DYNAMICS OF SOME RATIONAL DISCRETE DYNAMICAL SYSTEMS VIA INVARIANTS

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Abstract. We consider several discrete dynamical systems for which some invariants can be found. Our study includes complex Möbius transformations as well as the third order Lyness recurrence.

1. Introduction

Consider a discrete dynamical system (DDS for short)

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n), \tag{1}$$

where $F : \mathcal{U} \subset \mathbb{K}^m \longrightarrow \mathbb{K}^m$, being $\mathcal{U}$ an open subset, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

We will say that H is a *non–autonomous invariant* for (1) if H is a function defined in $\mathcal{V}$, an open and dense subset of $\mathcal{U}$, valued in $\mathbb{K}$ and satisfying:

$$H(F(\mathbf{x})) = \xi H(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathcal{V},$$

for some non zero $\xi \in \mathbb{K}$, which will be called the *multiplier* of H.

We introduce the above definition motivated from a similar concept used when studying non-autonomous (or time-dependent) first integrals for ordinary differential equations, see for instance [8, Chp. 2].

Notice that when a non-autonomous invariant H has multiplier $\xi = 1$ then we get that H is indeed an *invariant*, also called *first integral* for (1), *i.e.*,

$$H(F(\mathbf{x})) = H(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathcal{V}. \tag{2}$$

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The goal of the paper is to use first integrals and non–autonomous invariants to study the dynamics of several DDS coming from some rational difference equations. These invariants are obtained by using an extension of the method presented in [7], which is summarized in the Appendix. Observe that if H is a non–autonomous invariant for F, then $\xi^{-n}H(F^n(\mathbf{x}_0)) = H(\mathbf{x}_0)$, for all $n \in \mathbb{N}$. In particular, this property can be used to determine some properties of the limiting behaviour of a DDS in terms of the initial condition.

In Section 2, we consider Möbius (or linear fractional) transformations. Although the dynamics associated to these transformations is common knowledge, our approach based on the searching of non-autonomous invariants allows to give a clear interpretation of it. Our point of view also allows us to get some first integrals for them which, as far as we know, is a new result. More concretely, in Theorem 1 we obtain non–autonomous invariants for Möbius transformations. These invariants will be the key for the description of the dynamics, both in the periodic case (Corollary 3) and the non–periodic case (Corollary 4); see also Remark 5. Afterwards, first integrals for Möbius transformations are obtained in Theorem 2 and Corollary 3 (b), see also Figures 1 and 2. The description of the real case is done in Corollary 7. Finally, although it is some how far from the main goal of this paper, in Section 2.4, we apply this last result to describe the dynamics of Ricatti differential equations. We include this subject here because it is known than the maps given by the flow evaluated at time T are Möbius transformations, see Theorem 13, and our results on Möbius transformations give light to a geometrical interpretation of these differential equations.

As a new application of searching of invariants and from the results on the dynamics of real Möbius transformations obtained in Section 2, we get (in Section 3) some new results concerning the dynamics of the real difference equation

$$x_{n+3} = \frac{a + x_{n+2} + x_{n+1}}{x_n},$$

where a is a real parameter (also known as third order Lyness–type equation or Todd's equation) and the initial conditions are in $\mathbb{R}^3$. Our main result proves the existence of continua of initial conditions giving rise to periodic orbits of period 2 , 4 , 5 , 6, 7 and $4p$ for $p > 2$, as well as continua of initial conditions giving rise to dense orbits in $\mathbb{R}$ (see Theorem 12). As far as we know this result was only known for periodic orbits of period 4, see [6, Thm. 4]. We think that our results illustrate how complicated can be the solution of "Open problem 3" of [3, page 1046].

To end this introduction we give a geometrical interpretation of the way we have obtained the proof of Theorem 12 on the third-order Lyness equation. Firstly, by using the well known first integral of the DDS associated to the difference equation,

$$V(x,y,z) = \frac{(x+1)(y+1)(z+1)(a+x+y+z)}{xyz},$$

see [6] or [10], we can reduce our study to each of the level sets of V. Fixing one of these level sets, the dynamics can be studied using a new DDS, which fortunately, has again a first integral. The key point is that on each leaf of the foliation induced by this new first integral the dynamics is given by a real Möbius transformation.

2. Möbius Transformations

2.1. **Invariants for Möbius transformations.** In this section we find non–autonomous invariants and first integrals for the DDS defined by the complex Möbius transformations

$$f(z) = \frac{az+b}{cz+d}.$$

Observe that without loosing generality we can take $c = 1$, giving rise to

$$f(z) = \frac{az+b}{z+d}. \tag{3}$$

If $\Delta := (d-a)^2 + 4b \neq 0$, the Möbius transformation (3) has two different fixed points z_0 and z_1. In this case f can be written in the form

$$f(z) = \frac{(d+z_0+z_1)z - z_0 z_1}{z+d}. \tag{4}$$

If $\Delta = 0$, then (3) has a unique fixed point $z_0 = (a-d)/2$, and it can written as

$$f(z) = \frac{(d+2z_0)z - z_0^2}{z+d}. \tag{5}$$

The main results are the following

Theorem 1. (a) *If $\Delta \neq 0$, then $H_1(z) = (z-z_0)/(z-z_1)$ is a non–autonomous invariant with multiplier $\xi_1 = (d+z_1)/(d+z_0)$ for the DDS defined by (4).*

(b) *If $\Delta = 0$, then $H_2(z) = \exp(1/(z-z_0))$ is a non–autonomous invariant with multiplier $\xi_2 = \exp(1/(d+z_0))$ for the DDS defined by (5).*

Proof of Theorem 1. The proof of both statements (a) and (b) can be obtained just by checking that H_1 and H_2 are indeed non–autonomous invariants. However we prefer to show the way they have been obtained.

(a) In order to try to apply the method of integrability exposed in the Appendix is always useful to try to find simple functions R inducing invariant curves. Consider as candidates $R_i(z) = z - z_i$, $i = 0, 1$. Observe that if f is defined by (4), then $R_0(f(z)) = \dfrac{d+z_1}{z+d} R_0(z)$ and $R_1(f(z)) = \dfrac{d+z_0}{z+d} R_1(z)$. Hence setting $H_1(z) = R_0(z)/R_1(z)$ we obtain

$$H_1(f(z)) = \xi_1 H_1(z). \tag{6}$$

(b) The change $u = z - z_0$, brings f defined by (5) into the normal form

$$f_*(u) = \frac{u}{c_* u + 1},$$

where $c_* = 1/(z_0+d)$. It is well known (see [5, page 267, Exercice 7] that the change $w = 1/u$ conjugates f_* with the translation $g(w) = w + c_*$, which has a non–autonomous invariant $H(w) = e^w$ (since $H(g(w)) = e^{w+c_*} = e^{c_*} e^w = e^{c_*} H(w)$). Therefore $H_2(z) = H(1/(z - z_0)) = \exp(1/(z - z_0))$ is a non–autonomous invariant of the DDS generated by (5). ☐

Theorem 2 (First integrals of Möbius tranformations)**.** (a) *If* $\Delta \neq 0$ *and* $|\xi_1| = 1$*, then* $V_1(z) = |(z - z_0)/(z - z_1)|$ *is a real valued first integral of the DDS associated to (4).*

(b) *If* $\Delta \neq 0$*, and* $|\xi_1| = |(d + z_1)/(d + z_0)| \neq 1$*, then*

$$V_2(z) = \arg\left(\frac{z - z_0}{z - z_1}\right) - \frac{\arg(\xi_1)}{\ln|\xi_1|} \ln\left|\frac{z - z_0}{z - z_1}\right|$$

is a first integral of the DDS associated to (4).

(c) *If* $\Delta = 0$*, then*

$$V_3(z) = \mathrm{Im}\left(\frac{1}{z - z_0}\right) - \frac{\arg(\xi_2)}{\ln|\xi_2|} \mathrm{Re}\left(\frac{1}{z - z_0}\right)$$

where $\xi_2 = \exp(1/(d + z_0))$*, is a first integral of the DDS associated to (5).*

Proof. (a) From equation (6) we obtain that if $|\xi_1| = 1$ then H_1 is a first integral, which proves (a).

Now observe that if $H(z)$ is any non–autonomous invariant with multiplier ξ, then a Möbius transformation f is conjugated (in $\mathbb{C} \setminus \{z_0, z_1\}$) with the translation $g(w) = w + \ln(\xi)$, through $w = \Phi(z) = \ln(H(z))$. Indeed, observe that

$$H(f^n(z)) = \xi^n H(z) = \exp\left(\ln(H(z)) + n \ln(\xi)\right),$$

hence

$$\begin{aligned} f^n(z) &= H^{-1}\left(\exp\left(\ln(H(z))+n\ln(\xi)\right)\right)= \\ &= H^{-1}\left(\exp\left(w+n\ln(\xi)\right)\right)=\Phi^{-1}\left(g^n(w)\right). \end{aligned} \tag{7}$$

Therefore $\Phi^{-1}(w)=H^{-1}(e^w)$ conjugates the translation g with f.

If $|\xi|\neq 1$ is straightforward to check that the linear system g leaves invariant the straight lines given by

$$\mathrm{Im}(w)=\frac{\arg(\xi)}{\ln|\xi|}\mathrm{Re}(w)+C,$$

where C is a constant. Hence $W(w)=\mathrm{Im}(w)-\dfrac{\arg(\xi)}{\ln|\xi|}\mathrm{Re}(w)$ is a first integral for g, and therefore $V(z)=W\left(\Phi(z)\right)$ is a first integral of f.

If $\Delta\neq 0$ and $|\xi_1|\neq 1$, then we take

$$V_2=W\left(\ln\left(H_1(z)\right)\right)=W\left(\ln\left(\frac{z-z_0}{z-z_1}\right)\right),$$

which proves (b). If $\Delta=0$ then we take

$$V_3=W\left(\ln\left(H_2(z)\right)\right)=W\left(\frac{1}{z-z_0}\right),$$

which proves (c). Notice that in this case $|\xi|\neq 1$. □

If $\Delta\neq 0$ and $|\xi_1|=1$, then g leaves invariant the straight lines $\mathrm{Re}(w)=C$. Hence $\mathrm{Re}(\ln(H_1(z)))=\ln|H_1(z)|=\ln|(z-z_0)/(z-z_1)|=C$, and therefore we obtain again that $V_1=|(z-z_0)/(z-z_1)|$ is an invariant of the DDS generated by f, as shown in Theorem 2 (a).

In Figure 1 and 2 there are shown some level curves of V_2 for the map $f(z)=((2+i)z+2i)/(z-2+i)$, and V_3 for the map $f(z)=((5+i)z-2i)/(z+3-i)$.

2.2. Dynamics of the Möbius transformations.

Theorem 1 provides a tool for the study of the dynamics generated by (4) or (5).

Corollary 3 (Periodic solutions)**.**

(a) *Each orbit of the DDS defined by (4) is p–periodic if and only if the number* $\xi_1=(d+z_1)/(d+z_0)$ *is a p–root of the unity. Hence, there are p–periodic Möbius transformations for all* $p\in\mathbb{N}$.

(b) *Each orbit of the DDS defined by (4) is p–periodic if and only if* $V_4(z)=((z-z_0)/(z-z_1))^p$ *is a first integral of (4).*

(c) *There are no periodic orbits for the DDS defined by (5).*

Proof. Observe that each orbit of the DDS generated by (4), or (5) respectively, is p–periodic if and only if

$$H_i(f^p(z)) = \xi_i^p H_i(z) = H_i(z),\ i = 1, 2. \tag{8}$$

Hence all the orbits of the DDS generated by (4) or (5) are p–periodic if and only if $\xi_i^p = 1$, observe that this cannot occur in the case of ξ_2. This proves statements (a) and (c). Now observe $\xi_1^p = 1$ if and only if $V_4(f(z)) = H_1^p(f(z)) = H_1^p(z) = V_4(z)$, hence statement (b) also follows. □

To have a global sight of the dynamics of Möbius transformations it is useful to extend them to the Riemann's Sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ by setting:

$$f(z) = \begin{cases} \dfrac{az+b}{z+d} & \text{if } z \in \mathbb{C} \setminus \{-d\}, \\ \infty & \text{if } z = -d \\ a & \text{if } z = \infty \end{cases}$$

Corollary 4 (Dynamics of the Möbius transformations)**.** *Set* $\Delta = (d-a)^2 + 4b$, $\xi_2 = \exp\left(1/(d+z_0)\right)$ *and* $\xi_1 = (d+z_1)/(d+z_0)$.

(a) *Assuming that* $\Delta \neq 0$*, then*
  (a_1) *If* $|\xi_1| < 1$ *then* z_0 *is a global attractor in* $\mathbb{S}^2 \setminus \{z_1\}$ *of the DDS generated by (4).*
  (a_2) *If* $|\xi_1| > 1$ *then* z_1 *is a global attractor in* $\mathbb{S}^2 \setminus \{z_0\}$ *of the DDS generated by (4).*
  (a_3) *Suppose that* $|\xi_1| = 1$*. Let* $k > 0$*, and set* $\rho := \arg(\xi_1)/(2\pi) \mod (1)$*:*
    (*i*) *If* ξ_1 *is a p–root of unity then* $f_{|\{|H_1(z)|=k\}}$ *is conjugated to a p–periodic rotation with rotation number* ρ*.*
    (*ii*) *If* ξ_1 *is not a p–root of unity then* $f_{|\{|H_1(z)|=k\}}$ *is conjugated to a non–rational rotation with rotation number* ρ*, and* $\{f^n(z)\}_n$ *fills densely the circle* $|H_1(z)| = k$*.*

(b) *If* $\Delta = 0$*, then* z_0 *is a global attractor in* $\mathbb{S}^2$ *of the DDS generated by (5). Furthermore, since* $|\xi_2| \neq 1$*, setting* $z_{n+1} = f(z_n)$ *we have*
  (b_1) *If* $|\xi_2| < 1$ *then* $\lim\limits_{n\to\infty} \mathrm{Re}(z_n) - \mathrm{Re}(z_0) = 0^-$.
  (b_1) *If* $|\xi_2| > 1$ *then* $\lim\limits_{n\to\infty} \mathrm{Re}(z_n) - \mathrm{Re}(z_0) = 0^+$.

Proof. Observe that if H is any non–autonomous invariant of f with multiplier ξ, we have that for every initial condition $z_* \in \mathrm{Dom}(H)$

$$H(f^n(z_*)) = \xi^n H(z_*) \tag{9}$$

(a) Set $z_* \in \mathbb{S}^2 \setminus \{z_1\}$. Applying equation (9) for H_1 in the case $|\xi_1| < 1$, we have $\lim\limits_{n\to\infty} H_1(f^n(z_*)) = 0$. But this occurs if and only if $\lim_{n\to\infty} f^n(z_*) = z_0$, since $H_1(z) = 0$ if and only if $z = z_0$, this proves (a_1). To prove

(a_2) observe that setting $z_* \in \mathbb{S}^2 \setminus \{z_0\}$, and from equation (9) we have $\lim\limits_{n\to\infty} H_1(f^n(z_*)) = \infty$. But this occurs if and only if $\lim_{n\to\infty} f^n(z_*) = z_1$, since $H_1(z) = \infty$ if and only if $z = z_1$, so (a_2) follows.

Observe that the level curves $|H_1(z)| = k$ are circles containing z_0 if $k < 1$, containing z_1 if $k > 1$, and a straight line (splitting one family of circles from the other) if $k = 1$ (see Remark 5 for more details). Hence they all look as closed curves in the Riemann sphere $\mathbb{S}^2$.

Consider $z \neq z_0, z_1$; and observe that under the hypothesis of (a_3),

$$|H_1(f^n(z))| = |\xi^n H_1(z)| = |H_1(z))|,$$

hence $f_{|\{|H_1(z)|=k\}}$ is an application from the circle $|H_1(z)| = k$ of the Riemann sphere $\mathbb{S}^2$ to itself.

Now observe that from equation (9) we have

$$f^n(z) = H_1^{-1}(\xi_1^n H_1(z)). \tag{10}$$

If $|\xi_1| = 1$, then the map defined on the circle of radius k and center at the origin:

$$w \longrightarrow \xi_1 w \tag{11}$$

is a rotation of angle $\arg(\xi_1)$ (thus rotation number given by ρ). Equation (10) evidences that $f_{|\{|H_1(z)|=k\}}$ is conjugated with (11), through the (Möbius transformation) H_1. If ξ_1 is a p–root of the unity then the rotation (11) is p–periodic, so it is f on each level $|H_1(z)| = k$. If ξ_1 is not a p–root of unity then f is conjugated a non–rational rotation that fills densely each level $|H_1(z)| = k$. Therefore (a_3) is proved.

(b) Applying equation (9) for H_2 in the case $|\xi_2| < 1$, we have that $\lim\limits_{n\to\infty} H_2(f^n(z)) = 0$. But this occurs if and only if

$$\lim_{n\to\infty} \operatorname{Re}\left(\frac{1}{z_n - z_0}\right) = -\infty, \tag{12}$$

where $z_n = f^n(z)$. Since

$$\operatorname{Re}\left(\frac{1}{z_n - z_0}\right) = \frac{\operatorname{Re}(z_n) - \operatorname{Re}(z_0)}{|z_n - z_0|^2},$$

equation (12) holds if and only if $\lim\limits_{n\to\infty} z_n = z_0$ and $\lim\limits_{n\to\infty} \operatorname{Re}(z_n) - \operatorname{Re}(z_0) = 0^-$, hence statement (b_1) holds.

To prove statement (b_2) observe that from equation (9), for H_2 in the case $|\xi_2| > 1$, we have $\lim\limits_{n\to\infty} H_2(f^n(z)) = +\infty$. But this occurs if and only if

$$\lim_{n\to\infty} \operatorname{Re}\left(\frac{1}{z_n - z_0}\right) = +\infty.$$

Now, the proof of statement (b_2) follows by the same arguments as above. $\square$

Remark 5. *Setting* $z_0 = a_0 + ib_0$*, and* $z_1 = a_1 + ib_1$*, a straightforward computation shows that if* $k \neq 1$*, the level curves* $|H_1(z)| = k$ *are circles with center at* $\left(\dfrac{k^2a_1 - a_0}{k^2-1}, \dfrac{k^2b_1 - b_0}{k^2-1}\right)$ *(thus containing* z_0 *if* $k < 1$ *and containing* z_1 *if* $k > 1$*) and radius given by* $r^2 = \frac{k^2}{(k^2-1)^2}\left((a_0-a_1)^2 + (b_0-b_1)^2\right)$*, and* $|H_1(z)| = 1$ *is the straight line* $2(a_1-a_0)x + 2(b_1-b_0)y + a_0^2 + b_0^2 - a_1^2 - b_1^2 = 0$*.*

On the other hand, notice that the level curves $\arg(H_1(z)) = c$ *are orthogonal to the above ones because* $h(z) = \ln(H_1(z)) = \ln|H_1(z)| + i\arg(H_1(z))$ *is an analytic function on* $\mathbb{C} \setminus \{z_0, z_1\}$*, and it is well known that in this case the curves* $\mathrm{Re}(h) =$*constant are orthogonal to* $\mathrm{Im}(h) =$*constant.*

Since $H_1(f(z)) = \xi_1 H_1(z) = |\xi_1 H_1(z)| \exp\{i(\arg(H_1(z)) + \arg(\xi_1))\}$*, we get that if* $|\xi_1| \neq 1$ *each iteration of* f *jumps from a circle of level* k *to one of level* $|\xi_1|k$*, and from one level curve* $\arg(H_1(z)) = c$ *to one* $\arg(H_1(f(z))) = c + \arg(\xi_1)$*. The composition of the motion over this two orthogonal foliations of* $\mathbb{C} \setminus \{z_0, z_1\}$ *is done over the level curves of* V_2*, defined in Theorem 2 (b).*

Remark 6. *Notice that the inverse of a Möbius transformation is again a Möbius transformation. Hence the above Theorem can be as well applied to study the DDS generated by the inverse of the maps given in (4) and (5). For instance in the item (b) of Corollary 4 it can be proved that* z_0 *is also a global attractor for the DDS generated by the inverse map. The orbits in backward and forward iteration of any initial condition live on the level curves of the first integral given in Theorem 2; see again Figure 2 for a concrete example.*

2.3. **Real Möbius transformations.**

If both the coefficients and the initial conditions of (3) are real, then the dynamics summarize as follows:

Corollary 7 (Dynamics of the real Möbius transformations)**.** *Consider the transformation*

$$f(x) = \frac{ax+b}{x+d} \quad \text{where} \quad a, b, d, x \in \mathbb{R},$$

let $\Delta = (d-a)^2 + 4b$ *and consider the real iterations of* f*.*

(a) *For* $\Delta > 0$ *set* $\xi = \dfrac{a+d-\sqrt{\Delta}}{a+d+\sqrt{\Delta}}$, $x_0 = \dfrac{a-d+\sqrt{\Delta}}{2}$ *and* $x_1 = \dfrac{a-d-\sqrt{\Delta}}{2}$.

(a_1) *If* $|\xi| < 1$ *then* $\lim_{n\to\infty} f^n(x) = x_0$ *for all* $x \neq x_1$.

(a_2) *If* $|\xi| > 1$ *then* $\lim_{n\to\infty} f^n(x) = x_1$ *for all* $x \neq x_0$.

(a_3) $|\xi| = 1$ *if and only if* $\xi = -1$ *(i.e. when* $a = -d$*). In this case* $f^2(x) = x$ *for all* $x \in \mathbb{R}$, *i.e.,* f *is 2-periodic.*

(b) *For* $\Delta = 0$, *set* $\xi = \exp(2/(a+d))$ *and* $x_0 = (a-d)/2$. *Then,* $\lim_{n\to\infty} f^n(x) = x_0$ *for all* $x \in \mathbb{R}$, *i.e.* x_0 *is a global attractor.*

(c) *For* $\Delta < 0$, *consider* $\xi = \dfrac{a+d-i\sqrt{-\Delta}}{a+d+i\sqrt{-\Delta}}$ *(observe that* $|\xi| = 1$*).*

(c_1) *If* ξ *is a* p*–root of unity for* $p \in \mathbb{N}$, *then* f *is* p*–periodic.*

(c_2) *If* ξ *is not a* p*–root of unity for* $p \in \mathbb{N}$, *then for each* $x \in \mathbb{R}$ *the orbit of* x *is dense in* $\mathbb{R}$.

In fact in the both cases above, if we extend f *to* $\mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ *by defining* $f(-d) = \infty$ *and* $f(\infty) = a$, *then* f *is conjugated to a rotation with rotation number* $\arg(\xi)/(2\pi) \bmod (1)$.

2.4. **An application: dynamics of Ricatti equations.** This section is devoted to apply our results on Möbius transformations to understand the flow of the complex Ricatti equation

$$\frac{dw}{dt} = f_2(t)w^2 + f_1(t)w + f_0(t), \tag{13}$$

where $w \in \mathbb{C}$, $t \in \mathbb{R}$ and $f_j(t)$ are analytic functions, $j = 1, 2, 3$. The following result is well known, see for instance [9, Thm. 4.2.1]:

Theorem 8. *Let* $w(t) := \varphi(t; z)$ *be the solution of (13) that satisfies* $w(0) = z$. *For any* $z \in \mathbb{C}$ *and any* $t > 0$ *in its maximal interval of definition, it is given by*

$$\varphi(t; z) = \frac{A(t)z + B(t)}{C(t)z + D(t)},$$

for some analytic functions A, B, C *and* D.

In order to apply the above theorem to study the solutions of (13) we fix two values of t, for instance 0 and $T > 0$, and we consider the map

$$\Pi : P_0 = \{(z,t) : t = 0\} \longrightarrow P_T = \{(z,t) : t = T\},$$

given by $\Pi(z) = \varphi(z; T)$, wherever it is defined.

For instance, it is clear that when the functions f_j, $j = 1, 2, 3$ are $T-$periodic, we can identify the planes P_0 and P_T and then the fixed points of Π are periodic orbits of the Ricatti equation, the invariant curves of Π give rise to invariant tori for the Ricatti equation, and so on. A well known corollary of the fact that Π is a Möbius transformation is that a $T-$periodic Ricatti equation has either a continuum of $T-$periodic solutions or at most two of such solutions, see [11]. This result follows by studying the quadratic equation $A(T)z + B(T) = z(C(T)z + D(T))$. In order to go further in the dynamics of (13) we need to obtain the values of $A(T), B(T), C(T)$ and $D(T)$ from the functions f_j, $j = 1, 2, 3$. We do not know how to solve this problem in the general situation. In the sequel we will solve it in terms of a fixed particular solution of (13). Next lemma summarizes our results. Notice that in particular it provides a new proof of Theorem 8.

Lemma 9. *Let $w_p(t)$ be a solution of (13) defined in an open interval $I \ni 0$. Let J_z be the maximal interval of definition of the solution of (13), $\varphi(t; z)$. Then for $t \in I \cap J_z$,*

$$\varphi(t; z) = \frac{A(t)z + B(t)}{C(t)z + D(t)},$$

where

$$\begin{array}{ll} A(t) = \alpha(t) + w_p(t)\beta(t), & B(t) = w_p(t) - w_p(0)\alpha(t) - w_p(0)w_p(t)\beta(t) \\ C(t) = \beta(t), & D(t) = 1 - w_p(0)\beta(t), \end{array}$$

and

$$\begin{array}{l} \alpha(t) = \exp\left[\int_0^t 2f_2(s)w_p(s) + f_1(s)\, ds\right], \\ \beta(t) = -\int_0^t f_2(s) \exp\left[\int_0^s 2f_2(u)w_p(u) + f_1(u)\, du\right]\, ds. \end{array}$$

Proof. The proof is straightforward because by introducing the new variable $v(t) = 1/(w(t) - w_p(t))$ the Ricatti equation becomes the linear equation

$$dv/dt = -f_2(t) - [2f_2(t)w_p(t) + f_1(t)]\ v(t),$$

which can be easily integrated. Its solutions are $v(t) = (\beta(t) + k)/\alpha(t)$. Undoing the change of variables we get that $w(t) = w_p(t) + \alpha(t)/(\beta(t) + k)$, $k \in \mathbb{C}$. By imposing the initial condition $w(0) = z$ we obtain that $k = 1/(z - w_p(0))$. Making some computations the result follows. □

The above result can always be applied for instance when the Ricatti equation has the periodic solution $w = 0$ (in this case $f_0(t) \equiv 0$).

In short, by using Corollary 4 and Lemma 9 we can get explicit conditions for $T-$periodic Ricatti equations for which we know a particular solution in order to know the dynamic of the return map Π. In particular it is possible to check whether equation (13) has attractive or repulsive $T-$periodic orbits, or if it has $pT-$periodic solutions, for $p \in \mathbb{N}$, or invariant tori. We present a concrete example in the next result

Proposition 10. *Consider the Ricatti equation*

$$\frac{dw}{dt} = -Ef(t)w^2 + f(t)w, \tag{14}$$

where $0 \neq E \in \mathbb{C}$ *and* f *is a complex valued analytic* $T-$*periodic function. Then the solution starting when* $t=0$ *at* $z \in \mathbb{C}$ *is*

$$\varphi(T;z) = \frac{\alpha z}{E(\alpha-1)z+1}, \tag{15}$$

where $\alpha = e^{\Lambda}$, *with* $\Lambda = \int_0^T f(s)\,ds$. *Furthermore*

- (a_1) *If* $Re(\Lambda) < 0$ *then the periodic orbit* $w=0$ *attracts all the solutions of the Ricatti equation, except the one corresponding to the periodic orbit* $w = 1/E$.
- (a_2) *If* $Re(\Lambda) > 0$ *then the periodic orbit* $w = 1/E$ *attracts all the solutions of the Ricatti equation, except the one corresponding to the periodic orbit* $w=0$.
- (a_3) *If* $Re(\Lambda) = 0$ *and* $Im(\Lambda)/\pi \notin \mathbb{Q}$ *the periodic orbits* $w=0$ *and* $w=1/E$ *are surrounded by invariant tori and in each of these tori the orbits are dense.*
- (a_4) *If* $Re(\Lambda) = 0$ *and* $Im(\Lambda)/(2\pi) = q/p \in \mathbb{Q}$ *the periodic orbits* $w=0$ *and* $w=1/E$ *are surrounded by invariant tori and in each of these tori the orbits are* $pT-$*periodic.*

Proof. The expression given in (15) is a straightforward consequence of Lemma 9. In order to apply Corollary 4 to study the dynamics of this Möbius transformation notice that (15) can be written as (4) with $z_0 = 0$, $z_1 = 1/E$ and $d = z_1/(\alpha-1)$, *i.e.*

$$\varphi(T;z) = \frac{(z_1/(\alpha-1)+z_1)\,z}{z+z_1/(\alpha-1)}.$$

The key number to get the dynamics generated by $\varphi(T;z)$ is $\xi_1 = \dfrac{d+z_1}{d+z_0} = \alpha$. Thus $|\xi_1| = \exp(\mathrm{Re}(\Lambda))$ and the result follows by using the corollary quoted above. □

The above example looks specially simple (indeed it is given by an equation with separable variables) and it could have been also studied with other methods, but we have chosen it for the sake of simplicity. More complicated Ricatti equations (even with $f_0(t) \equiv 0$) for which a particular solution is known can also be studied with our approach.

Remark 11. *An interesting behaviour around a periodic orbit* γ *of a complex periodic Ricatti equation appears when the flow map* $z \to \varphi(T;z)$ *is a Möbius transformation with a double fix point (i.e.* $\Delta = 0$*). In this case the Poincaré Section* P_0 *associated to* γ *has invariant curves like the ones plotted in Figure 2. Each one of these invariant curves (except the one corresponding to a straight line) gives rise to a homoclinic manifold which* *together with γ is a torus. These tori are invariant by the flow of the Ricatti equation and all the orbits in them have γ as α and ω limit.*

3. The third order Lyness-type equation

Consider the third order difference equation

$$x_{n+3} = \frac{a + x_{n+2} + x_{n+1}}{x_n}, \tag{16}$$

where a is a real parameter and the initial condition x_0 , x_1 , x_2 are also real numbers.

It is a well known fact that if $a = 1$, then (16) is an 8–periodic recurrence. It means that for any election of x_0 , x_1 , x_2 the sequence generated by (16)– whenever it is defined– is a cycle of eight points which is indefinite repeated. For $a \neq 1$, the dynamics of (16) is much more complicated. We will prove that for $a \neq 1$ the recurrence has continua of initial conditions giving rise to periodic orbits of periods $2, 4, 5, 6, 7$ and $4p$, for $p > 2$. For some values of a the recurrence also has isolated fixed and 3 periodic orbits. Curiously enough, when $a \neq 1$, the first integer for which we have not found periodic points is precisely[*] 8.

On the other hand we also prove that for $a \neq 1$ the recurrence has continua of initial conditions giving rise to dense orbits in $\mathbb{R}$. All together we have proved the following Theorem:

Theorem 12. *Consider the third order Lyness equation*

$$x_{n+3} = \frac{a + x_{n+2} + x_{n+1}}{x_n},$$

for $a \in \mathbb{R}$ and real initial conditions. Then the following statements hold:

(a) *For any $a \in \mathbb{R}$, there exist a continuum of initial conditions giving rise to periodic orbits of periods 2 and 4,*

(b) *if $a \neq 1$ there exists a continuum of initial conditions giving rise to periodic orbits of periods $5, 6$ and 7.*

(c) *If $a \neq 1$, there exists a continuum of initial conditions giving rise to $4p$–periodic orbits, for all natural $p > 2$.*

(d) *If $a \neq 1$, there exists a continuum of initial conditions such that the sequence generated by the third order Lyness recurrence is dense in $\mathbb{R}$.*

(e) *For $a \geq -1$ there are isolated fixed points and for $a > 1$ there is an isolated periodic orbit of period 3.*

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[*] When one considers the usual Lyness recurrence $x_{n+2} = (a + x_{n+1})/x_n$ it is also true that it is 5-periodic for $a = 1$ and for $a \neq 1$ it can be seen that 5 is never one of its periods.

Let $T : \mathbb{R}^3 \setminus \{x = 0\} \longrightarrow \mathbb{R}^3$ be the DDS associated to (16), defined by

$$T(x, y, z) = \left(y, z, \frac{a + y + z}{x}\right)$$

Notice for $a = 1$ the map T is 8–periodic but it indeed has two fix points, a continua of 2–periodic points and a continua of 4–periodic points. The existence for all a of continua of initial conditions giving rise to periodic orbits of period 4 has been already proved in [6, Thm. 4].

It is known (see [6] or [10], for instance) that the real function

$$V(x, y, z) = \frac{(x + 1)(y + 1)(z + 1)(a + x + y + z)}{xyz}$$

is a first integral of $T(x, y, z)$. Therefore each orbit of T lies on a level surface $\{V = \text{constant}\}$.

The proof of some of the statements of Theorem 12 is obtained studying the dynamics of T on the 0–level surface of V. This dynamics is related with the dynamics of a 2–dimensional map (given below by (17), see Propositions 16 and 17), which already appears in [6]. The key point is that this new map has also a rational first integral (Proposition 13), and the dynamics on each level set of this first integral is determined by the one of a real Möbius transformation. The rest of the statements are obtained by standard arguments.

Clearly $V = 0$ on the four planes of $\mathbb{R}^3$ determined by $\{x + 1 = 0\}$, $\{y + 1 = 0\}$, $\{z + 1 = 0\}$ and $\{a + x + y + z = 0\}$. It is known (see again ([6])) that these planes are mapped one into the other by the map T, and so they are invariant under $T^4(x, y, z)$. In fact, the cyclical order determined by T is:

$$\{x + 1 = 0\} \to \{a + x + y + z = 0\} \to \{z + 1 = 0\} \to \{y + 1 = 0\}.$$

The restriction of T^4 on $\{x + 1 = 0\}$ is given by

$$T^4(-1, y, z) = \left(-1, -\frac{1 + y + z}{z}, \frac{1 + y + (2 - a)z}{z(a + y + z)}\right)$$

and so we can consider the two dimensional map

$$f(y, z) = \left(-\frac{1 + y + z}{z}, \frac{1 + y + (2 - a)z}{z(a + y + z)}\right). \tag{17}$$

By the same manner we could also consider the restrictions of $T^4(x, y, z)$ on the other three cyclical planes. It is clear that the dynamics of each of these restrictions is essentially given by the one of (17). The map T^4 starting on the plane $\{x+1 = 0\}$ writes as $\Psi(-1, y, z) = T^4(-1, y, z) = (-1, f(y, z))$. Similarly, if for instance, we start on points belonging to the plane $\{z + 1 = 0\}$ and we denote by $\Phi(x, y, -1) = T^4(x, y, -1) = (h(x, y), -1)$, for some function h, it holds that $\Psi(T^2) = T^2(\Phi)$, so f and h are almost conjugate. Notice that the problem to get an actual conjugacy is the fact that T^2 is not an invertible map. So, from now on, we restrict our attention to the plane $\{x+1=0\}$, parameterized by (y,z), and the iterations of the map $f(y,z)$ defined in (17).

The proof of the following result is just a computation, but we prefer to show the way in which the first integral has been obtained.

Proposition 13. *The function $H_f(y,z)$ given by*

$$H_f(y,z) = \frac{(1+y+z)(y+a-1)}{z}$$

is a first integral of f. In other words $H_f(f(y,z)) = H_f(y,z)$ for $z \neq 0$, $a+y+z \neq 0$ and $z(1-a)+(1+y+z) \neq 0$.

Proof. To apply the method exposed in the Appendix, we try to obtain a closed set of functions invariant under $f(y,z)$. Consider $R_1 = 1+y+z$ as a candidate to induce an invariant curve (see Remark 14, for the details concerning this election). A computation shows that

$$R_1(f(y,z)) = \left(-\frac{y+a-1}{z\,(a+y+z)}\right)(1+y+z), \tag{18}$$

so it induces an invariant curve. Now, we keep track of one of the new elements introduced in the right hand of expression (18), such as $R_2 = z$, obtaining

$$R_2(f(y,z)) = \left(\frac{1+y+(2-a)z}{(a+y+z)}\right)\frac{1}{z}.$$

Hence R_2 does not induce an invariant curve, but its seems to be a good candidate to be in a close set for f. We continue keeping track of one more element appearing in expression (18), we set $R_3 = y+a-1$, thus we obtain:

$$R_3(f(y,z)) = -\frac{1+y+(2-a)z}{z}.$$

Again R_3 does not induce an invariant curve, but we can observe that $\mathcal{R} = \{1+y+z, z, a+y+z\}$ is closed under $f(y,z)$, and therefore we test

$$H(y,z) = (1+y+z)^{\alpha}(y+a-1)^{\beta}z^{\gamma},$$

as a first integral of f. Indeed

$$\begin{aligned} H(f(y,z)) &= (-1)^{\alpha+\beta}(1+y+z)^{\alpha}(y+a-1)^{\alpha}z^{-(\alpha+\beta+\gamma)}\cdot \\ &\cdot(a+y+z)^{-(\alpha+\gamma)}\,(1+y+(2-a)z)^{\;beta+\gamma} = H(y,z), \end{aligned}$$

if and only if $\alpha = -\gamma$, and $\beta = -\gamma$. Setting $\gamma = -1$ we obtain $H_f(y,z)$. $\square$

Remark 14. *Why starting from $R_1 = 1 + y + z$ in the above proof? From (17) is easy to see that $\{z = -1\}$ is a line of fixed points of f, so to simplify its expression we can take the change $(y, u) = \psi(y, z) = (y, z + 1)$. In this new coordinates f is rewritten as*

$$\tilde{f}(y,u) = \psi \circ f \circ \psi^{-1}(y,u) = \left(\frac{y+u}{1-u}, \frac{u(y+u)}{(1-u)(1-a-y-u)} \right).$$

Its easy to see that $y + u$ is a common factor which induces an invariant curve of $\tilde{f}$, so $y + z + 1$ induces an invariant curve of f.

Since we have a first integral for f, the first step to understand its dynamics is to describe the foliation determined by $\{H_f = k,\ k \in \mathbb{R}\}$. It turns out that there are two critical values of k such that for each of these values the level curves are given by two straight lines. For the other values of k, the level curve are hyperbolae. Let us denote by $H_{f,k}$ the set $\{H_f = k\}$.

Proposition 15. *Let $H_{f,k} = \{(y,z) \in \mathbb{R}^2 : (1 + y + z)(y + a - 1) = kz\}$. Then,*

- (i) *For $k = 0$, $H_{f,k}$ is determined by the two straight lines $1+y+z = 0$ and $y + a - 1 = 0$.*
- (ii) *For $k = a - 2$, $H_{f,k}$ is determined by the two straight lines $y + z - 1 + a = 0$ and $y + 1 = 0$.*
- (iii) *For $k \neq 0$ and $k \neq a - 2$, $H_{f,k}$ is a hyperbola.*

Proof. We only prove (iii). Set $y_0 = k + 1 - a$, $z_0 = a - 2k - 2$ and $B_k = 1 - a - ay_0 - y_0^2$. Then the equation $(1 + y + z)(y + a - 1) = kz$ is equivalent to $u(u + v) = B_k$ where $u = y - y_0$ and $v = z - z_0$. □

From now on we are interested in describe the dynamics on each $H_{f,k}$. We begin by searching the fixed points of f and studying the behavior of f at the critical levels.

Proposition 16. *Consider the map $f(y, z)$ given by (17) and defined on $\mathbb{R}^2 \setminus \{z = 0\,,\ a + y + z = 0\}$ and take $a \neq 1$.*

- (i) *The points $(y, -1)$ with $y \in \mathbb{R} \setminus \{1 - a\}$ and the point $(-1, 2 - a)$ are fixed by f.*
- (ii) *The straight line $1+y+z = 0$ is invariant by f and $f(y, z) = (0, -1)$ for all (y, z) such that $1 + y + z = 0$. The points such that $y = 1 - a$ satisfy $f^2(1 - a, z) = (0, -1)$ for all $z \neq 0$.*
- (iii) *The straight line $y + 1 = 0$ is invariant by f. The restriction of f on $y + 1 = 0$ has two fixed points (resp. one fixed point) for $a \neq 3$ (resp. for $a = 3$), one attractor and the other one repellor (resp. an attractor point). The same happens for the straight line $y + z + a - 1 = 0$.*

Proof. We only prove (iii). On $y+1=0\,,\ z\neq 0:$

$$f(-1,z)=\left(-1,\frac{2-a}{a-1+z}\right),$$

and so we get a 1-dimensional iteration determined by the Möbius transformation:

$$z\longrightarrow\frac{2-a}{a-1+z}.$$

The fixed points are determined by the roots of $z^2+(a-1)z+a-2=0$. We see that for $a\neq 3$ we get the two fixed points $z_0=-1$ and $z_1=2-a$ while for $a=3$ we get only one. In order to apply Corollary 7 in the case $a\neq 3$ we have that

$$\frac{2-a}{a-1+z}=\frac{(d+z_0+z_1)z-z_0z_1}{d+z}$$

and so $d=a-1$ and

$$\xi=\frac{d+z_1}{d+z_0}=\frac{1}{a-2}.$$

Hence, from Corollary 7, we get that $|\xi|>1$ if and only if $a\in(1,3)$ if and only if $\lim_{n\to\infty}f^n(-1,z)=(-1,z_1)$ for all $z\neq z_0$. On the other hand $|\xi|<1$ if and only if $a>3$ or $a<1$ if and only if $\lim_{n\to\infty}f^n(-1,z)=(-1,z_0)$ for all $z\neq z_1$. If $a=3$, from Corollary 7 again, we can assert that the unique fixed point is a global attractor on the line $y+1=0$.

If we consider the line $y+z+a-1=0$ we see that

$$f(y,1-a-y)=(t,1-a-t)\quad\text{where}\quad t=\frac{2-a}{a-1-y},$$

so the dynamics of f in this line is completely determined by the dynamics of the Möbius transformation

$$t=\frac{2-a}{a-1-y}.$$

Hence we get similar results as before. □

Proposition 17. *Consider the map $f(y,z)$ given by (17) and defined on $\mathbb{R}^2\setminus\{z=0\,,\,a+y+z=0\}$ and the sets $H_{f,k}=\{(y,z)\in\mathbb{R}^2:(1+y+z)(y+a-1)=kz\}$ with $k\neq 0\,,\ k\neq a-2$ and $a\neq 1$.*

(i) *If $k<(a-1)^2/4$ then the restriction of f on $H_{f,k}$ has two fixed points, one attractor and the other one repellor.*

(ii) *If $k=(a-1)^2/4$ then the restriction of f on $H_{f,k}$ has a unique fixed point which is globally attractor on $H_{f,k}$.*

(iii) *If $k>(a-1)^2/4$ then there is a countable and dense collection of values of k such that f has a $p-$periodic orbit on $H_{f,k}$ for all natural $p>2$ and for the other values of k the orbit of f is dense on $H_{f,k}$.*

For k a non-critical value, the equality $(1+y+z)(y+a-1)=kz$ can be written as

$$z=h_{a,k}(y) \quad \text{where} \quad h_{a,k}(y)=\frac{(y+1)(1-a-y)}{y+a-1-k} \tag{19}$$

and

$$f(y,h_{a,k}(y))=(t,h_{a,k}(t)) \quad \text{with} \quad t=\frac{k}{1-a-y}.$$

Therefore the dynamics on each $H_{f,k}$ is determined by the Möbius transformation:

$$t=\frac{k}{1-a-y}.$$

The fixed points satisfy the second degree equation $y^2+(a-1)y+k=0$ with discriminant $\Delta=(a-1)^2-4k$. Now, (i) and (ii) follow directly from Corollary 7. In order to see (iii), and using the same notation as in Corollary 7 (c) for $t=k/(1-a-y)$, we obtain

$$\xi_a(k)=\frac{a-1-i\sqrt{-\Delta}}{a-1+i\sqrt{-\Delta}}=\frac{(a-1)^2-2k}{2k}+\frac{(1-a)\sqrt{4k-(a-1)^2}}{2k}\,i,$$

which always has modulus one. A simple computation shows that while k runs from $(a-1)^2/4$ to $+\infty$, the function $\arg(\xi_a(k))$ grows monotonically from 0 to π if $a<1$, and decreases monotonically from 0 to $-\pi$ if $a>1$ (see Figure 3). Therefore $\xi_a(k)$ runs from 1 to -1 (covering the upper-half unit-circle for $a<1$, and the lower-half unit-circle for $a>1$). This means that the map $t=k/(1-a-y)$ (extended to $\mathbb{S}^1\cong\mathbb{R}\cup\{\infty\}$, as explained in Corollary 7 (c)) is a rotation with rotation number $\rho_a(k):=\arg(\xi_a(k))/(2\pi)$. The allowed values for $\rho_a(k)$ are in $[0,1/2)$ if $a<1$, and in $(-1/2,0]$ if $a>1$. Fixed $a<1$, observe that for each $p\in\mathbb{N}\setminus\{2\}$ it exists $q\in\mathbb{N}$ coprime with p such that $q/p\in[0,1/2)$, hence there exist a level $k_{q/p}$ such that the hyperbola $H_f(y,z)=k_{q/p}$ is filled with periodic orbits of f with period p. For the other values of k the orbit of f is dense in the hyperbola, since it is conjugated to a non–rational rotation. Notice that we have not periodic orbits of period $p=2$ because this situation corresponds with the case $k=+\infty$. The case $a>1$ is analogous. □

Notice that we have found a similar scenario to the one appearing in the first quadrant, when the Lyness equation $x_{n+2}=(a+x_{n+1})/x_n$ is considered; see the papers of Zeeman [12], Beukers and Cushman [1] and Bastien and Rogalski [2]. In this case the dynamics on the level sets are "conjugated" to rotations and the rotation number increases (or decreases) when the value of the first integral increases. Fortunately for us, in our case this conjugacy is much easier to be found.

Proof of Theorem 12. (a) For all $a\,, x \in \mathbb{R}$ and $x \neq 0, 1$, we have that

$$T^2\left(x, \frac{x+a}{x-1}, x\right) \;=\; \left(x, \frac{x+a}{x-1}, x\right)$$

and so we get a curve of 2–periodic points. Also

$$T^4(-1, y, -1) \;=\; (-1, y, -1)$$

for all $a, y \in \mathbb{R}\,,\; y \neq 0$.

(b) To study the existence of infinite sequences of period five, a tedious but straightforward computation shows that the curves

$$\left(w_i(y), y, \frac{ayw_i(y) + ay + y^2 + aw_i(y)}{(w_i(y)+1)(yw_i(y) - y - 1)}\right) \tag{20}$$

are invariant under T^5 if $w_i(y)$ is any real solution of the equation

$$yw^2 + (a - 1 + ay + y^2)w + (a-1)(y+1) = 0.$$

Observe that its discriminant Δ can be factorized as

$$\Delta = (y+1)(y^3 + (2a-1)y^2 + (a^2 - 4a + 3)y + (a-1)^2).$$

So, $\Delta = 0$ has the root $y = -1$ and at least another real root. Let $\bar{y}$ be the greatest real root of $y^3 + (2a-1)y^2 + (a^2-4a+3)y + (a-1)^2 = 0$. It is clear that if $y > -1$ and $y > \bar{y}$, then $\Delta > 0$ and the above second degree equation has two roots $w_1(y)\,,\; w_2(y)$ depending on y. Choosing $y > \bar{y}$, and $y > 0$, we obtain that the curves (20) are invariant under T^5, and therefore T is 5–periodic on these curves.

On the other hand, for all $a, x \in \mathbb{R}\,,\; a \neq 1$ and $x \neq 0$

$$T^6\left(x, \frac{a-1}{x}, -\frac{x+a-1}{x}\right) \;=\; \left(x, \frac{a-1}{x}, -\frac{x+a-1}{x}\right)$$

which gives the curve of six periodic points.

Similarly, to study the period seven case, given z, consider the equation

$$(z+a-1)w^2 + (z^2 + (-a^2+5a-3)z - 2 + 2a)w + (a-1)z^2 + (-2+2a)z + a - 1 = 0. \tag{21}$$

Its discriminant

$$\Delta = z^4 + (6a-2-2a^2)z^3 + (27a^2 - 10a^3 + 9 - 26a + a^4)z^2 + (8 - 20a - 4a^3 + 16a^2)z$$

is positive for z big enough and consequently it has two roots $w_1(z)$ and $w_2(z)$ for these values of z. Now consider the curves $(x_i(z), w_i(z), z)$, where $w_i(z)$ is a solution of (21) and $x_i(z) = N(w_i(z), z)/D(w_i(z), z)$, where

$$\begin{aligned} N(y,z) = \quad & -z^3 + \Big(3a^2 + 3 - a^3 - 6a + (-2 - a^2 + 2a)y\Big)z^2 \\ + \quad & \Big((-15a + a^4 + 15a^2 + 6 - 7a^3)y - 3a^3 + 3 + 7a^2 - 7a\Big)z \\ + \quad & (a^4 - 13a + 4 + 15a^2 - 7a^3)y + 9a^2 + 3 - 9a - 3a^3, \end{aligned}$$

and

$$D(y,z) = \Big(z^2 + (-2 + 3a + ya)z - 1 + 2ya + a^2 - 2y\Big)(z + a - 1).$$

It is clear that for z big enough they are well defined and a tedious computation, done with an algebraic manipulator, proves that

$$T^7(x_i(z), w_i(z), z) = (x_i(z), w_i(z), z).$$

Thus the proof of (b) is finished.

(c) The statement concerning periodicity is consequence of Proposition 16 (*iii*). In fact, take a natural $p > 2$ and consider the leaves $H_{f,k}$ with $k > (a-1)^2)/4$ and such that f restricted to $H_{f,k}$ is p–periodic. We note that there are a finite number of leaves where the dynamic is conjugated– after extending the dynamics to infinity– to a rotation of angle $2k\pi/p$ (in fact, there are as many of such leaves as $\varphi(p)/2$, where φ is the Euler function). Moreover, on these leaves all the points are periodic of period p. So, for all $(x_1, x_2) \in H_{f,k}$,

$$f^p(x_1, x_2) = (x_1, x_2)$$

or equivalently

$$T^{4p}(-1, x_1, x_2) = (-1, f(x_1, x_2)) = (-1, x_1, x_2).$$

Hence the sequence $-1, x_1,\ x_2, x_3, \ldots$ is $4p$–periodic. Notice that the sequence $-1, x_1,\ x_2, x_3, \ldots$ generated by the third order Lyness equation satisfies that $x_{4k} = -1$ for all k. Moreover the subsequences $\{x_{4k+1}\}_k$, $\{x_{4k+2}\}_k$ and $\{x_{4k+3}\}_k$ are $4p$–periodic. It is because

$$\begin{aligned} T^{4p}(x_{4k+1}, x_{4k+2}, x_{4k+3}) \quad & = T^{4p}(T(-1, x_{4k+1}, x_{4k+2})) = \\ & = T(T^{4p}(-1, x_{4k+1}, x_{4k+2})) = \\ & = T(-1, x_{4k+1}, x_{4k+2}) = (x_{4k+1}, x_{4k+2}, x_{4k+3}). \end{aligned}$$

To see (d) consider $x_0 = -1$, x_1, x_2 arbitrary numbers and the sequence $-1, x_1$, x_2, x_3, $\ldots$, generated by the third order Lyness recurrence, as above. Then, again

$$T^4(-1, x_1, x_2) = (-1, f(x_1, x_2)).$$

Hence $x_{4k} = -1$ for all $k \in \mathbb{N}$ and $f(x_1, x_2) = (x_5, x_6)$, $f(x_5, x_6) = (x_9, x_{10})$ and so on. Now from (*iii*) of Proposition 16 and taking $k > \frac{(a-1)^2}{4}$, there is a non-countable collection of such a values k such that the orbit of f is dense in the corresponding hyperbola $H_{f,k}$. Then the projections of each one of the points $(-1, x_{4k+1}, x_{4k+2})$, contained in the plane $\{x = -1\}$, on the y-axis is dense in it. In particular the subsequence x_{4k+1} is dense in $\mathbb{R}$ and the result follows.

(e) It is easy to check that the fixed points for T are given by $(1 \pm \sqrt{1+a}, 1 \pm \sqrt{1+a}, 1 \pm \sqrt{1+a})$ (for $a = 0$ only the plus sign gives an actual fixed point). Similarly the points of period 3 are $(\pm\sqrt{a-1}, \pm\sqrt{a-1}, -1 \mp \sqrt{a-1})$ and its images by T and T^2. Thus the result follows. $\square$

From the function $\arg(\xi_a(k))$, presented in the proof of Proposition 17 we also can extract which are the curves on the phase portrait of f containing some prescribed behaviour. In fact given an admissible rotation number ρ we can find its prescribed dynamics in the hyperbolae $\left(y, \frac{(y+1)(1-a-y)}{y+a-1-k}\right)$, where $k = (a-1)^2/2$ if $\rho = \pm 1/4$ or

$$k = (a-1)^2 \left[\frac{1 + \tan^2(2\pi\rho) + \operatorname{sgn}(a,\rho)\sqrt{1+\tan^2(2\pi\rho)}}{2\tan^2(2\pi\rho)} \right], \text{ if } \rho \neq \pm 1/4,$$

where $\operatorname{sgn}(a,\rho) = \operatorname{sign}(\rho - 1/4)$ when $a < 1$ and $\operatorname{sgn}(a,\rho) = -\operatorname{sign}(\rho + 1/4)$ when $a > 1$. In next remark we present some applications of the above facts to study the dynamics of the third order Lyness recurrence.

Remark 18. (Some examples of periodic and non-periodic behaviour)

(a) In Figure 4 there is a detail of the curves $H_f = k$ *filled with* 15*−periodic points for* f*, when* $a = 1/2$. *Notice that for (minimal) period* $p = 15$*, the admissible values for* $\rho_{1/2}$ *are* $j/15$ *for* $j \in \{1, 2, 4, 7\}$. *In Figure 5 there is a piece of the hyperbolae corresponding to* $\rho_{1/2} = 4/15$*, and a periodic orbit over it. Indeed this initial condition corresponds to* $(1, h_{1/2,k^*}(1))$*, where*

$$k^* = \frac{1 + \tan^2(8\pi/15) + \sqrt{1 + \tan^2(8\pi/15)}}{2\tan^2(8\pi/15)} \simeq -2.7746...,$$

and h *is given in (19). Thus the initial condition* $x_0 = -1, x_1 = 1$ *and* $x_2 = k^*$ *gives rise to a 60-periodic orbit for the third order Lyness equation (16). In Figure 6 we plot the first 120 points of this recurrence (2 complete periods). Notice that, apart from the expected property* $x_{4k} = -1$ *for all* k*, within a period the orbit does not seem to exhibit any special feature. On the other hand if we restrict our attention to the subsequence defined by* $t_k = x_{4k+1}$*, as expected it has a very regular structure; namely, the points are distributed in the real axis as*

$$t_{33} < t_{49} < t_5 < t_{21} < t_{37} < t_{53} < t_9 < t_{25} <$$
$$< t_{41} < t_{57} < t_{13} < t_{29} < t_{45} < t_1 = t_{61} < t_{17}.$$

Notice that the effect of advancing 4 points in the recurrence (i.e. of applying $f = T^4$*) corresponds to jump 4 values among the 15 values of the set* $\{t_k\}_k$*, thinking these values on a circle by identifying infinity to a point. This jump of 4 values corresponds to the fact that* $\rho = 4/15$*. Something similar happens with the subsequences* $\{x_{4k+2}\}_k$ *and* $\{x_{4k+3}\}_k$*.*

(b) We can plot the hyperbolae corresponding to initial conditions giving rise to periodic orbits of period smaller than some value, say for instance 8. When $a < 1$*, these periodic orbits have rotations numbers which can be ordered as the first half of the Farey sequence associated to 7, i.e.* $\frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{2}{7} < \frac{1}{3} < \frac{2}{5} < \frac{3}{7}$*, and the corresponding hyperbolae follow the same ordering, see Figure 7. In that figure, the points at the intersection of all the curves are points for which the iteration is not well defined.*

(c) Let us see how complicated can look a orbit of the third order Lyness recurrence which is dense in $\mathbb{R}$*. We choose a irrational number very "far" from the rational numbers,* $\rho = (\sqrt{5}-1)/2 = [0,1,1,1,\ldots,]$ *(i.e. the golden mean minus 1) and again* $a = 1/2$*. Like in (a), these values give rise to an initial condition which dynamics for* f *is conjugated to a rotation –in this case an irrational rotation. Concretely we choose the initial conditions* $x_0 = -1, x_1 = 1$ *and* $x_2 \simeq -41.58476446...$ *For these initial conditions we plot in Figure 8 the points* $\{x_k\}_{k=11,\ldots,120}$*. We have stopped there because* $x_{129} \simeq 76.4472...$ *is much bigger than the previous values and makes the picture useless. We also remark that for instance* $x_{358} \simeq -1818.8835...$

4. Appendix: Finding invariants for DDS

In this section we briefly describe a constructive method for finding non-autonomous invariants based on the method developed in [7], which has been used through this paper.

Set $F : \mathcal{U} \subseteq \mathbb{K}^n \to \mathbb{K}^n$, where $\mathcal{U}$ is an open set. We say that a set of functions $\mathcal{R} = \{R_i\}_{i\in\{1,\ldots,m\}}$ is *closed under* F if for all $i \in \{1,\ldots,m\}$, there exist functions K_i and constants $\alpha_{i,j}$, such that

$$R_i(F) = K_i\left(\prod_{j=1}^{m} R_j^{\alpha_{i,j}}\right), \tag{22}$$

with $\prod_{j=1}^{m} R_j^{\alpha_{i,j}} \neq 1$. We call each function K_i the cofactor[†] of R_i.

Notice that it is very easy to construct "artificial" sets of closed functions. For instance, any function R can be seen as a closed set just by taking $K = R(F)/R^{\alpha}$. It is intuitively clear that this kind of choices can not give interesting information about the dynamics of F. Thus it seems natural to

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[†] Notice that this is an extension of the usual definition of cofactor. Usually, a cofactor is a function $\tilde{K}_i$ such that $R_i(F) = \tilde{K}_i R_i$.

introduce what can be called as *a natural closed set of functions*. Since in our paper we deal with rational maps, we will restrict to this situation. So, for F a rational map, we will say that a set of rational functions $\mathcal{R}$ is *a natural closed set* for F if each of the cofactors K_i appearing in (22) does not contain any power of functions of $\mathcal{R}$ as a factor. Although it is not clear for us how to introduce the concept of natural close set in a bigger class of functions, the next theorem holds even for the continuous class:

Proposition 19. *Let* $\mathbf{x}_{n+1} = F(\mathbf{x}_n)$, *be a discrete dynamic system defined in an open set* $\mathcal{U} \subset \mathbb{K}^n$. *Let* $R_i : \mathcal{V} \longrightarrow \mathbb{K}$, $i = 1, 2, \cdots, m$ *(where* $\mathcal{V}$ *is an open and dense subset of* $\mathcal{U}$*), such that* $\{R_i\}_{i\in\{1,\ldots,m\}}$ *is closed under* F, *and* $\alpha_{i,j}$ *and* K_i *are defined as in (22). If there exists* $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{K}$ *such that*

(a) $\prod_{i=1}^m K_i(\mathbf{x})^{\beta_i} = \xi \in \mathbb{K}$, *and*

(b) *these values are also a solution of the linear system* $\sum_{i=1}^m \beta_i \alpha_{i,j} = \beta_j$, $j = 1, \ldots, m$,

then

$$H(\mathbf{x}) := \prod_{i=1}^{m} R_i(\mathbf{x})^{\beta_i},$$

is a non-autonomous invariant of the DDS associated to F, *with multiplier* ξ *and defined in* $\mathcal{V}$.

Hypothesis (b) automatically holds if $\{R_i\}_{i\in\{1,\ldots,m\}}$ only contains function inducing invariant curves or exponential factors for F, that is functions satisfying equation $R_i(F(\mathbf{x})) = K_i(\mathbf{x})R_i(\mathbf{x})$, for all $\mathbf{x} \in \mathcal{V}$.

Proof. Observe that from hypotheses (a) and (b):

$$\begin{aligned} H(F) = {} & \prod_{i=1}^m R_i(F)^{\beta_i} = \left(\prod_{i=1}^m K_i^{\beta_i}\right) \left[\prod_{i=1}^m \left(\prod_{j=1}^m R_j^{\alpha_{i,j}}\right)^{\beta_i}\right] = \\ = {} & \xi \prod_{j=1}^m R_j^{\sum_{i=1}^m \beta_i \alpha_{i,j}} = \xi \prod_{j=1}^m R_j^{\beta_j} = \xi H. \end{aligned}$$

□

This result provides the following constructive procedure for finding non-autonomous invariants or first integrals : If $\mathcal{R} = \{R_i\}_{i\in\{1,\ldots,m\}}$ is *closed under* F, then a natural candidate is:

$$H(\mathbf{x}) = \prod_{i=1}^{m} R_i(\mathbf{x})^{\beta}_{i},$$

where $\beta_i \in \mathbb{K}$ and ξ have to be determined by imposing $H(F(\mathbf{x})) = \xi H(\mathbf{x})$.

It is not easy to find useful closed sets of functions. Of course these closed sets have to be chosen among the natural sets. As it is proposed in [7] we will search the possible functions R_i looking for functions inducing invariant curves and exponential factors. In all the cases in this paper this is enough, but in general it is also necessary to look for functions inducing invariant curves for F^k, and for the functions defining the *singularities* of F and its pre-images (we say that a set $\mathcal{M}$ is a *singularity* of F if $F(\mathbf{x})$ is not defined for all $\mathbf{x} \in \mathcal{M}$). We refer the reader to the proof of Proposition 13 for an explicit example of how the method works for searching a *natural* set of closed functions.

5. Figures

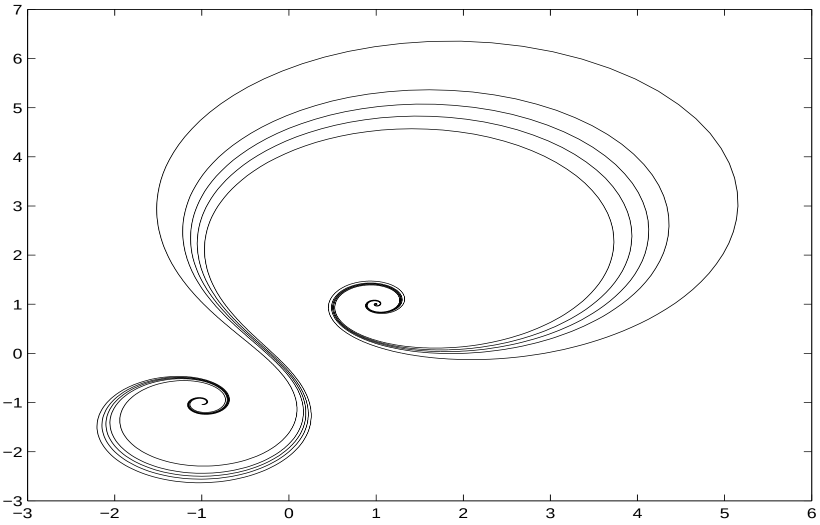

Figure 1. Some level curves of the first integral V_2 defined in Theorem 2, for the map $f(z) = ((2+i)z+2i)/(z-2+i)$.

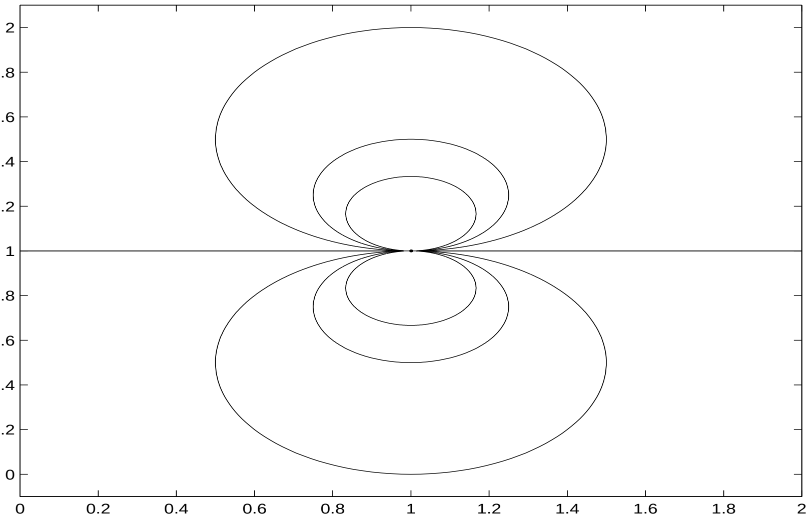

FIGURE 2. Some level curves of the first integral V_3 defined in Theorem 2, for the map $f(z) = ((5+i)z-2i)/(z+3-i)$.

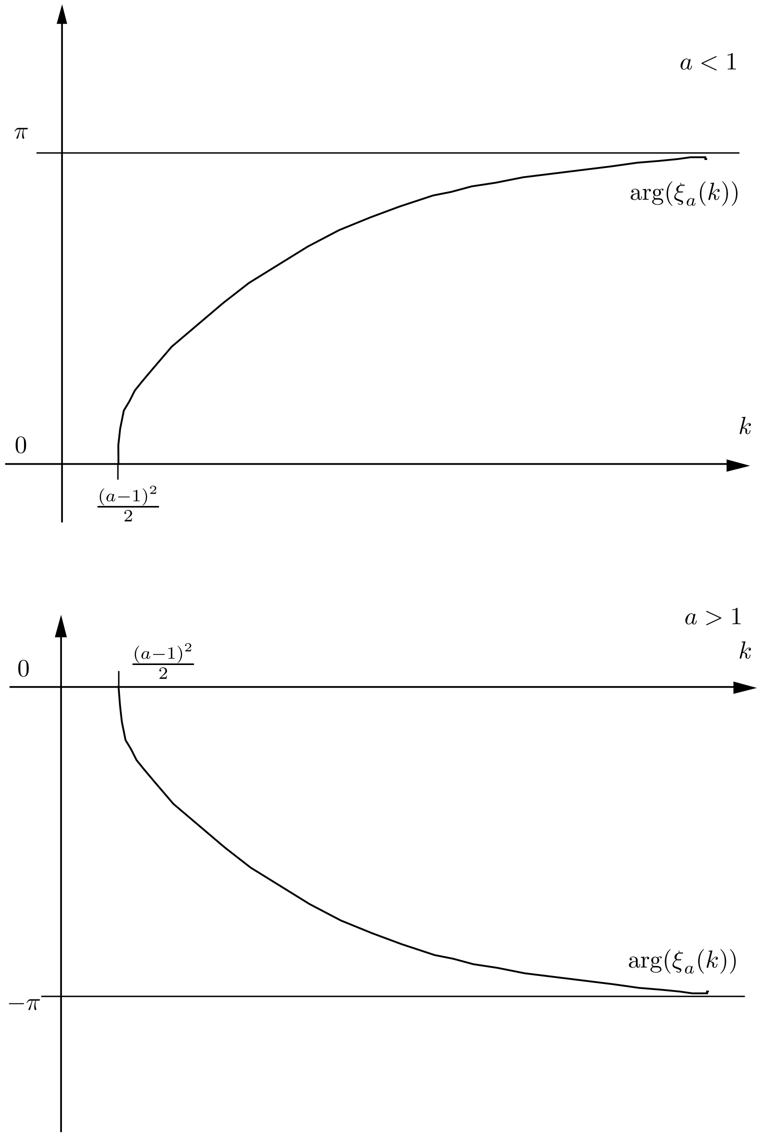

FIGURE 3. Graphic of the function $\arg(\xi_a(k))$ appearing in the proof of Proposition 17.

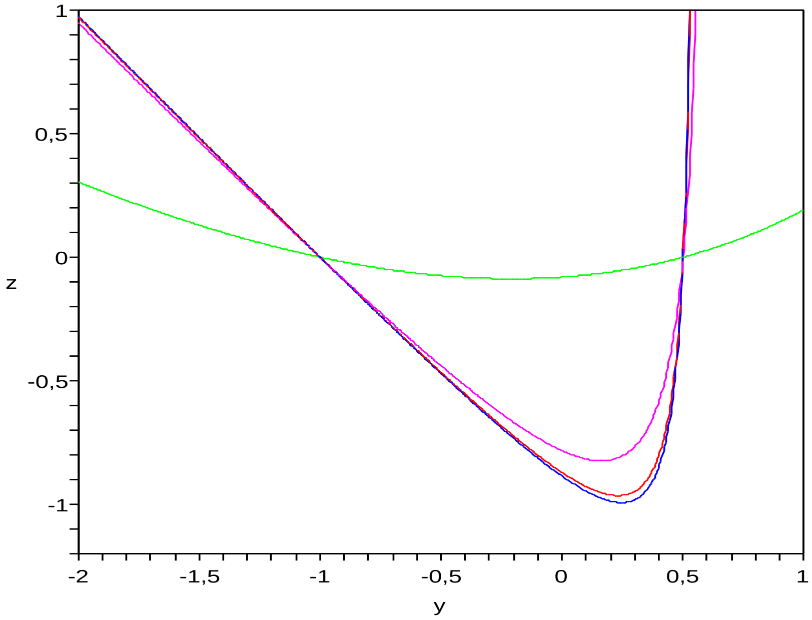

Figure 4. Detail of the curves $H_f = k$ filled with 15–periodic points for f, when $a = 1/2$.

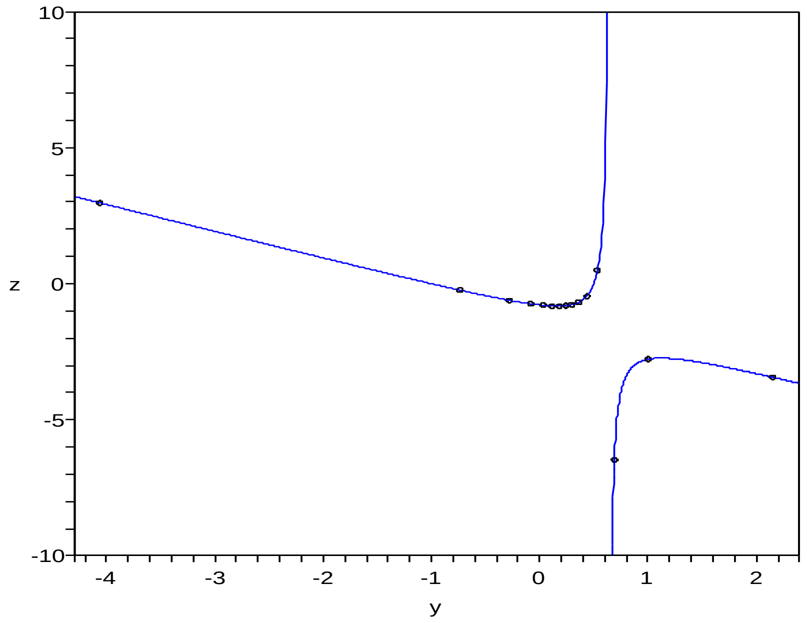

Figure 5. Detail of one of the hyperbolae corresponding to $\rho_{1/2} = 4/15$, and a periodic orbit of f over it.

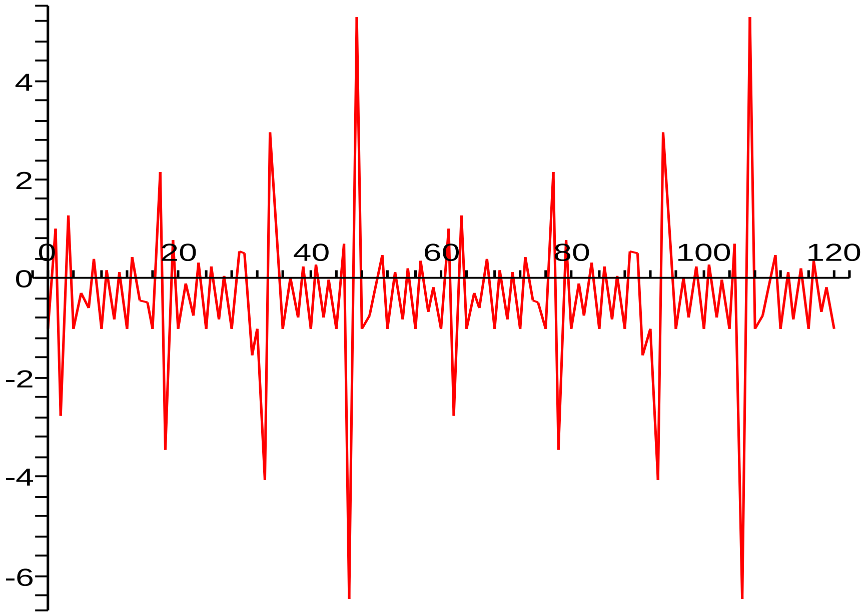

Figure 6. The first 120 iterates of a 60-periodic orbit for the third order Lyness recurrence, corresponding to the 15-periodic orbit of f given in Figure 5

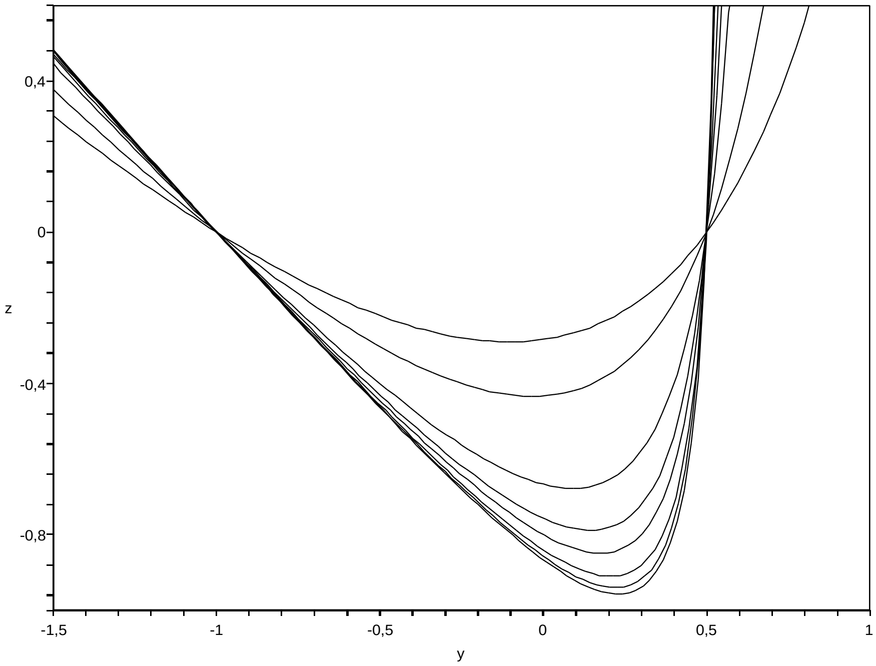

FIGURE 7. Detail of the initial conditions corresponding to periodic points of f of period smaller than 8. The curves correspond to rotation numbers $\frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{2}{7} < \frac{1}{3} < \frac{2}{5} < \frac{3}{7}$, from bellow to the top. The points at the intersection of all the curves are points for which the iteration is not well defined

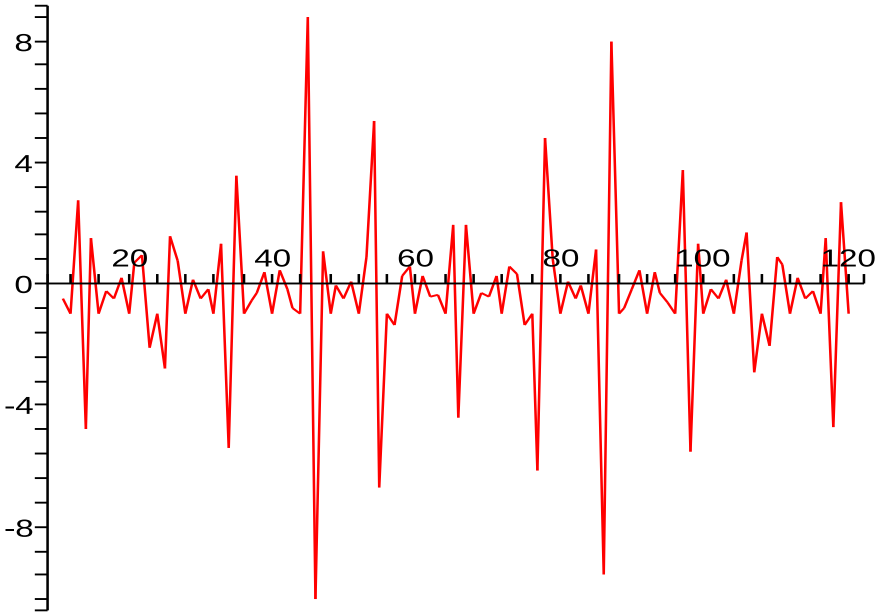

FIGURE 8. The iterates 11th to 120th for a dense orbit of the third order Lyness recurrence

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