2k-1)} + \frac{2}{k(2k-1)^2} - \frac{2}{k(2k-1)^3} + \frac{2}{k(2k-1)^4}.$$

□

Proposition 8.4 and Proposition 8.7 immediately imply:

Corollary 8.8. *Let F be a finitely generated free group of rank $k \geq 2$. Then for any free basis A of F there exists an automorphism ϕ of F such that $\lambda_A(\phi) \neq \lambda_A(\phi^{-1})$.*

9. Interpreting the intersection form as the distortion of a random geodesic

Proposition 7.3 yields a geometric interpretation of the value of the intersection form $I(\ell, n_A)$ where $\ell \in FLen(F)$ is arbitrary and n_A is the uniform current corresponding to a free basis A of F. This is similar to Bonahon's interpretation of the intersection number between Liouville currents corresponding to two hyperbolic structures on a compact surface as the generic distortion of a long random geodesic in first hyperbolic structure with respect to the second hyperbolic structure.

Proposition 9.1. *Let A be a free basis of F and let $\ell \in FLen(F)$.*
Then

$$I(\ell, n_A) = \lambda_A(\ell).$$

In particular, for an arbitrary $\phi \in Out(F)$

$$I(\phi^{-1}l_A, n_A) = I(\ell_A, \phi n_A) = I(\ell_A, n_{\phi(A)}) = \lambda_A(\phi).$$

Proof. Let $\zeta \in \partial F$ be a m_A-random point. Then by Proposition 7.3 $\lim_{n\to\infty} \frac{\eta_{\zeta_A(n)}}{n} = n_A$. By continuity of I it follows that

$$I(\ell, n_A) = \lim_{n\to\infty} \frac{1}{n} I(\ell, \eta_{\zeta_A(n)}) = \lim_{n\to\infty} \frac{\ell(\zeta_A(n))}{n} = \lambda_A(\ell).$$

□

We should also note that the generic stretching factor $\lambda_A(\ell)$ can be interpreted in terms of the Hausdorff dimension of the measure m_A with respect to the metric on ∂F corresponding to F. We refer the reader to [14, 15] for more details.

Recall that for the hyperbolic surface case Bonahon's notion of an intersection number between two geodesic currents is symmetric. We can now prove that in the free group case such symmetry is essentially impossible.

Theorem 9.2. *Suppose $\hbar : Len(F) \to Curr(F)$ is an $Out(F)$-equivariant map such that for some free basis A of F we have $\hbar l_A = n_A$.*

Then there does not exist a symmetric $Out(F)$-equivariant map

$$\hat{I} : Curr(F) \times Curr(F) \to \mathbb{R}$$

such that for every $\ell \in FLen(F)$ and for every $\nu \in Curr(F)$ we have $\hat{I}(\hbar l, \nu) = I(\ell, \nu)$.

Proof. Suppose, on the contrary, that such a map $\hat{I}$ exists. Recall that for every free basis A of F and for every $\phi \in Aut(F)$ we have $\phi\ell_A = \ell_{\phi(A)}$ and, by Lemma 7.4, we have $\phi n_A = n_{\phi(A)}$. Since by assumption $\hbar l_A = n_A$ for some free basis A, the equivariance of κ implies that $\hbar l_B = n_B$ for every free basis B of F.

Let A be a free basis of F and let $\phi \in Out(F)$. By equivariance and symmetry we have

$$I(\ell_A, \phi n_A) = \hat{I}(\hbar l_A, \phi n_A) = \hat{I}(n_A, \phi n_A) = \hat{I}(\phi^{-1} n_A, n_A) = \text{ by symmetry}$$
$$\hat{I}(n_A, \phi^{-1} n_A) = \hat{I}(\hbar l_A, \phi^{-1} n_A) = I(\ell_A, \phi^{-1} n_A).$$

However, by Proposition 9.1 we have $I(\ell_A, \phi n_A) = \lambda_A(\phi)$ and $I(\ell_A, \phi^{-1} n_A) = \lambda_A(\phi^{-1})$. Hence for every $\phi \in Aut(F)$ we have $\lambda_A(\phi) = \lambda_A(\phi^{-1})$, which contradicts Corollary 8.8. □

10. Finite-dimensional approximations

Notation 10.1. Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart for F. Let $E\Gamma$ denote the set of oriented edges of Γ. If u is an edge-path of positive length in Γ, we denote by $a(u)$ the set of all $e \in E\Gamma$ such that eu is a reduced edge-path in Γ. Similarly, denote by $b(u)$ the set of all $e \in E\Gamma$ such that ue is a reduced edge-path in Γ.

Definition 10.2. Let $m \geq 1$ be an integer. Let $S(m) = S_\Gamma(m)$ denote set of all $v \in \mathcal{P}(\Gamma)$ with $|v| = m$ and let $D(m) = D_\Gamma(m)$ be the number of elements in $S(m)$. We will think of points of $\mathbb{R}^{D(m)}$ as tuples $x = (x_v)_{v \in S(m)}$.

Put

$$R_m = R_m(\Gamma) := \{x = (x_v)_{v\in S(m)} \in \mathbb{R}^{D(m)} : \\ x_v \ge 0 \text{ for each } v \in S(m), \text{ and} \\ \sum_{e\in a(u)} x_{eu} = \sum_{e\in b(u)} x_{ue} \text{ for each } u \in S_\Gamma(m-1).\}$$

For a point $x \in R_m(\Gamma)$ denote $\omega(x) := \sum_{v\in S(m)} x_v$ and call it the *weight* of x.

Put $Q_m = Q_m(\Gamma) := \{x \in R_m(\Gamma) : \omega(x) = 1\}$.

Thus both Q_m and R_m are finite-dimensional convex polyhedra and, in addition, Q_m is compact.

Lemma-Definition 10.3. Define $\pi_m : \mathbb{R}^{D(m)} \to \mathbb{R}^{D(m-1)}$ as follows:

$$\pi_m : (x_v)_{v\in S(m)} \mapsto (x_u)_{u\in S(m-1)}$$

where for every $u \in S(m-1)$ $x_u := \sum_{e\in a(u)} x_{eu}$.

Then:

(1) We have $\pi_m(R_m) = R_{m-1}$ and $\pi_m(Q_m) = Q_{m-1}$.
(2) We have $\omega(\pi_m(x)) = \omega(x)$ for every $x \in R_m$.

Lemma 10.4. *Suppose $\nu \in Curr(F)$. For each $v \in \mathcal{P}(\Gamma)$ denote $x_v := \langle v, \nu\rangle$. Then for each $m \ge 1$ the point $x = (x_v)_{v\in S(m)}$ belongs to R_m and $\omega_\alpha(\nu) = \omega(x)$. In particular, if ν is α-normalized then $x \in Q_m$.*

Proof. It is clear that all $x_v \ge 0$. Let $u \in S(m-1)$ and let $\gamma \in \mathcal{P}(X)$ be a lift of u.

Then

$$Cyl(\gamma) = \sqcup_{e\in a(\gamma)} Cyl(e\gamma) = \sqcup_{e\in b(\gamma)} Cyl(\gamma e)$$

Since ν is finitely-additive,

$$\nu(Cyl(\gamma)) = \sum_{e\in a(\gamma)} \nu(Cyl(e\gamma)) = \sum_{e\in b(\gamma)} \nu(Cyl(\gamma e)),$$

that is $x_u = \sum_{e\in a(u)} x_{eu} = \sum_{e\in b(u)} x_{ue}$.

Since $u \in S(m-1)$ was arbitrary, this means that $x = (x_v)_{v\in S(m)}$ belongs to R_m, as claimed. □

Convention 10.5. We denote by $j_m : Curr(F) \to R_m$ the map that sends each $\nu \in Curr(F)$ to $(\langle v, \nu\rangle)_{v\in S(m)}$. We denote by $\overline{j}_m : \mathbb{P}Curr(F) \to Q_m$ the quotient of the map j_m at the level of projectivizations. That is $\overline{j}_m([\nu]) = j_m(\nu_\alpha)$.

The following is essentially proved in [12]:

Proposition 10.6. *We have canonical homeomorphisms* $Curr(F) \cong \varprojlim(R_m, \pi_m)$ *and* $\mathbb{P}Curr(F) \cong \varprojlim(Q_m, \pi_m)$.

Moreover, the maps $j_m : Curr(F) \to R_m$ *and* $\overline{j}_m : \mathbb{P}Curr(F) \to Q_m$ *are "onto" for* $m \geq 2$.

Because of the above proposition we think of R_m and Q_m as finite-dimensional approximations to $Curr(F)$ and $\mathbb{P}Curr(F)$ respectively.

Notation 10.7. Let $v \in \mathcal{P}(\Gamma)$ be a path of length $m \geq 2$, so that $x \in S(m)$. We denote by $v-$ the initial segment of v of length $m-1$ and we denote by $v+$ the terminal segment of v of length $m-1$. Thus $v-, v+ \in S(m-1)$.

Definition 10.8 (Initial graph)**.** Let $m \geq 2$ and let $x = (x_v)_{v\in S(m)}$ be a point in $R_m(\Gamma)$. The *initial graph* $\Delta(x)$ *of* x is defined as follows: $\Delta(x)$ is a directed labelled graph with $V\Delta(x) := S(m-1)$ and $E\Delta(x) = S(m)$. For an edge $v \in S(m)$ of $\Delta(x)$ the initial vertex of v is $v-$ and the terminal vertex of v is $v+$. The edge v is labelled by the number x_v.

We also define the *improved initial graph* $\Delta'(x)$ *of* x as the union of all edges of $\Delta(x)$ with positive labels, together with the end-vertices of these edges.

For a vertex u of $\Delta(x)$ the sum of the labels on the incoming edges at u is equal to the sum of the labels on the outgoing edges from u. We denote this sum by $d_x(u)$.

Note that for any $x \in S(m)$ we have $\sum_{u\in S(m-1)} d_x(u) = \omega(x) = \sum_{v\in S(m)} x_v$.

If $g \in F$ is a nontrivial element, then for the point $x = \alpha_m(\eta_g)$ all coordinates x_v are integers. Namely, x_v is the number of occurrences of v in the cyclic path $c(g)$ in Γ representing $[g]$. It is natural to ask which points of $R(m)$ with integer coordinates arise in this way. It turns out that one can provide an explicit answer in terms of initial graphs.

Proposition 10.9. *Let* $m \geq 2$ *and let* $x = (x_v)_{v\in S(m)}$ *be a nonzero point in* $R_m(\Gamma)$. *Then there is* $g \in F$ *with* $x = \alpha_m(\eta_g)$ *if and only if all the coordinates* x_v *of* v *are integers and the improved initial graph* $\Delta'(x)$ *of* x *is topologically connected.*

Proof. This statement is essentially proved in [12] (where the problem of which rational points of Q_m are realized as the images of normalized rational currents is considered) and the proof is exactly the same here.

We will sketch the argument for the "if" direction. Suppose all x_v are integers and $\Delta'(x)$ is connected. Let $N = \omega(x)$. Thus $N > 0$ is an integers. If we think of $\Delta'(x)$ as a directed multi-graph, where each edge with a label $n > 0$ is thought of as n multiple edges, then for each vertex u the out-degree is equal to the in-degree at u. Since $\Delta'(x)$ is connected, there exists an *Euler circuit* in $\Delta'(x)$, that is a circuit γ such that for each directed edge v of $\Delta'(x)$ the circuit γ passes through the edge v exactly x_v times. Note that $|\gamma| = N$. We obtain a cyclic path c in Γ from the circuit γ as follows. We replace each edge $v \in S(m)$ in γ by the last edge e of v, when v is considered as a path in Γ. The result is a reduced cyclic path c in Γ and, as is easily seen, for every $v \in S_\Gamma(m)$ the number of occurrences of v in c is equal to x_v. Thus if $g \in F$ is an element represented by c, then $x = \alpha_m(\eta_g)$, as required. □

Proposition 10.10. *Let x be a nonzero point of Q_m for $m \geq 2$. Then x is extremal if and only if the improved initial graph $\Delta'(x)$ is isomorphic to a directed simplicial circle where the labels of all edges are equal.*

In particular, $\Delta'(x)$ is connected for extremal points, and hence all extremal points of Q_m are j_m-images of rational currents.

Proof. Recall that Q_m is a convex finite-dimensional compact polyhedron.

Let x be an extremal point of Q_m.

Choose a subgraph Λ of $\Delta'(x)$ such that Λ is a directed simplicial circle. Then there exists $y \in R_m$ such that $\Delta'(y)$ is Λ where each edge of Λ is given label 1. We claim that x is a scalar multiple of y. Suppose not.

Then by looking at the definition of R_m we see that for a sufficiently small $\epsilon > 0$ the points $x + \epsilon y$ and $x - \epsilon y$ of $\mathbb{R}^{D(m)}$ belong to R_m. Note that $\omega(x) = 1 = \frac{1}{2}(\omega(x+\epsilon y) + \omega(x - \epsilon y))$. Since some edge of $\Delta'(x)$ is not contained in Λ, the points $x \pm \epsilon y$ are not scalar multiples of x. Thus x is a convex linear combination of two points of R_m that are not scalar multiples of x:

$$x = (\omega(x - \epsilon y)/2)\frac{x - \epsilon y}{\omega(x - \epsilon y)} + (\omega(x + \epsilon y)/2)\frac{x + \epsilon y}{\omega(x + \epsilon y)}.$$

It follows that x is a convex linear combination of two points of Q_m different from x, contrary to our assumption that x is an extremal point of Q_m. This proves the "only if" implication of the proposition. We leave the "if" direction to the reader. □

The above proof is due to a UIUC Geometric Group Theory REU student Tyler Smith.

We have seen before that there are natural continuous linear maps $\pi_m : R_m \to R_{m-1}$. It turns out that there are also canonical continuous (but not linear) maps going in the opposite direction. They arise from performing certain kinds of random walks in initial graphs.

Definition 10.11. Let $m \geq 1$ and let $x = (x_v)_{v \in S(m)}$ be a nonzero point in $R_m(\Gamma)$. We consider the following random walk on the initial graph $\Delta(x)$.

The initial distribution $\theta = \theta_x$ on the vertex set $S(m-1)$ of $\Delta(x)$ is given as $\theta(u) := d_x(u)/\omega(x)$.

For two vertices $u, u' \in S(m-1)$ of $\Delta(x)$ the transition probability $\rho(u, u')$ is set to be 0 if there is no directed edge from u to u' in $\Delta(x)$ and it is defined as $\rho_x(u, u') := x_v / d_x(u)$ if there is an edge $v \in S(m)$ from u to u' in $\Delta(x)$.

Note that for an edge $v \in S(m)$ of $\Delta(x)$ the P_θ-probability that the trajectory of the random walk begins with $v-, v+$ is $\theta_x(v-)\rho_x(v-, v+) = d_x(v-)\frac{x_v}{d_x(v-)} = x_v$.

Lemma 10.12. *Let $m \geq 1$ and let $x = (x_v)_{v \in S(m)}$ be a nonzero point in $R_m(\Gamma)$. Consider the initial distribution $\theta = \theta_x$ on the vertex set $S(m-1)$ of $\Delta(x)$ and the random walk on $\Delta(x)$ as in Definition 10.11.*

For each $z \in S_\Gamma(m+1)$ put

$$y_z := x_{z-}\rho_x(z-+, z++) = x_{z-}\frac{x_{z+}}{d_x(z-+)}.$$

Thus, in view of the above remark, y_z is the P_θ-probability that the trajectory of the random walk in $\Delta(x)$ begins with $z--, z-+, z++$, that is, the walk begins by going through the edge-sequence $z-, z+$ of $\Delta(x)$.

Then the point $y = (y_z)_{z \in S_\Gamma(m+1)}$ belongs to $R_m(\Gamma)$ and $\pi_{m+1}(y) = x$.

Proof. Let $v \in S(m)$. We need to verify that $x_v = \sum_{e \in a(v)} y_{ev} = \sum_{h \in b(v)} y_{vh}$.

Then v is an edge from $v-$ to $v+$ in $\Delta(x)$. Let $t_1, \dots, t_l$ be the labels of the incoming edges for the vertex $v-$ of $\Delta(x)$ and let $s_1, \dots, s_n$ be the labels of the outgoing edges from $v+$ in $\Delta(x)$. Thus $t_1 + \cdots + t_l = d_x(v-)$ and $s_1 + \cdots + t_n = d_x(v+)$ since $x \in R_m$.

Then

$$\sum_{e \in a(v)} y_{ev} = \sum_{i=1}^{l} t_i \frac{x_v}{d_x(v-)} = x_v$$

and

$$\sum_{h \in b(v)} y_{vh} = \sum_{i=1}^{n} x_v \frac{s_i}{d_x(v+)} = x_v,$$

as required. □

Definition 10.13. For a point $x \in R_m(\Gamma)$ we denote the point $y \in R_{m+1}(\Gamma)$ defined in Lemma 10.12 by $\iota_m(x)$. Thus, in view of Lemma 10.12 we have a map $\iota_m : R_m(\Gamma) \to R_{m+1}(\Gamma)$ such that $\pi_{m+1} \circ \iota_m = Id_{R_m}$ (that is, ι_m is a section of π_{m+1}).

We list some of the basic properties of ι_m in the following statement whose proof is left to the reader:

Proposition 10.14. *Let $m \geq 1$. Then:*

(a) The map $\iota_m : R_m(\Gamma) \to R_{m+1}(\Gamma)$ is continuous.

(b) Let $x \in R_m(\Gamma)$, $n \geq m$ and let $y = (\iota_n \cdots \circ \iota_m)(x) \in R_{n+1}$. Let $z \in S(n+1)$. Then y_z can be computed as follows. For $i = 0, \dots, n+1-m$ let v_i be the sub-path of z of length m such that $z \equiv z_i v_i z_i'$ where $|z_i| = i$. Thus v_0 is the initial segment of length m of z and v_{n+1-m} is the terminal segment of length m of z. Let $u_i = v_{i-1}+ = v_i-$ for $1 \leq i \leq n+1-m$ and let $u_0 = v_0-, u_{n+2-m} = v_{n+1-m}+$, so that u_i are vertices of $\Delta(x)$.

Then

$$y_z = x_{v_0} \frac{x_{v_1}}{d_x(u_1)} \frac{x_{v_2}}{d_x(u_2)} \cdots \frac{x_{v_{n+1-m}}}{d_x(u_{n+1-m})}.$$

That is y_z is equal to the P_{θ_x}-probability that the trajectory of the random walk in $\Delta(x)$ begins with the vertex sequence

$$u_0, u_1, \dots, u_{n+2-m}.$$

(c) Let $m \geq 1$. For each $x \in R_m(\Gamma)$ and $v \in \mathcal{P}(\Gamma)$ with $|v| = n$ denote:

$$x_v = \begin{cases} (\iota_{n-1} \cdots \circ \iota_m)(x)_v, & \text{if } n > m \\ x_v, & \text{if } n = m \\ (\pi_{n+1} \cdots \circ \pi_m)(x)_v, & \text{if } n < m. \end{cases}$$

Then $\varepsilon_m(x) := (x_v)_{v \in P(\Gamma)} \in \varprojlim(R_n, \pi_n)$ and $\varepsilon_m : R_m(\Gamma) \to \varprojlim(R_n, \pi_n)$ is a topological embedding that provides a section for the map $\alpha_m : \varprojlim(R_n, \pi_n) \to R_m$, so that $\varepsilon_m \circ \iota = Id_{R_m}$.

11. The distortion functional

Definition 11.1. [The distortion functional] Let $\alpha : F \to \pi_1(\Gamma, p)$ be a simplicial chart and let $X = \widetilde{\Gamma}$. We equip X and Γ with simplicial metrics, where every edge has length 1. Let $\ell = \ell_\alpha \in FLen(F)$ be the length function corresponding to the action of F on X via α. Let $\ell' \in FLen(F)$ be arbitrary.

Put

$$\delta(\nu) = \delta_{\ell',\ell}(\nu) := I(\ell', \nu)/I(\ell, \nu)$$

for any nonzero current $\nu \in Curr(F)$.

Note that $\delta(\nu) = \delta(r\nu)$ for any nonzero scalar r. Thus we can define $\delta_{\ell',\ell}([\nu]) := \delta_{\ell',\ell}(\nu)$ for any $\nu \in [\nu]$ so that $\delta_{\ell',\ell}$ is now defined on $\mathbb{P}Curr(F)$.

Recall that $I(\ell', \eta_g) = \ell'(g)$ for every nontrivial $g \in F$. Thus for a nontrivial $g \in F$ we have

$$\delta_{\ell',\ell}(\eta_g) = I(\ell', \eta_g)/I(\ell, \eta_g) = \ell'(g)/\ell(g).$$

Recall also that the α-normalized representative ν_α of $[\nu]$ has the property that $\omega(\nu_\alpha) = I(\ell, \nu_\alpha) = 1$ and hence $\delta(\nu_\alpha) = I(\ell', \nu_\alpha)$. Since $\mathbb{P}Curr(F)$ can be thought of (via normalization) as a convex subset of $Curr(F)$, this implies:

Lemma 11.2. *In the notations of Definition 11.1 we have that δ is a continuous linear functional on $\mathbb{P}Curr(F)$.*

The following is an important corollary of the local formulas established earlier:

Proposition 11.3. *In the notations of Definition 11.1 the maximum and the minimum values of $\delta : \mathbb{P}Curr(F) \to \mathbb{R}$ are achieved and they are realized by rational currents.*

Proof. The local formulas obtained in Theorem 6.2 imply that there exists $m \geq 1$ and a linear functional $\bar\delta : Q_m(\Gamma) \to \mathbb{R}$ such that $\bar\delta(\bar{j}_m(\nu)) = \delta(\nu_\alpha)$ for every nonzero ν. Since $\bar{j}_m$ is "onto" Q_m, it follows that $\sup\delta = \sup\bar\delta$ and $\inf\delta = \inf\bar\delta$. Since Q_m is finite-dimensional convex polyhedron, the linear functional $\bar\delta$ does achieve its extremal values on Q_m and they are attained at extremal points of Q_m. By Proposition 10.10 the extremal points of Q_m have connected improved initial graphs and hence correspond to the $\bar{j}_m$-images of rational currents. Thus the result follows. □

12. Currents and monomorphisms

Recall that the action of $Aut(F)$ on $Curr(F)$ was defined as follows. For any $\phi \in Aut(F)$, for any $\nu \in Curr(F)$ and for any $S \subseteq \partial^2 F$ we have $(\phi\nu)(S) := \nu(\phi^{-1}S)$. We have verified earlier that this indeed defines an action on $Curr(F)$ and that, moreover, by Proposition 5.5, for every nontrivial $g \in F$ we have $\phi(\eta_g) = \eta_{\phi(g)}$.

Suppose F' is another finitely generated free group and that $\phi : F' \to F$ is an injective homomorphism. Then ϕ is a quasi-isometric embedding and hence it defines an equivariant topological embedding $\hat\phi : \partial^2 F' \to \partial^2 F$. It is therefore natural to want to emulate the automorphism case here and to define a map $\phi_* : Curr(F') \to Curr(F)$ in a similar way. Thus in [21] R.Martin claims to define a map $\phi_* : Curr(F') \to Curr(F)$ by the same formula as in the automorphism case: for any $\nu \in Curr(F')$ and for any Borel $S \subseteq \partial^2 F$

$$(\dagger) \qquad (\phi\nu)(S) := \nu(\hat\phi^{-1} S)$$

It is then claimed in [21] (Lemma 13, Section 5.5) that this defines an injective proper map $\phi_* : Curr(F') \to Curr(F)$.

Unfortunately, this approach is quite incorrect, as are the statements of Lemma 13 and Lemma 14 in Section 5.5 of [21]. The problem is that that the measure $\phi\nu$ on $\partial^2 F$ defined by (†), will not, generally, be F-invariant unless the map ϕ is "onto" (that is, unless ϕ is an isomorphism). For example, suppose $F = F' * F''$ and ϕ is the inclusion of F' to F. Let $\nu = \eta_g$ for $g \in F'$. Let $h \in F''$ be a nontrivial element and let $S = \{(g^{-\infty}, g^{\infty})\}$. Then $hS \cap \hat{\phi}(\partial^2 F') = \varnothing$ and $\hat{\phi}^{-1}(hS) = \varnothing$. Thus $(\phi_*\eta_g)(S) = 1$ while $(\phi_*\eta_g)(hS) = 0$, so that $\phi_*\eta_g$ is not F-invariant and so does not belong to $Curr(F)$.

Nevertheless, we show that the following approach works for monomorphisms and that one should generalized Proposition 5.5 rather than the definition of the action of $Aut(F)$ on $Curr(F)$:

Proposition-Definition 12.1 (Maps determined by monomorphisms)**.** Let F', F be finitely generated nonabelain free groups and let $\phi : F' \to F$ be an injective homomorphism.

Then there exists a unique continuous linear map $\phi_* : Curr(F') \to Curr(F)$ such that for every nontrivial $g \in F'$ we have

$$\phi_*(\eta_g) = \eta_{\phi(g)}.$$

Proof. We will sketch the argument and leave the details to the reader.

Note that if such ϕ_* exists then it is unique since rational currents are dense in $Curr(F')$.

Choose a simplicial chart $\alpha : F \to \pi_1(\Gamma, p)$ for F and let $X = \widetilde{\Gamma}$.

Consider the action of F' on X via ϕ.

There exists a unique minimal F'-tree Y such that the quotient graph $\Delta := Y/F'$ provides a simplicial chart β for F' in the obvious way. Note that each edge of Δ is "labelled" by some path in Γ and that this labelling is "folded" in the obvious way.

Hence the same argument as in the proof of Proposition 6.1 implies that for each $v \in \mathcal{P}\Gamma$ there are some integers $c(u) = c(u, v) \geq 0$, where $|u| \leq K$ and $u \in \mathcal{P}(\Delta)$ such that for every cyclic path w' in Δ defining a cyclic path w in Γ we have

$$\langle v, w\rangle_\alpha = \sum_{|u|\leq K, u\in\mathcal{P}\Gamma} c(u, v)\langle u, w'\rangle_\beta.$$

This means that for every nontrivial $g \in F'$ we have

$$\langle v, \phi(g)\rangle_\alpha = \sum_{|u|\leq K, u\in\mathcal{P}\Gamma} c(u, v)\langle u, g\rangle_\beta.$$

We now define a map $\phi_* : Curr(F') \to Curr(F)$ by the following formula: if $\nu \in Curr(F)$ then for every $v \in \mathcal{P}\Gamma$

$$(!) \qquad \langle v, \phi_*\nu\rangle_\alpha := \sum_{|u|\le K, u\in\mathcal{P}\Gamma} c(u,v)\langle u, \nu\rangle_\beta.$$

If these formulas indeed define a geodesic current on F then the map ϕ_* is continuous and has the property that for every nontrivial $g \in F'$ we have $\phi_*(\eta_g) = \eta_{\phi(g)}$.

Thus it remains to check that $\phi_*\nu$ is indeed a geodesic current on F. Put $x_v = \langle v, \phi_*\nu\rangle_\alpha$ for every $v \in \mathcal{P}\Gamma$.

By Lemma 10.6 it suffices to show that the infinite tuple $(x_v)_v$ defines an element of $\varprojlim(R_m(\Gamma), \pi_m)$. That is we need to show that

(1) for every $m \ge 1$ the tuple $x_m := (x_v)_{|v|=m}$ is an element of $R_m(\Gamma)$ and

(2) for every $m \ge 2$ we have $\pi_m(x_m) = x_{m-1}$.

In view of (!) both (1) and (2) reduce to verifying that some explicitly defined continuous linear functions on $Curr(F)$ are equal to each other. Since we do have that $\phi_*(\eta_g) = \eta_{\phi(g)}$ for every nontrivial $g \in F'$, this equalities hold on a dense subset of $Curr(F')$ and hence on the entire $Curr(F')$ as well. □

However, in general one cannot expect the map ϕ_* to be injective since the map from the set of conjugacy classes of F' to the set of conjugacy classes of F induced by ϕ need not be injective. For example, let $F' = F = F(a,b)$ and let $\phi(a) = a, \phi(b) = bab^{-1}$. Then ϕ is injective but $\phi(a)$ is conjugate to $\phi(b)$ and hence $\phi_*(\eta_a) = \phi_*(\eta_b)$.

13. Translation Equivalence

The following notion was introduced and studied in detail by Kapovich, Levitt, Schupp and Shpilrain in [13].

Definition 13.1 (Translation Equivalence of Elements)**.** Elements $g, h \in F$ are said to be *translation equivalent in* F, denoted $g \equiv_t h$, if for every $\ell \in FLen(F)$ we have

$$\ell(g) = \ell(h).$$

In view of the properties of the intersection form I it makes sense to generalize it as follows:

Definition 13.2 (Translation Equivalence of Currents)**.** Currents $\nu_1, \nu_2 \in Curr(F)$ are said to be *translation equivalent in* $Curr(F)$, denoted $\nu_1 \equiv_t \nu_2$, if for every $\ell \in FLen(F)$ we have

$$I(\ell, \nu_1) = I(\ell, \nu_2).$$

Thus for nontrivial elements $g, h \in F$ we have $g \equiv_t h$ in F iff $\eta_g \equiv_t \eta_h$ in $Curr(F)$. The notion of translation equivalence can be thought of as measuring the "degeneracy" of the intersection form I.

The following is proved in [13] (recall that by convention F denotes a free group of finite rank $k \geq 2$.)

Proposition 13.3. *Let $F(a,b)$ be free of rank two nd let $\vartheta : F(a,b) \to F(a,b)$ be the automorphism defined as $\vartheta(a) = a^{-1}$ and $\vartheta(b) = b^{-1}$. Then for every $w \in F(a,b)$ and for any homomorphism $\phi : F(a,b) \to F$ we have $\phi(w) \equiv_t \phi(\vartheta(w))$ in F.*

In other words, for any $w(a,b)$ and for any $g, h \in F$ we have $w(g,h) \equiv_t w(g^{-1}, h^{-1})$ in F. Note that ϑ represents the only nontrivial central element in $Out(F(a,b)$.

Theorem 13.4. *Let $\phi : F(a,b) \to F$ be any injective homomorphism. Let $\vartheta \in Aut(F(a,b))$ be defined as $\vartheta(a) = a^{-1}$ and $\vartheta(b) = b^{-1}$. Then for every $\nu \in Curr(F(a,b)$ we have*

$$\phi_*(\nu) \equiv_t \phi_*(\vartheta\nu) \text{ in } Curr(F).$$

Proof. Let $\nu_n \in Curr(F(a,b))$ be a sequence of rational currents such that $\lim_{n\to\infty} \nu_n = \nu$. Then $\lim_{n\to\infty} \phi_*\nu_n = \phi_*(\nu)$ and $\lim_{n\to\infty} \phi_*\vartheta\nu_n = \phi_*(\vartheta\nu)$. Let $\ell \in FLen(F)$ be arbitrary. By Proposition 13.3 $\phi_*\nu_n \equiv_t \phi_*\vartheta\nu_n$ in $Curr(F)$. Therefore

$$I(\ell, \phi_*(\nu)) = \lim_{n\to\infty} I(\ell, \phi_*(\nu_n)) = \lim_{n\to\infty} I(\ell, \phi_*(\vartheta\nu_n)) = I(\ell, \phi_*(\vartheta\nu)),$$

as required. □

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Ilya Kapovich
Department of Mathematics,
University of Illinois at Urbana-Champaign,
1409 West Green Street, Urbana, IL 61801, USA
http://www.math.uiuc.edu/~kapovich/
E-mail: kapovich@math.uiuc.edu