THERMAL STRESSES IN MICROSTRETCH ELASTIC PLATES

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Abstract. We consider the linear theory of homogeneous and isotropic thermo-micro- stretch elastic solids. First, we present the basic equations which characterize the bending of thin plates. Then we establish a uniqueness result with no definiteness assumption on constitutive coefficients. Existence of solutions is proved under assumption that the internal energy density is positive definite. Finally, the asymptotic behavior is analyzed.

The theory of microstretch elastic solids has been introduced by Eringen [1-3]. This theory is a special case of the micromorphic theory. In the framework of micromorphic theory a material point is endowed with three deformable directors. When the directors are constrained to have only breathing-type microdeformations, then the body is a microstretch continuum [3]. The material points of this continua can stretch and contract independently of their translations and rotations. The theory is expected to find applications in the treatment of the mechanics of composite materials reinforced with chopped fibers and various porous materials. The theory of microstrech continua is a generalization of the theory of micropolar continua. In [4], Eringen presented a micropolar plate theory and established basic theorems in the context of isothermal case. These results have been used to investigate the isothermal bending of microstretch elastic plates [5]. A detailed discussion of the theory of micropolar plates was presented in [3,4].

In [2], Eringen has extended the theory of microstretch elastic solids to include the heat conduction. In this paper we use the results of Eringen [2,4], Nowacki [6] and Inan [7] to derive a theory of bending of thermo-microstretch elastic thin plates. In Section 2 we present the basic equations of the linear theory of thermo-microstretch elastic solids. The Section 3 is devoted to the equations of the bending theory of thermoelastic thin plates. In Section 4 we establish a uniqueness result with no definiteness assumption on constitutive coefficients. In Section 5 we prove an existence result under assumption that the internal energy density is positive definite. The asymptotic behavior is studied in the last section.

BASIC EQUATIONS

We refer the motion of the continuum to a fixed system of rectangular Cartesian axes $Ox_k (k = 1, 2, 3)$. We consider a body that at time t_0 occupies the regular region B of Euclidean three-dimensional space and is bounded by the surface ∂B. We designate by $\mathbf{n}$ the outward unit normal of ∂B. Letters in boldface stand for tensor of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots s}$(p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate system. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

We assume that B is occupied by a homogeneous and isotropic thermo-microstretch elastic solid. We denote by u_i the components of the displacement vector and by φ_i the components of the microrotation vector. Let ψ be the microstretch function. The strain measures in the context of the linear theory are defined by

$$\varepsilon_{ij} = u_{j,i} + \varepsilon_{jir}\varphi_r, \quad \kappa_{ij} = \varphi_{j,i}, \quad \gamma_i = \psi_{,i}, \tag{1}$$

where ε_{ijk} is the alternating symbol.

Let t_{ij} be the stress tensor and let m_{ij} be the couple stress tensor over B. We denote by σ_i the microstress vector. The equations of motion of a microstretch continua can be expressed as

$$\begin{aligned} &t_{ji,j} + f_i = \rho\ddot{u}_i, \\ &m_{ji,j} + \varepsilon_{irs}t_{rs} + g_i = J\ddot{\varphi}_i, \\ &\sigma_{i,i} - \tau + \ell = j\ddot{\psi}, \end{aligned} \tag{2}$$

where $\mathbf{f}, \mathbf{g}$ and ℓ are body loads, ρ is the reference mass density, j and J are coefficients of inertia $(j = 3J/2)$ and τ is the microstress function. The energy equation is given by

$$\rho T_0\dot{\eta} = q_{i,i} + s, \tag{3}$$

where η is entropy, $\mathbf{q}$ is the heat flux vector, s is the heat supply, and T_0 is the constant absolute temperature of the body in the reference configuration.

In the context of the linear theory of isotropic and homogeneous elastic bodies the constitutive equations have the form

$$
\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu+\kappa)e_{ij} + \mu e_{ji} + \lambda_0\psi\delta_{ij} - \beta_0\theta\delta_{ij},\\
m_{ij} &= \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + b_0\varepsilon_{sji}\psi_{,s},\\
\sigma_i &= a_0\psi_{,i} + b_0\varepsilon_{irs}\kappa_{sr},\\
\tau &= \lambda_0 e_{rr} + \lambda_1\psi - \beta_1\theta,\\
\rho\eta &= \beta_0 e_{rr} + a\theta + \beta_1\psi,\\
q_i &= k\theta_{,i},
\end{aligned}
\tag{4}
$$

where θ is the temperature measured from the constant temperature T_0, and $\lambda,\mu,\kappa,\alpha,\beta$,
$\gamma, a, a_0, \beta_0, \beta_1, \lambda_0, \lambda_1, b_0$ and k are constitutive constants.

The components of surface traction, the components of surface couple, the microsurface traction and the heat flux at regular points of ∂B are defined by

$$
t_i = t_{ji}n_j,\quad m_i = m_{ji}n_j,\quad p = \sigma_i n_i,\quad q = q_i n_i,
\tag{5}
$$

respectively.

Let M and N be non-negative integers and $\mathcal{T}$ a given interval of time. We say that f is of class $C^{M,N}$ on $B\times\mathcal{T}$ if f is continuous on $B\times\mathcal{T}$ and the functions

$$
\frac{\partial^m}{\partial x_i\partial x_j\ldots\partial x_s}\left(\frac{\partial^n f}{\partial t^n}\right),\quad m\in\{0,1,2,\ldots,M\}, n\in\{0,1,2,,\ldots,N\},
$$
$$
m+n\le\max\{M,N\},
$$

exist and are continuous on $B\times\mathcal{T}$. We write C^M for $C^{M,M}$.

Let $\mathcal{P}=(u_i,\varphi_i,\psi,\theta)$. We say that $\mathcal{P}$ is an admissible process on $B\times\mathcal{T}$ provided: (i) u_s,φ_s and ψ are of class C^2 on $B\times\mathcal{T}$; (ii) θ is of class $C^{2,1}$ on $B\times\mathcal{T}$; (iii) u_i,φ_i,ψ and θ are of class C^1 on $\overline{B}\times\mathcal{T}$.

We assume that $\mathbf{f},\mathbf{g},\ell$ and s are continuous on $B\times\mathcal{T}$ and that ρ and J are positive constants.

THERMOELASTIC PLATES

In what follows we assume that the region B refers to the interior of a right cylinder of length $2h$ with open cross-section Σ and the smooth lateral boundary Π. Let Γ be the boundary of Σ. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the plane x_1Ox_2 is middle plane. Thus, we have

$$
B=\{\mathbf{x}:(x_1,x_2)\in\Sigma,-h<x_3<h\},\quad \Pi=\{\mathbf{x}:(x_1,x_2)\in\Gamma,-h<x_3<h\}.
$$

An admissible process $\mathcal{P} = (u_i, \varphi_i, \psi, \theta)$ is a *state of bending* on $B \times \mathcal{T}$ provided [4]

$$
\begin{aligned}
&u_\alpha(x_\rho, x_3, t) = -u_\alpha(x_\rho, -x_3, t), \quad && u_3(x_\rho, x_3, t) = u_3(x_\rho, -x_3, t),\\
&\varphi_\alpha(x_\rho, x_3, t) = \varphi_\alpha(x_\rho, -x_3, t), && \varphi_3(x_\rho, x_3, t) = -\varphi_3(x_\rho, -x_3, t),\\
&\psi(x_\rho, x_3, t) = -\psi(x_\rho, -x_3, t), &&\\
&\theta(x_\rho, x_3, t) = -\theta(x_\rho, -x_3, t), && (x_\rho, x_3) \in B,\ t \in \mathcal{T}.
\end{aligned} \tag{6}
$$

Here and in what follows the Greek subscripts are confined to the range $1, 2$. In view of (1), (4) and (6) we find that

(7)

$$
\begin{aligned}
&t_{\alpha\beta}(x_\rho, x_3, t) = -t_{\alpha\beta}(x_\rho, -x_3, t), && t_{33}(x_\rho, x_3, t) = -t_{33}(x_\rho, -x_3, t),\\
&t_{\alpha 3}(x_\rho, x_3, t) = t_{\alpha 3}(x_\rho, -x_3, t), && t_{3\alpha}(x_\rho, x_3, t) = t_{3\alpha}(x_\rho, -x_3, t),\\
&m_{\alpha\beta}(x_\rho, x_3, t) = m_{\alpha\beta}(x_\rho, -x_3, t), && m_{33}(x_\rho, x_3, t) = m_{33}(x_\rho, -x_3, t),\\
&m_{\alpha 3}(x_\rho, x_3, t) = -m_{\alpha 3}(x_\rho, -x_3, t), && m_{3\alpha}(x_\rho, x_3, t) = -m_{3\alpha}(x_\rho, -x_3, t),\\
&\tau(x_\rho, x_3, t) = -\tau(x_\rho, -x_3, t), && \sigma_\alpha(x_\rho, x_3, t) = -\sigma_\alpha(x_\rho, -x_3, t),\\
&\sigma_3(x_\rho, x_3, t) = \sigma_3(x_\rho, -x_3, t), && \eta(x_\rho, x_3, t) = -\eta(x_\rho, -x_3, t),\\
&q_\alpha(x_\rho, x_3, t) = -q_\alpha(x_\rho, -x_3, t), && q_3(x_\rho, x_3, t) = q_3(x_\rho, -x_3, t).
\end{aligned}
$$

We say that the system of body loads $(\mathbf{f}, \mathbf{g}, \ell, s)$ is compatible with a state of bending if

$$
\begin{aligned}
&f_\alpha(x_\rho, -x_3, t) = -f_\alpha(x_\rho, x_3, t), \quad && f_3(x_\rho, -x_3, t) = f_3(x_\rho, x_3, t),\\
&g_\alpha(x_\rho, -x_3, t) = g_\alpha(x_\rho, x_3, t), && g_3(x_\rho, -x_3, t) = -g_3(x_\rho, x_3, t),\\
&\ell(x_\rho, -x_3, t) = -\ell(x_\rho, x_3, t), && s(x_\rho, -x_3, t) = -s(x_\rho, x_3, t).
\end{aligned} \tag{8}
$$

In what follows we assume that the body loads satisfy the restrictions (8).

Following [4], we derive a theory of thermoelastic thin plates of uniform thickness assuming that the fields u_i, φ_i, ψ and θ do not vary violently with respect to x_3. We denote

(9)

$$
\begin{aligned}
&\tau_{ij} = \frac{1}{2h}\int_{-h}^{h} t_{ij}dx_3, \quad \mu_{ij} = \frac{1}{2h}\int_{-h}^{h} m_{ij}dx_3, \quad \pi_i = \frac{1}{2h}\int_{-h}^{h} \sigma_i dx_3,\\
&H_i = \frac{1}{2h}\int_{-h}^{h} q_i dx_3, \quad \eta^* = \frac{1}{2h}\int_{-h}^{h} \eta dx_3, \quad \tau^* = \frac{1}{2h}\int_{-h}^{h} \tau dx_3,\\
&f_i^* = \frac{1}{2h}\int_{-h}^{h} f_i dx_3, \quad g_i^* = \frac{1}{2h}\int_{-h}^{h} g_i dx_3, \quad \ell^* = \frac{1}{2h}\int_{-h}^{h} \ell dx_3, \quad s^* = \frac{1}{2h}\int_{-h}^{h} s dx_3.
\end{aligned}
$$

From (7) and (8) we get

$$\begin{aligned}&\tau_{\alpha\beta}=0,\ \ \tau_{33}=0,\ \ \mu_{\alpha 3}=0,\ \ \pi_\alpha=0,\ \ H_\alpha=0,\ \ \eta^*=0,\\ &\tau^*=0,\ \ f^*_\alpha=0,\ \ g^*_3=0,\ \ \ell^*=0,\ \ s^*=0.\end{aligned}\tag{10}$$

We assume that the functions t_i, m_i, p and q are prescribed on the surfaces $x_3=\pm h$. We integrate the equations (2) with respect to x_3 between the limits $-h$ and h. On the basis of (6) and (10) we obtain the equations

$$\begin{aligned}&\tau_{\alpha 3,\alpha}+F=\rho\ddot{w},\\ &\mu_{\beta\alpha,\beta}+\varepsilon_{3\rho\alpha}(\tau_{3\rho}-\tau_{\rho 3})+G_\alpha=J\ddot{\psi}_\alpha,\ \ \text{on } \Sigma\times\mathcal{T},\end{aligned}\tag{11}$$

where

$$\begin{aligned}&w=\frac{1}{2h}\int_{-h}^{h}u_3dx_3,\ \ \psi_\alpha=\frac{1}{2h}\int_{-h}^{h}\varphi_\alpha dx_3,\\ &F=f^*_3+\frac{1}{h}t_{33}(x_1,x_2,h,t)\ \ G_\alpha=g^*_\alpha+\frac{1}{h}m_{3\alpha}(x_1,x_2,h,t).\end{aligned}\tag{12}$$

If we multiply by x_3 the equations $(2)_1$ and $(2)_3$ and integrate from $x_3=-h$ to $x_3=h$, then we obtain

$$\begin{aligned}&M_{\beta\alpha,\beta}-2h\tau_{3\alpha}+L_\alpha=\rho I\ddot{v}_\alpha,\\ &\Lambda_{\alpha,\alpha}-2h\pi_3-P+H=\zeta\ddot{u},\ \ \text{on }\ \Sigma\times\mathcal{T},\end{aligned}\tag{13}$$

where

$$\begin{aligned}&M_{\alpha\beta}=\int_{-h}^{h}x_3t_{\alpha\beta}dx_3,\ \ \Lambda_\alpha=\int_{-h}^{h}x_3\sigma_\alpha dx_3,\ \ P=\int_{-h}^{h}x_3\tau dx_3,\\ &Iv_\alpha=\int_{-h}^{h}x_3u_\alpha dx_3,\ \ Iu=\int_{-h}^{h}x_3\psi dx_3,\ \ I=\frac{2}{3}h^3,\ \ \zeta=jI,\\ &L_\alpha=2ht_{3\alpha}(x_1,x_2,h,t)+\int_{-h}^{h}x_3f_\alpha dx_3,\\ &H=2h\sigma_3(x_1,x_2,h,t)+\int_{-h}^{h}x_3\ell dx_3.\end{aligned}\tag{14}$$

Now multiply by x_3 the equation (3) and integrate from $x_3=-h$ to $x_3=h$. In view of (7)-(9) we obtain

$$\rho T_0\dot{\sigma}=Q_{\alpha,\alpha}-2hR+S,\tag{15}$$

where

$$\begin{aligned}&\sigma=\int_{-h}^{h}x_3\eta dx_3,\ \ Q_\alpha=\int_{-h}^{h}x_3q_\alpha dx_3,\ \ R=H_3,\\ &S=2hq_3(x_1,x_2,h,t)+\int_{-h}^{h}x_3sdx_3.\end{aligned}\tag{16}$$

The functions F, G_α, L_α, H and S are prescribed.

Following [3,4,7] we restrict our attention to the state of bending characterized by

$$
\begin{aligned}
&u_\alpha = x_3 v_\alpha(x_1, x_2, t), \quad u_3 = w(x_1, x_2, t),\\
&\varphi_\alpha = \psi_\alpha(x_1, x_2, t), \quad \varphi_3 = 0, \quad \psi = x_3 u(x_1, x_2, t),\\
&\theta = x_3 T(x_1, x_2, t), \quad \text{on} \quad B \times \mathcal{T}.
\end{aligned} \tag{17}
$$

In the isothermal theory of micropolar elasticity the representation (17) has been introduced by Eringen [4].

In view of (17) we have

$$
\begin{aligned}
&e_{\alpha\beta} = x_3 \varepsilon_{\alpha\beta}, \quad e_{\alpha 3} = \varepsilon_{\alpha 3}, \quad e_{3\alpha} = \varepsilon_{3\alpha}, \quad e_{33} = 0,\\
&\kappa_{\alpha\beta} = \eta_{\alpha\beta}, \quad \kappa_{\alpha 3} = \kappa_{3\alpha} = 0, \quad \kappa_{33} = 0, \quad \gamma_\alpha = x_3 \xi_\alpha, \quad \gamma_3 = u,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
&\varepsilon_{\alpha\beta} = v_{\beta,\alpha}, \quad \varepsilon_{\alpha 3} = w_{,\alpha} + \varepsilon_{3\alpha\beta}\psi_\beta, \quad \varepsilon_{3\alpha} = v_\alpha - \varepsilon_{3\alpha\beta}\psi_\beta,\\
&\eta_{\alpha\beta} = \psi_{\beta,\alpha}, \quad \xi_\alpha = u_{,\alpha}.
\end{aligned} \tag{19}
$$

From (4) and (18) it follows that

$$
\begin{aligned}
&t_{\alpha\beta} = x_3[\lambda \varepsilon_{\rho\rho}\delta_{\alpha\beta} + (\mu+\kappa)\varepsilon_{\alpha\beta} + \mu\varepsilon_{\beta\alpha} + \lambda_0 u \delta_{\alpha\beta} - \beta_0 T \delta_{\alpha\beta}],\\
&t_{\alpha 3} = (\mu+\kappa)\varepsilon_{\alpha 3} + \mu\varepsilon_{3\alpha}, \quad t_{3\alpha} = (\mu+\kappa)\varepsilon_{3\alpha} + \mu\varepsilon_{\alpha 3},\\
&t_{33} = x_3(\lambda\varepsilon_{\rho\rho} + \lambda_0 u - \beta_0 T),\\
&m_{\kappa\nu} = \alpha\eta_{\rho\rho}\delta_{\kappa\nu} + \beta\eta_{\nu\kappa} + \gamma\eta_{\kappa\nu} + b_0\varepsilon_{3\nu\kappa}u,\\
&m_{\alpha 3} = b_0 x_3 \varepsilon_{\beta 3\alpha} u_{,\beta}, \quad m_{3\alpha} = b_0 x_3 \varepsilon_{\beta\alpha 3} u_{,\beta},\\
&m_{33} = \alpha\eta_{\rho\rho}, \quad q_\alpha = k x_3 T_{,\alpha}, \quad q_3 = kT,\\
&\rho\eta = x_3(\beta_0\varepsilon_{\rho\rho} + aT + \beta_1 u), \quad \sigma_\alpha = x_3 a_0 u_{,\alpha},\\
&\sigma_3 = a_0 u + b_0 \varepsilon_{3\alpha\beta}\eta_{\beta\alpha}, \quad \tau = (\lambda_0\varepsilon_{\rho\rho} + \lambda_1 u - \beta_1 T)x_3.
\end{aligned} \tag{20}
$$

It follows from (9), (14), (16) and (20) that

$$
\begin{aligned}
&\tau_{\alpha 3} = (\mu+\kappa)\varepsilon_{\alpha 3} + \mu\varepsilon_{3\alpha}, \quad \tau_{3\alpha} = (\mu+\kappa)\varepsilon_{3\alpha} + \mu\varepsilon_{\alpha 3},\\
&\mu_{\kappa\nu} = \alpha\eta_{\rho\rho} + \beta\eta_{\nu\kappa} + \gamma\eta_{\kappa\nu} + b_0\varepsilon_{3\nu\kappa}u,\\
&\pi_3 = a_0 u + b_0\varepsilon_{3\alpha\beta}\eta_{\beta\alpha}, \quad \rho\sigma = I(\beta_0\varepsilon_{\rho\rho} + aT + \beta_1 u),\\
&M_{\alpha\beta} = I[\lambda\varepsilon_{\rho\rho}\delta_{\alpha\beta} + (\mu+\kappa)\varepsilon_{\alpha\beta} + \mu\varepsilon_{\beta\alpha} + \lambda_0 u\delta_{\alpha\beta} - \beta_0 T\delta_{\alpha\beta}],\\
&\Lambda_\alpha = I a_0 u_{,\alpha}, \quad Q_\alpha = I k T_{,\alpha}, \quad R = kT,\\
&P = I(\lambda_0\varepsilon_{\rho\rho} + \lambda_1 u - \beta_1 T).
\end{aligned} \tag{21}
$$

The equations of the theory consist of the equations of motion (11), (13), the energy equation (15), the constitutive equations (21) and the geometrical equations (19). These equations can be expressed in terms of the functions $v_\alpha, w, \psi_\alpha, u$ and T. Thus, we obtain

$$
\begin{aligned}
&I[(\mu+\kappa)\Delta v_\alpha + (\lambda+\mu)v_{\rho,\rho\alpha} - \beta_0 T_{,\alpha} + \lambda_0 u_{,\alpha}]- \\
&\quad -2h[(\mu+\kappa)v_\alpha + \mu w_{,\alpha}] + 2h\kappa\varepsilon_{3\alpha\beta}\psi_\beta + L_\alpha = \rho I \ddot{v}_\alpha, \\
&(\mu+\kappa)\Delta w + \mu v_{\rho,\rho} + \kappa\varepsilon_{3\alpha\beta}\psi_{\beta,\alpha} + F = \rho\ddot{w}, \\
&\gamma\Delta\psi_\alpha + (\alpha+\beta)\psi_{\rho,\rho\alpha} + \kappa\varepsilon_{3\rho\alpha}(v_\rho - w_{,\rho}) - 2\kappa\psi_\alpha + \\
&\quad +b_0\varepsilon_{3\alpha\nu}u_{,\nu} + G_\alpha = J\ddot{\psi}_\alpha, \\
&Ia_0\Delta u - 2hb_0\varepsilon_{3\alpha\beta}\psi_{\alpha,\beta} - I(\lambda_0 v_{\rho,\rho} - \beta_1 T)- \\
&\quad -(2ha_0 + I\lambda_1)u + H = \zeta\ddot{u}, \\
&Ik\Delta T - IT_0(\beta_0\dot{v}_{\rho,\rho} + \beta_1\dot{u} + a\dot{T}) - 2hkT = -S, \ \text{ on } \Sigma\times\mathcal{T}.
\end{aligned} \tag{22}
$$

To the field equations we adjoin initial conditions and boundary conditions. The initial conditions are

$$
\begin{aligned}
&v_\alpha(x_1,x_2,0) = v_\alpha^0(x_1,x_2), \ \ w(x_1,x_2,0) = w^0(x_1,x_2), \\
&\psi_\alpha(x_1,x_2,0) = \psi_\alpha^0(x_1,x_2), \\
&u(x_1,x_2,0) = u^0(x_1,x_2), \ \ T(x_1,x_2,0) = T^0(x_1,x_2), \\
&\dot{v}_\alpha(x_1,x_2,0) = \nu_\alpha(x_1,x_2), \ \ \dot{w}(x_1,x_2,0) = \omega(x_1,x_2), \\
&\dot{\psi}_\alpha(x_1,x_2,0) = \chi_\alpha(x_1,x_2), \\
&\dot{u}(x_1,x_2,0) = \xi^0(x_1,x_2), \ \ (x_1,x_2)\in\overline{\Sigma},
\end{aligned} \tag{23}
$$

where $v_\alpha^0, w^0, \psi_\alpha^0, u^0, T^0, \nu_\alpha, \omega, \chi_\alpha$ and ξ^0 are prescribed functions.

We consider the boundary conditions

$$
\begin{aligned}
&M_{\beta\alpha}n_\beta = \widetilde{M}_\alpha, \ \ \tau_{\alpha 3}n_\alpha = \widetilde{\tau}, \ \ \mu_{\beta\alpha}n_\beta = \widetilde{\mu}_\alpha, \\
&\Lambda_\alpha n_\alpha = \widetilde{\Lambda}, \ \ Q_\alpha n_\alpha = \widetilde{Q}, \ \text{ on } \Gamma\times\mathcal{T},
\end{aligned} \tag{24}
$$

where the functions $\widetilde{M}_\alpha, \widetilde{\tau}, \widetilde{\mu}_\alpha, \widetilde{\Lambda}$ and $\widetilde{Q}$ are given.

RECIPROCITY AND UNIQUENESS

In this section we use the method presented in [8] to establish reciprocity and uniqueness results. Let F and G be scalar fields on $\Sigma\times\mathcal{T}$ that are continuous in time. We denote by $F*G$ the convolution of F and G, i.e.

$$
[F*G](\mathbf{x},t) = \int_0^t F(\mathbf{x},t-s)G(\mathbf{x},s)ds, \ \ \mathbf{x}\in\Sigma, t\in\mathcal{T}.
$$

We suppose that $\mathcal{T} = (0,\infty)$. Let f and g be functions on $\mathcal{T}$ defined by

$$
f(t) = 1, \ \ g(t) = t, \ \ t\in\mathcal{T}. \tag{25}
$$

If F is a continuous function on $\Sigma\times\mathcal{T}$, then we write $\widehat{F}$ for $f*F$, that is

$$\widehat{F}(\mathbf{x},t)=\int_0^t F(\mathbf{x},s)ds,\quad \mathbf{x}\in\Sigma, t\in\mathcal{T}$$

We define the function W^0 on $\Sigma\times\mathcal{T}$ by

$$W=\widehat{S}+\rho T_0\sigma^0. \tag{26}$$

where

$$\rho\sigma^0=I(\beta_0 v^0_{\rho,\rho}+aT^0+\beta_1 u^0). \tag{27}$$

In view of (15), (21), (23), (26) and (27) we get

Lemma 1. *The functions $\sigma\in C^{0,1}, Q_\alpha\in C^{1,0}$ and $R\in C^{0,0}$ satisfy the equation (15) and the initial condition $\sigma(\mathbf{x},0)=\sigma^0(x), \mathbf{x}\in\Sigma$, if and only if*

$$\rho T_0\sigma=\widehat{Q}_{\alpha,\alpha}-2h\widehat{R}+W,\ \text{ on }\ \Sigma\times[0,\infty). \tag{28}$$

The proof is immediate.

We now consider two external data systems $Z^{(\alpha)}=\{F^{(\alpha)},G^{(\alpha)}_\rho,L^{(\alpha)}_\rho,H^{(\alpha)}, S^{(\alpha)},\widetilde{M}^{(\alpha)}_\rho,\widetilde{\tau}^{(\alpha)},\widetilde{\mu}^{(\alpha)}_\rho,\widetilde{\Lambda}^{(\alpha)},\widetilde{Q}^{(\alpha)},v^{(0)(\alpha)}_\rho,w^{(0)(\alpha)},\ldots,\xi^{(0)(\alpha)}\},(\alpha=1,2)$. Let $A^{(\alpha)}=\{w^{(\alpha)},\psi^{(\alpha)}_\rho,v^{(\alpha)}_\rho,u^{(\alpha)},T^{(\alpha)},\varepsilon^{(\alpha)}_{ij},\eta^{(\alpha)}_{\nu\rho},\tau^{(\alpha)}_{ij},\mu^{(\alpha)}_{\kappa\nu},\pi^{(\alpha)}_3,\sigma^{(\alpha)},M^{(\alpha)}_{\nu\kappa},\Lambda^{(\alpha)}_\nu, Q^{(\alpha)}_\nu,R^{(\alpha)},P^{(\alpha)}\}$ be a solution corresponding to $Z^{(\alpha)}$. We denote

$$\begin{aligned}&M^{(\rho)}_\alpha=M^{(\rho)}_{\beta\alpha}n_\beta,\ \ \tau^{(\rho)}=\tau^{(\rho)}_{\alpha 3}n_\alpha,\ \ \mu^{(\rho)}_\alpha=\mu^{(\rho)}_{\beta\alpha}n_\beta,\\ &\Lambda^{(\rho)}=\Lambda^{(\rho)}_\alpha n_\alpha,\ \ Q^{(\rho)}=Q^{(\rho)}_\alpha n_\alpha,\ \ W^{(\alpha)}=\widehat{S}^{(\alpha)}+\rho T_0\sigma^{0(\alpha)}\end{aligned} \tag{29}$$

and introduce the notations

(30)

$$\begin{aligned}\Pi_{\kappa\nu}(r,s)\ =&\int_\Gamma[M^{(\kappa)}_\alpha(r)v^{(\nu)}_\alpha(s)+2h\tau^{(\kappa)}(r)w^{(\nu)}(s)+2h\mu^{(\kappa)}_\alpha(r)\psi^{(\nu)}_\alpha(s)+\\ &+\Lambda^{(\kappa)}(r)u^{(\nu)}(s)-\frac{1}{T_0}\widehat{Q}^{(\kappa)}(r)T^{(\nu)}(s)]d\ell+\int_\Sigma[2hF^{(\kappa)}(r)w^{(\nu)}(s)+\\ &+2hG^{(\kappa)}_\alpha(r)\psi^{(\nu)}_\alpha(s)+L^{(\kappa)}_\alpha(r)v^{(\nu)}_\alpha(s)+H^{(\kappa)}(r)u^{(\nu)}(s)-\\ &-\frac{1}{T_0}W^{(\kappa)}(r)T^{(\nu)}(s)]da,\\ K_{\kappa\nu}(r,s)\ =&\int_\Sigma\{\rho I\ddot{v}^{(\kappa)}_\alpha(r)v^{(\nu)}_\alpha(s)+\zeta\ddot{u}^{(\kappa)}(r)u^{(\nu)}(s)+2h\rho\ddot{w}^{(\kappa)}(r)w^{(\nu)}(s)+\\ &+2hJ\ddot{\psi}^{(\kappa)}_\alpha(r)\psi^{(\nu)}_\alpha(s)-\frac{1}{T_0}k[I\widehat{T}^{(\kappa)}_{,\alpha}(r)T^{(\nu)}_{,\alpha}(s)+\\ &+2h\widehat{T}^{(\kappa)}(r)T^{(\nu)}(s)]\}da,\end{aligned}$$

where, for convenience, we have suppressed the argument $\mathbf{x}$.

A general reciprocity relation is expressed by the following theorem.

Theorem 1. *Let*

$$E_{\kappa\nu}(r,s)=\Pi_{\kappa\nu}(r,s)-K_{\kappa\nu}(r,s), \tag{31}$$

for all $r,s\in\mathcal{T}$. *Then*

$$E_{\kappa\nu}(r,s)=E_{\nu\kappa}(s,r). \tag{32}$$

Proof. Let

$$\begin{aligned} I_{\kappa\nu}(r,s)\ &=M^{(\kappa)}_{\alpha\beta}(r)\varepsilon^{(\nu)}_{\alpha\beta}(s)+\Lambda^{(\kappa)}_{\alpha}(r)u^{(\nu)}_{,\alpha}(s)+\\ &+P^{(\kappa)}(r)u^{(\nu)}(s)-\rho\sigma^{(\kappa)}(r)T^{(\nu)}(s)+2h[\tau^{(\kappa)}_{3\alpha}(r)\varepsilon^{(\nu)}_{3\alpha}(s)+\\ &+\tau^{(\kappa)}_{\alpha3}(r)\varepsilon^{(\nu)}_{\alpha3}(s)+\mu^{(\kappa)}_{\alpha\beta}(r)\eta^{(\nu)}_{\alpha\beta}(s)+\pi^{(\kappa)}_{3}(r)u^{(\nu)}(s)]. \end{aligned} \tag{33}$$

In view of (21) we obtain

(34)

$$\begin{aligned} I_{\kappa\nu}(r,s)\ &=I\{\lambda\varepsilon^{(\kappa)}_{\rho\rho}(r)\varepsilon^{(\nu)}_{\eta\eta}(s)+(\mu+\kappa)\varepsilon^{(\kappa)}_{\alpha\beta}(r)\varepsilon^{(\nu)}_{\alpha\beta}(s)+\\ &+\mu\varepsilon^{(\kappa)}_{\beta\alpha}(r)\varepsilon^{(\nu)}_{\alpha\beta}(s)+a_0u^{(\kappa)}_{,\alpha}(r)u^{(\nu)}_{,\alpha}(s)+\\ &+\lambda_0[\varepsilon^{(\nu)}_{\rho\rho}(s)u^{(\kappa)}(r)+\varepsilon^{(\kappa)}_{\rho\rho}(r)u^{(\nu)}(s)]-\beta_0[\varepsilon^{(\nu)}_{\rho\rho}(s)T^{(\kappa)}(r)+\\ &+\varepsilon^{(\kappa)}_{\rho\rho}(r)T^{(\nu)}(s)]-\beta_1[u^{(\kappa)}(r)T^{(\nu)}(s)+u^{(\nu)}(s)T^{(\kappa)}(r)]+\\ &+\lambda_1u^{(\kappa)}(r)u^{(\nu)}(s)\}+2h\{(\mu+\kappa)\varepsilon^{(\kappa)}_{3\alpha}(r)\varepsilon^{(\nu)}_{3\alpha}(s)+\\ &+\mu[\varepsilon^{(\kappa)}_{\alpha3}(r)\varepsilon^{(\nu)}_{3\alpha}(s)+\varepsilon^{(\nu)}_{\alpha3}(s)\varepsilon^{(\kappa)}_{3\alpha}(r)]+\\ &+(\mu+\kappa)\varepsilon^{(\kappa)}_{\alpha3}(r)\varepsilon^{(\nu)}_{\alpha3}(s)+\alpha\eta^{(\kappa)}_{\rho\rho}(r)\eta^{(\nu)}_{\beta\beta}(s)+\beta\eta^{(\kappa)}_{\beta\alpha}(r)\eta^{(\nu)}_{\alpha\beta}(s)+\\ &+\gamma\eta^{(\kappa)}_{\alpha\beta}(r)\eta^{(\nu)}_{\alpha\beta}(s)+b_0\varepsilon_{3\alpha\beta}[\eta^{(\kappa)}_{\beta\alpha}(r)u^{(\nu)}(s)+\eta^{(\nu)}_{\beta\alpha}(s)u^{(\kappa)}(r)]+\\ &+a_0u^{(\kappa)}(r)u^{(\nu)}(s)\}. \end{aligned}$$

It follows from (34) that

$$I_{\kappa\nu}(r,s)=I_{\nu\kappa}(s,r). \tag{35}$$

On the other hand, in view of (11), (13), (28) and (19), we obtain

(36)

$$\begin{aligned} I_{\kappa\nu}(r,s)\ &=\{M^{(\kappa)}_{\alpha\beta}(r)v^{(\nu)}_{\beta}(s)+\Lambda^{(\kappa)}_{\alpha}(r)u^{(\nu)}(s)-\frac{1}{T_0}\widehat{Q}^{(\kappa)}_{\alpha}(r)T^{(\nu)}(s)+\\ &+2h[\mu^{(\kappa)}_{\alpha\beta}(r)\psi^{(\nu)}_{\beta}(s)+\tau^{(\kappa)}_{\alpha3}(r)w^{(\nu)}(s)]\}_{,\alpha}+\\ &+2h[F^{(\kappa)}(r)w^{(\nu)}(s)+G^{(\kappa)}_{\alpha}(r)\psi^{(\nu)}_{\alpha}(s)]+L^{(\kappa)}_{\alpha}(r)v^{(\nu)}_{\alpha}(s)+\\ &+H^{(\kappa)}(r)u^{(\nu)}(s)-\frac{1}{T_0}W^{(\kappa)}(r)T^{(\nu)}(s)-\rho I\ddot{v}^{(\kappa)}_{\alpha}(r)v^{(\nu)}_{\alpha}(s)-\\ &-\rho\ddot{u}^{(\kappa)}(r)u^{(\nu)}(s)-2h[\rho\ddot{w}^{(\kappa)}(r)w^{(\nu)}(s)+J\ddot{\psi}^{(\kappa)}_{\alpha}(r)\psi^{(\nu)}_{\alpha}(s)]+\\ &+\frac{1}{T_0}k[I\widehat{T}^{(\kappa)}_{,\alpha}(r)T^{(\nu)}_{,\alpha}(s)+2h\widehat{T}^{(\kappa)}(r)T^{(\nu)}(s)]. \end{aligned}$$

By using the divergence theorem, from (30) and (31) we find that

$$\int_{\Sigma} I_{\kappa\nu}(r,s)da = E_{\kappa\nu}(r,s).$$

In view of (35) we conclude that (32) holds.

A consequence of Theorem 1 is the following proposition.

Corollary 1. *Let*

(37)

$$\begin{aligned}\Pi(r,s) \ &= \int_{\Gamma}\{M_\alpha(r)v_\alpha(s)+\Lambda(r)u(s)+2h\tau(r)w(s)+\\ &+2h\mu_\alpha(r)\psi_\alpha(s)-\frac{1}{T_0}\widehat{Q}(r)T(s)\}d\ell+\int_{\Sigma}[L_\alpha(r)v_\alpha(s)+\\ &+H(r)u(s)+2hF(r)w(s)+2hG_\alpha(r)\psi_\alpha(s)-\frac{1}{T_0}W(r)T(s)]da,\end{aligned}$$

for all $r,s\in\mathcal{T}$, *where* $\{w,\psi_\alpha,v_\alpha,u,T\}$ *is a solution of the boundary-initial-value problem (22)-(24). Then*

(38)

$$\begin{aligned}&\frac{d}{dt}\{\int_{\Sigma}[\rho I v_\alpha v_\alpha+\zeta u^2+2h(\rho w^2+J\psi_\alpha\psi_\alpha)]da+\\ &+\frac{1}{T_0}k\int_0^t\int_{\Sigma}(I\widehat{T}_{,\alpha}\widehat{T}_{,\alpha}+2h\widehat{T}^2)dtda\}=\\ &=\int_0^t[\Pi(t-s,t+s)-\Pi(t+s,t-s)]ds+\int_{\Sigma}\{\rho I[\dot{v}_\alpha(0)v_\alpha(2t)+\\ &+\dot{v}_\alpha(2t)v_\alpha(0)]+\zeta[\dot{u}(2t)u(0)+\dot{u}(0)u(2t)]+2hJ[\dot{\psi}_\alpha(0)\psi_\alpha(2t)+\\ &+\dot{\psi}_\alpha(2t)\psi_\alpha(0)]+2h\rho[\dot{w}(0)w(2t)+\dot{w}(2t)w(0)]\}da.\end{aligned}$$

Proof. On the basis of (32) we can write

$$\int_0^t E_{11}(t+s,t-s)ds=\int_0^t E_{11}(t-s,t+s)ds. \tag{39}$$

We apply this relation for $w^{(1)}=w,\psi_\alpha^{(1)}=\psi_\alpha,v_\alpha^{(1)}=v_\alpha,u^{(1)}=u,T^{(1)}=T$. In view of (30), (31) and (37) we obtain

(40)

$$\begin{aligned}&\int_0^t E_{11}(t+s,t-s)ds=\int_0^t\Pi(t+s,t-s)ds-\int_{\Sigma}[\rho I\ddot{v}_\alpha(t+s)v_\alpha(t-s)+\\ &+\zeta\ddot{u}(t+s)u(t-s)+2hJ\ddot{\psi}_\alpha(t+s)\psi_\alpha(t-s)+2h\rho\ddot{w}(t+s)w(t-s)]da+\\ &+\frac{1}{T_0}k\int_{\Sigma}[I\widehat{T}_{,\alpha}(t+s)T_{,\alpha}(t-s)+2h\widehat{T}(t+s)T(t-s)]da\end{aligned}$$

and
(41)
$$\int_0^t E_{11}(t-s,t+s)ds = \int_0^t \Pi(t-s,t+s)ds - \int_\Sigma [\rho I \ddot{v}_\alpha(t-s)v_\alpha(t+s)+$$
$$+\zeta\ddot{u}(t-s)u(t+s) + 2hJ\ddot{\psi}_\alpha(t-s)\psi_\alpha(t+s) + 2h\rho\ddot{w}(t-s)w(t+s)]da+$$
$$+\frac{1}{T_0}k\int_\Sigma [I\widehat{T}_{,\alpha}(t-s)T_{,\alpha}(t+s) + 2h\widehat{T}(t-s)T(t+s)]da.$$

We note that
(42)
$$\int_0^t f(t+s)\dot{h}(t-s)ds = -h(0)f(2t) + f(t)h(t) + \int_0^t \dot{f}(t+s)h(t-s)ds,$$
$$\int_0^t \ddot{f}(t+s)h(t-s)ds = \dot{f}(2t)h(0) - \dot{f}(t)h(t) + \int_0^t \dot{h}(t-s)\dot{f}(t+s)ds,$$
$$\int_0^t \ddot{h}(t-s)f(t+s)ds = \dot{h}(t)f(t) - \dot{h}(0)f(2t) + \int_0^t \dot{h}(t-s)\dot{f}(t+s)ds.$$

If we substitute (40) and (41) in (39) and take into account the relations (42) then we conclude that (38) holds.

Corollary 1 forms the basis of the following uniqueness result.

Theorem 2. *Assume that*
(i) ρ and J are strictly positive;
(ii) k is strictly positive.

Then the boundary-initial-value problem has at most one solution.

Proof. If there are two solutions, then their difference $\{w, \psi_\alpha, v_\alpha, u, T\}$ corresponds to null data. In view of (38) and the initial conditions we get

$$\int_\Sigma [\rho I v_\alpha v_\alpha + \zeta u^2 + 2h(\rho w^2 + J\psi_\alpha\psi_\alpha)]da+ \quad (43)$$
$$+\frac{1}{T_0}k\int_0^t\int_\Sigma (I\widehat{T}_{,\alpha}\widehat{T}_{,\alpha} + 2h\widehat{T}^2)dtda = 0.$$

By the hypotheses of theorem and (43) we conclude that $w = 0, \psi_\alpha = 0, v_\alpha = 0, u = 0$ and $T = 0$ on $\Sigma \times \mathcal{T}$. □

Theorem 1 implies the following reciprocal theorem.

Theorem 3. *Let $A^{(\alpha)}$ be a solution corresponding to the external data system $Z^{(\alpha)}(\alpha=1,2)$. Then*

(44)
$$\int_\Sigma [\mathcal{M}_\alpha^{(1)} * v_\alpha^{(2)} + 2h\mathcal{H}^{(1)} * w^{(2)} + 2h\mathcal{N}_\alpha^{(1)} * \psi_\alpha^{(2)} + Y^{(1)} * u^{(2)} - \\ -\frac{1}{T_0} g * W^{(1)} * T^{(2)}]da + \int_\Gamma g * [M_\alpha^{(1)} * v_\alpha^{(2)} + 2h\tau^{(1)} * w^{(2)} + 2h\mu_\alpha^{(1)} * \psi_\alpha^{(2)} + \\ +\Lambda^{(1)} * u^{(2)} - \frac{1}{T_0} f * Q^{(1)} * T^{(2)}]ds = \int_\Sigma [\mathcal{M}_\alpha^{(2)} * v_\alpha^{(1)} + \\ +2h\mathcal{H}^{(2)} * w^{(1)} + 2h\mathcal{N}_\alpha^{(2)} * \psi_\alpha^{(1)} + Y^{(2)} * u^{(1)} - \frac{1}{T_0} g * W^{(2)} * T^{(1)}]da + \\ + \int_\Gamma g * [M_\alpha^{(2)} * v_\alpha^{(1)} + 2h\tau^{(2)} * w^{(1)} + 2h\mu_\alpha^{(2)} * \psi_\alpha^{(1)} + \Lambda^{(2)} * u^{(1)} - \\ -\frac{1}{T_0} f * Q^{(2)} * T^{(1)}]ds,$$

where

(45)
$$\mathcal{M}_\alpha^{(\nu)} = g * L_\alpha^{(\nu)} + \rho I(t\nu_\alpha^{(\nu)} + v_\alpha^{0(\nu)}), \ \ \mathcal{N}_\alpha^{(\nu)} = g * G_\alpha^{(\nu)} + J(t\chi_\alpha^{(\nu)} + \psi_\alpha^{0(\nu)}), \\ \mathcal{H}^{(\nu)} = g * F^{(\nu)} + \rho(t\omega^{(\nu)} + w^{0(\nu)}), \ \ Y^{(\nu)} = g * H^{(\nu)} + \zeta(t\xi^{0(\nu)} + u^{0(\nu)}).$$

Proof. If we take in (32) $r = y$ and $s = t - y$, and integrate with respect to y from 0 to t, we arrive at

$$\int_\Sigma [M_\alpha^{(1)} * v_\alpha^{(2)} + 2h\tau^{(1)} * w^{(2)} + 2h\mu_\alpha^{(1)} * \psi_\alpha^{(2)} + \\ +\Lambda^{(1)} * u^{(2)} - \frac{1}{T_0} f * Q^{(1)} * T^{(2)}]d\ell + \int_\Sigma [2hF^{(1)} * w^{(2)} + \\ +2hG_\alpha^{(1)} * \psi_\alpha^{(2)} + L_\alpha^{(1)} * v_\alpha^{(2)} + H^{(1)} * u^{(2)} - \frac{1}{T_0} W^{(1)} * T^{(2)}]da - \\ - \int_\Sigma \{\rho I \ddot{v}_\alpha^{(1)} * v_\alpha^{(2)} + \zeta \ddot{u}^{(1)} * u^{(2)} + 2h\rho \ddot{w}^{(1)} * w^{(2)} + \\ +2hJ\ddot{\psi}_\alpha^{(1)} * \psi_\alpha^{(2)} - \frac{1}{T_0} k[If * T_{,\alpha}^{(1)} * T_{,\alpha}^{(2)} + 2hf * T^{(1)} * T^{(2)}]\}da = \\ = \int_\Sigma [M_\alpha^{(2)} * v_\alpha^{(1)} + 2h\tau^{(2)} * w^{(1)} + 2h\mu_\alpha^{(2)} * \psi_\alpha^{(1)} + \\ +\Lambda^{(2)} * u^{(1)} - \frac{1}{T_0} f * Q^{(2)} * T^{(1)}]d\ell + \int_\Sigma [2hF^{(2)} * w^{(1)} + \\ +2hG_\alpha^{(2)} * \psi_\alpha^{(1)} + L_\alpha^{(2)} * v_\alpha^{(1)} + H^{(2)} * u^{(1)} - \frac{1}{T_0} W^{(2)} * T^{(1)}]da \tag{46}$$

$$-\int_{\Sigma}\{\rho I\ddot{v}_{\alpha}^{(2)}*v_{\alpha}^{(1)}+\zeta\ddot{u}^{(2)}*u^{(1)}+2h\rho\ddot{w}^{(2)}*w^{(1)}+$$
$$+2hJ\ddot{\psi}_{\alpha}^{(2)}*\psi_{\alpha}^{(1)}-\frac{1}{T_0}k[If*T_{,\alpha}^{(2)}*T_{,\alpha}^{(1)}+2hf*T^{(2)}*T^{(1)}]\}da$$

We note that

$$g*\ddot{u}=u-t\dot{u}(x,0)-u(x,0). \tag{47}$$

If we take the convolution of the relation (46) with g and use (47) and (45) then we obtain (44).□

As in [6], we can obtain various applications of the reciprocal theorem. A study of the thermal stresses in microstretch elastic cylinders has been presented in [9].

EXISTENCE OF SOLUTION

In this section we use the results of the semigroup of linear operators theory to obtain an existence theorem. Thought other boundary conditions could be proposed, we restrict our attention to the following boundary conditions

$$v_{\alpha}=0,\ w=0,\ \psi_{\alpha}=0,\ u=0,\ T=0,\ \text{on } \Gamma\times\mathcal{T}. \tag{48}$$

In what follows we assume that the internal energy density is a positive definite quadratic form. We can see that the necessary and sufficient conditions that the internal energy density be positive are

$$\begin{aligned}&2\mu+\kappa>0,\quad \kappa>0,\quad 2\alpha+\beta+\gamma>0,\ \ \gamma+\beta>0,\ \ a_0>0,\\&\lambda_1+\vartheta a_0>0,\quad a>0,(2\lambda+2\mu+\kappa)(\lambda_1+\vartheta a_0)-2\lambda_0^2>0,\\&(\gamma-\beta)[(\lambda_1+\vartheta a_0)(2\lambda+2\mu+\kappa)-2\lambda_0^2]-2b_0^2\vartheta(2\lambda+2\mu+\kappa)>0,\end{aligned} \tag{49}$$

where $\vartheta=2h/I$. The positive definiteness of the internal energy density implies the existence of a positive constant C such that

$$\begin{aligned}&I\left[\lambda\varepsilon_{\rho\rho}\varepsilon_{\nu\nu}+(\mu+\kappa)\varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta}+\mu\varepsilon_{\beta\alpha}\varepsilon_{\alpha\beta}+2\lambda_0\varepsilon_{\rho\rho}u\right]\\&+2h\big[(\mu+\kappa)(\varepsilon_{3\alpha}\varepsilon_{3\alpha}+\varepsilon_{\alpha3}\varepsilon_{\alpha3})+2\mu\varepsilon_{3\alpha}\varepsilon_{\alpha3}+\alpha\eta_{\rho\rho}\eta_{\nu\nu}+\beta\eta_{\nu\rho}\eta_{\rho\nu}\\&+\gamma\eta_{\alpha\beta}\eta_{\alpha\beta}+2b_0\varepsilon_{3\alpha\beta}u\eta_{\alpha\beta}\big]+Ia_0u_{,\alpha}u_{\alpha}+(I\lambda_1+2ha_0)u^2+IaT^2\\&\geq C\left[\varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta}+\varepsilon_{3\alpha}\varepsilon_{3\alpha}+\varepsilon_{\alpha3}\varepsilon_{\alpha3}+\eta_{\alpha\beta}\eta_{\alpha\beta}+u^2+u_{,\alpha}u_{\alpha}+T^2\right],\end{aligned} \tag{50}$$

for any $\varepsilon_{ij},\eta_{\alpha\beta},u,u_{,\alpha}$ and T.

We also suppose that the constants ρ,J,I,ζ and k are strictly positive.

We now transform the boundary initial value problem defined by the system (22), the initial conditions (23) and the boundary conditions (48) into an abstract problem on a suitable Hilbert space. We denote

$$\mathcal{Z} = \{\mathbf{U} = (\mathbf{v}, \mathbf{z}, w, y, \Psi, \Phi, u, s, T);\quad v_\alpha, \psi_\alpha, w, u \in W_0^{1,2}(\Sigma), \\ z_\alpha, \phi_\alpha, y, s, T \in L^2(\Sigma)\}, \tag{51}$$

where $W_0^{1,2}(\Sigma)$ and $L^2(\Sigma)$ are the usual Sobolev spaces.

Let us consider the operators

$$\begin{aligned}
&A_\alpha(\mathbf{v}) = \frac{1}{\rho}\left[(\mu+\kappa)\Delta v_\alpha + (\lambda+\mu)v_{\rho,\rho\alpha}\right] - \frac{2h(\mu+\kappa)}{I\rho}v_\alpha,\\
&B_\alpha(w) = -\frac{2h\mu}{I\rho}w_{,\alpha},\quad C_\alpha(\Psi) = \frac{2h\kappa}{I\rho}\varepsilon_{3\alpha\beta}\psi_\beta,\\
&D_\alpha(u) = \frac{\lambda_0}{\rho}u_{,\alpha},\ E_\alpha(T) = -\frac{\beta_0}{\rho}T_{,\alpha},\\
&F_1(w) = \frac{(\mu+\kappa)}{\rho}\Delta w,\ Gv = \frac{\mu}{\rho}v_{\nu,\nu},\quad H_1(\Psi) = \frac{\kappa}{\rho}\varepsilon_{3\alpha\beta}\psi_{\beta,\alpha},\\
&K_\alpha(\Psi) = \frac{\gamma}{J}\Delta\psi_\alpha + \frac{(\alpha+\beta)}{J}\psi_{\rho,\rho\alpha} - \frac{2\kappa}{J}\psi_\alpha,\ Z_\alpha(\mathbf{v}) = \frac{\kappa}{J}\varepsilon_{3\rho\alpha}v_\rho,\\
&M_\alpha(w) = -\frac{\kappa}{J}\varepsilon_{3\rho\alpha}w_{,\rho},\quad N_\alpha(u) = \frac{b_0}{J}\varepsilon_{3\alpha\beta}u_{,\beta},\\
&P_1(u) = \frac{Ia_0}{\zeta}\Delta u - \frac{(2ha_0 + I\lambda_1)}{\zeta}u,\ Q(\Psi) = -\frac{2hb_0}{\zeta}\varepsilon_{3\alpha\beta}\psi_{\alpha,\beta},\\
&R_1(v) = -\frac{I\lambda_0}{\zeta}v_{\rho,\rho},\ S_1(T) = \frac{I\beta_1}{\zeta}T,\\
&U_1(T) = \frac{k}{aT_0}\Delta T - \frac{2hk}{aIT_0}T,\ V_1(\mathbf{z}) = -\frac{\beta_0}{a}z_{\rho,\rho},\ W_1(s) = -\frac{\beta_1}{a}s.
\end{aligned}$$

We denote

$$\mathcal{D} = \left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}\times\left(W_0^{1,2}\cap W^{2,2}\right)\times W_0^{1,2}\times \\ \times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}\times\left(W^{2,2}\cap W_0^{1,2}\right)\times W_0^{1,2}\times\left(W^{2,2}\cap W_0^{1,2}\right). \tag{52}$$

Let $\mathcal{A}$ the matrix operator defined on $\mathcal{D}$ by

$$\mathcal{A} = \begin{pmatrix}
\mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{A} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{C} & \mathbf{0} & \mathbf{D} & \mathbf{0} & \mathbf{E}\\
0 & 0 & 0 & Id & 0 & 0 & 0 & 0 & 0\\
F_1 & 0 & G & 0 & H_1 & 0 & 0 & 0 & 0\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{Z} & \mathbf{0} & \mathbf{M} & \mathbf{0} & \mathbf{K} & \mathbf{0} & \mathbf{N} & \mathbf{0} & \mathbf{0}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Id & 0\\
R_1 & 0 & 0 & 0 & Q & 0 & P_1 & 0 & S_1\\
0 & V_1 & 0 & 0 & 0 & 0 & 0 & W_1 & U_1
\end{pmatrix}, \tag{53}$$

where $\mathbf{A} = (A_\alpha), \mathbf{B} = (B_\alpha), \mathbf{C} = (C_\alpha), \mathbf{D} = (D_\alpha), \mathbf{E} = (E_\alpha), \mathbf{Z} = (Z_\alpha), \mathbf{M} = (M_\alpha), \mathbf{N} = (N_\alpha), \mathbf{K} = (K_\alpha)$ and $Id, \mathbf{Id}$ represent the identity in the respective spaces. We note that that the domain $\mathcal{D}$ is dense in $\mathcal{Z}$.

The initial boundary value problem (22), (23), (48) can be transformed into the following abstract equation in the Hilbert space $\mathcal{Z}$,

$$\frac{d\mathbf{U}}{dt} = \mathcal{A}\mathbf{U}(t) + \mathbf{F}(t), \quad \mathbf{U}(0) = \mathbf{U}_0, \tag{54}$$

where

(55)

$$\mathbf{F} = \left(\mathbf{0}, \frac{L_\alpha}{\rho I}, 0, \frac{F}{\rho}, \mathbf{0}, \frac{G_\alpha}{J}, 0, \frac{H}{\zeta}, \frac{S}{aIT_0}\right), \mathbf{U}_0 = \left(v^0_\alpha, \nu_\alpha, w^0, \omega, \psi^0_\alpha, \phi^0_\alpha, u^0, \xi^0, T^0\right).$$

We introduce in $\mathcal{Z}$ the inner product

(56)

$$< \mathbf{U}, \mathbf{V} >= \\ = \frac{1}{2}\int_\Sigma \left(\rho I z_\alpha z^*_\alpha + 2h\rho y y^* + 2hJ\phi_\alpha\phi^*_\alpha + \zeta s s^* + IaTT^* + \mathcal{M}\left[\mathbf{U}^0, \mathbf{V}^0\right]\right) da,$$

where

$$\mathbf{U} = (\mathbf{v}, \mathbf{z}, w, y, \Psi, \Phi, u, s, T), \ \mathbf{V} = (\mathbf{v}^*, \mathbf{z}^*, w^*, y^*, \Psi^*, \Phi^*, u^*, s^*, T^*),$$
$$\mathbf{U}^0 = (v_\alpha, w, \psi_\alpha, u), \ \mathbf{V}^0 = (v^*_\alpha, w^*, \psi^*_\alpha, u^*)$$

and

$$\mathcal{M}\left[\mathbf{U}^0, \mathbf{V}^0\right] = I\left[\lambda\varepsilon_{\rho\rho}\varepsilon^*_{\nu\nu} + (\mu+\kappa)\varepsilon_{\alpha\beta}\varepsilon^*_{\alpha\beta} + \mu\varepsilon_{\alpha\beta}\varepsilon^*_{\beta\alpha} + \lambda_0(\varepsilon_{\rho\rho}u^* + \varepsilon^*_{\rho\rho}u)\right] \\ +2h[(\mu+\kappa)(\varepsilon_{3\alpha}\varepsilon^*_{3\alpha} + \varepsilon_{\alpha 3}\varepsilon^*_{\alpha 3}) + \mu(\varepsilon_{\alpha 3}\varepsilon^*_{3\alpha} + \varepsilon_{3\alpha}\varepsilon^*_{\alpha 3}) + \alpha\eta_{\rho\rho}\eta^*_{\nu\nu} + \beta\eta_{\nu\rho}\eta^*_{\rho\nu} \\ +\gamma\eta_{\nu\rho}\eta^*_{\nu\rho} + b_0\varepsilon_{3\alpha\beta}(u\eta^*_{\alpha\beta} + u^*\eta_{\alpha\beta})] + Ia_0u_{,\alpha}u^*_{,\alpha} + (I\lambda_1 + 2ha_0)uu^*.$$

In the above relations we have used the notation

$$\varepsilon^*_{\alpha\beta} = v^*_{\beta,\alpha}, \ \ \varepsilon^*_{\alpha 3} = w^*_{,\alpha} + \varepsilon_{3\alpha\beta}\psi^*_\beta, \ \ \varepsilon^*_{3\alpha} = v^*_\alpha - \varepsilon_{3\alpha\beta}\psi^*_\beta, \ \ \eta^*_{\alpha\beta} = \psi^*_{\beta,\alpha}.$$

We note that on the basis of the inequality (50) and the Korn inequality the norm induced in $\mathcal{Z}$ by the product (56) is equivalent to the usual one in $\mathcal{Z}$.

Lemma 2. *The operator $\mathcal{A}$ has the property*

$$< \mathcal{A}\mathbf{U}, \mathbf{U} > \leq 0, \tag{57}$$

for any $\mathbf{U} \in \mathcal{D}$.

Proof. Let $\mathbf{U} = (\mathbf{v}, \mathbf{z}, w, y, \Psi, \Phi, u, s, T) \in \mathcal{D}$. A use of the divergence theorem and the boundary conditions gives

$$< \mathcal{A}\mathbf{U}, \mathbf{U} >= \int_\Sigma \big(- z_{\alpha,\beta}M_{\beta\alpha} - 2hz_\alpha\tau_{3\alpha} - 2hy_{,\alpha}\tau_{\alpha 3} - 2h\phi_{\alpha,\beta}\mu_{\beta\alpha} \tag{58}$$

$$+2h\phi_\alpha\varepsilon_{3\rho\alpha}(\tau_{3\rho}-\tau_{\rho 3})-s_\alpha\Lambda_\alpha-2hs\Pi_3-Ps-\frac{Ik}{T_0}T_{,\alpha}T_{,\alpha}-\frac{2hk}{T_0}T^2)da$$
$$-\int_\Sigma\big(I\beta_0 z_{\rho,\rho}T+I\beta_1 sT-\mathcal{M}\left[\mathbf{U}^0,\mathbf{U}^0\right]\big)da$$
$$=-\int_\Sigma(\frac{Ik}{T_0}T_{,\alpha}T_{,\alpha}+\frac{2hk}{T_0}T^2)da.$$

In view of the assumptions on the constitutive coefficients the lemma is proved.□

Lemma 3. *The operator $\mathcal{A}$ satisfies the range condition*

$$\text{Range }(\mathcal{I}-\mathcal{A})=\mathcal{Z}. \tag{59}$$

Proof. Let $\mathbf{U}^*=(\mathbf{v}^*,\mathbf{z}^*,w^*,y^*,\Psi^*,\Phi^*,u^*,s^*,T^*)\in\mathcal{Z}$. We must show that the equation

$$\mathbf{U}-\mathcal{A}\mathbf{U}=\mathbf{U}^*, \tag{60}$$

has a solution $\mathbf{U}=(\mathbf{v},\mathbf{z},w,y,\Psi,\Phi,u,s,T)\in\mathcal{D}$. If we take into account the operator $\mathcal{A}$, then from (60) we find the system

$$\begin{aligned}
&\mathbf{v}-\mathbf{z}=\mathbf{v}^*,\quad w-y=w^*,\quad \mathbf{\Psi}-\mathbf{\Phi}=\mathbf{\Psi}^*,\quad u-s=u^*,\\
&\mathbf{z}-\mathbf{A}\mathbf{v}-\mathbf{B}w-\mathbf{C}\mathbf{\Psi}-\mathbf{D}u-\mathbf{E}T=\mathbf{z}^*,\\
&y-F_1\mathbf{v}-Gw-H_1\mathbf{\Psi}=y^*,\\
&\mathbf{\Phi}-\mathbf{Z}\mathbf{v}-\mathbf{M}w-\mathbf{K}\mathbf{\Psi}-\mathbf{N}u=\mathbf{\Phi}^*,\\
&s-R_1\mathbf{v}-Q\mathbf{\Psi}-P_1u-S_1T=s^*,\\
&T-V_1\mathbf{z}-W_1 s-U_1T=T^*.
\end{aligned} \tag{61}$$

Substituting the first fourth equations in the others, we obtain the following system with unknowns $(\mathbf{v},w,\mathbf{\Psi},u,T)$

$$\begin{aligned}
&\mathbf{v}-\mathbf{A}\mathbf{v}-\mathbf{B}w-\mathbf{C}\mathbf{\Psi}-\mathbf{D}u-\mathbf{E}T=\mathbf{z}^*+\mathbf{v}^*,\\
&w-F_1\mathbf{v}-Gw-H_1\mathbf{\Psi}=y^*+w^*,\\
&\mathbf{\Psi}-\mathbf{Z}\mathbf{v}-\mathbf{M}w-\mathbf{K}\mathbf{\Psi}-\mathbf{N}u=\mathbf{\Phi}^*+\mathbf{\Psi}^*,\\
&u-R_1\mathbf{v}-Q\mathbf{\Psi}-P_1u-S_1T=s^*+u^*,\\
&T-V_1\mathbf{v}-W_1u-U_1T=T^*-V_1\mathbf{v}^*-W_1u^*.
\end{aligned} \tag{62}$$

To study this system, we introduce the following bilinear form

$$\Lambda\left[\Gamma_1,\Gamma_2\right]=<\mathcal{N}^*,\mathcal{P}>_{\mathbf{L^2}}, \tag{63}$$

where

$$\Gamma_1=(\mathbf{v},w,\mathbf{\Psi},u,T),\;\;\Gamma_2=(\hat{\mathbf{v}},\hat{w},\hat{\mathbf{\Psi}},\hat{u},\hat{T})$$
$$\mathcal{N}^*=(\mathbf{v}^{(1)},w^{(1)},\mathbf{\Psi}^{(1)},u^{(1)},T^{(1)}),\;\;\mathcal{P}=(\rho I\hat{\mathbf{v}},2h\hat{w},2hJ\hat{\mathbf{\Psi}},\zeta\hat{u},aI\hat{T}),$$

and

$$\begin{aligned}
\mathbf{v}^{(1)} &= \mathbf{v} - \mathbf{A}\mathbf{v} - \mathbf{B}w - \mathbf{C}\mathbf{\Psi} - \mathbf{D}u - \mathbf{E}T,\\
w^{(1)} &= w - F_1\mathbf{v} - Gw - H_1\mathbf{\Psi},\\
\mathbf{\Psi}^{(1)} &= \mathbf{\Psi} - \mathbf{Z}\mathbf{v} - \mathbf{M}w - \mathbf{K}\mathbf{\Psi} - \mathbf{N}u,\\
u^{(1)} &= u - R_1\mathbf{v} - Q\mathbf{\Psi} - P_1u - S_1T,\\
T^{(1)} &= T - V_1\mathbf{v} - W_1u - U_1T.
\end{aligned}$$

It is easy to see that Λ is a bilinear form defined on $\mathbf{W}_0^{1,2}$ and that Λ is bounded in each variable. We note that

(64)
$$\Lambda\left[\Gamma_1, \Gamma_1\right] =$$
$$= \int_\Sigma \left(\rho I v_i v_i + 2h\rho w^2 + 2hJ\psi_i\psi_i + \zeta u^2 + aIT^2 + \frac{k}{T_0}(IT_{,\alpha}T_{,\alpha} + 2hT^2)\right) da$$
$$+ \int_\Sigma \mathcal{M}\left[(\mathbf{U}^0, \mathbf{U}^0\right] da.$$

In view of our assumptions on the constitutive constants, we see that Λ is coercive on $\mathbf{W}_0^{1,2}$. On the other hand, it is easy to see that the vector

(65)
$$\left(\mathbf{z}^* + \mathbf{v}^*, y^* + w^*, \mathbf{\Phi}^* + \mathbf{\Psi}^*, s^* + u^*, T^* - V_1\mathbf{v}^* - W_1u^*\right),$$

lies in $\mathbf{W}^{-1,2}$. Hence the Lax- Milgram theorem implies that there exists a solution $(\mathbf{v}, w, \mathbf{\Psi}, u, T) \in \mathbf{W}_0^{1,2}$ of the system (62). Thus, the system (61) has also a solution.□

Theorem 4. *The operator $\mathcal{A}$ generates a semigroup of contractions in $\mathcal{Z}$.*

The proof follows from the above lemmas and Lumer-Phillips corollary to the Hille-Yosida theorem.□

Theorem 5. *Assume that $L_\alpha, F, G_\alpha, H, S \in C^1([0,\infty), L^2)$ and $\mathbf{U}_0 \in \mathcal{D}$. Then, there exists a unique solution $\mathbf{U}(t) \in C^1([0,\infty), \mathcal{Z}) \cap C^0([0,\infty), \mathcal{D})$ to the problem (54).*

Since the semigroup defined by the operator $\mathcal{A}$ is contractive, we obtain the estimate

(66)

$$||\mathbf{U}(t)||_\mathcal{Z} \leq$$
$$\leq ||\mathbf{U}_0||_\mathcal{Z} + \int_0^t (||\mathbf{L}(\tau)||_{L^2} + ||F(\tau)||_{L^2} + ||\mathbf{G}(\tau)||_{L^2} + ||H(\tau)||_{L^2} + ||S(\tau)||_{L^2})\, d\tau,$$

which proves the continuous dependence of the solutions upon initial data and body loads. Thus, the problem of bending of thermo-microstretch elastic plates is well posed.

ASYMPTOTIC BEHAVIOR

In this section we study the asymptotic behavior of solutions when the body loads are zero. We will continue to assume that the hypotheses on the constitutive coefficients considered in the previous section hold.

We recall that for any semigroup of contractions such that its generator $\mathcal{A}$ has only the fixed point $\mathbf{0}$ and whose orbits are precompact the orbits tend to the ω-limit sets (see for instance [10]). The structure of the ω-limit sets is determined by the eigen-vectors of eigen-value $i\xi$ (where ξ is a real number) in the closed subspace

$$\mathcal{L} =<< \{\mathbf{U} \in \mathcal{Z}; < \mathcal{A}\mathbf{U}, \mathbf{U} >= 0\} >>, \tag{67}$$

where $<< \mathcal{C} >>$ denotes the closed vectorial subspace generated by the set $\mathcal{C}$.

Lemma 4. *The operator $\mathcal{A}$ has the property*

$$(\mathcal{I} - \mathcal{A})^{-1} \text{ is compact.} \tag{68}$$

Proof. Let $(\hat{\mathbf{v}}_n, \hat{\mathbf{z}}_n, \hat{w}_n, \hat{y}_n, \hat{\mathbf{\Psi}}_n, \hat{\mathbf{\Phi}}_n, \hat{u}_n, \hat{s}_n, \hat{T}_n)$ be a bounded sequence in $\mathcal{Z}$ and $(\mathbf{v}_n, \mathbf{z}_n, w_n, y_n, \mathbf{\Psi}_n, \mathbf{\Phi}_n, u_n, s_n, T_n)$ the sequence of the respective solutions of the system (61). We have

$$\Lambda\left[\Gamma_n, \Gamma_n\right] = \langle \mathcal{N}_n^*, \mathcal{P}_n \rangle_{\mathbf{L^2}}, \tag{69}$$

where

$$\Gamma_n = (\mathbf{v}_n, w_n, \mathbf{\Psi}_n, u_n, T_n), \;\; \mathcal{N}_n^* = (\mathbf{v}_n^{(1)}, w_n^{(1)}, \mathbf{\Psi}_n^{(1)}, u_n^{(1)}, T_n^{(1)}),$$
$$\mathcal{P}_n = (\rho I \hat{\mathbf{v}}, 2h\hat{w}, 2hJ\hat{\mathbf{\Psi}}, \zeta\hat{u}, aI\hat{T}),$$

and

$$\begin{aligned}
\mathbf{v}_n^{(1)} &= \mathbf{v}_n - \mathbf{A}\mathbf{v}_n - \mathbf{B}w_n - \mathbf{C}\mathbf{\Psi}_n - \mathbf{D}u_n - \mathbf{E}T_n,\\
w_n^{(1)} &= w_n - F_1\mathbf{v}_n - Gw_n - H_1\mathbf{\Psi}_n,\\
\mathbf{\Psi}_n^{(1)} &= \mathbf{\Psi}_n - \mathbf{Z}\mathbf{v}_n - \mathbf{M}w_n - \mathbf{K}\mathbf{\Psi}_n - \mathbf{N}u,\\
u_n^{(1)} &= u_n - R_1\mathbf{v}_n - Q\mathbf{\Psi}_n - P_1u_n - S_1T_n,\\
T_n^{(1)} &= T_n - V_1\mathbf{v}_n - W_1u_n - U_1T_n.
\end{aligned}$$

In view of the definition of $(\mathbf{v}_n^{(1)}, w_n^{(1)}, \mathbf{\Psi}_n^{(1)}, u_n^{(1)}, T_n^{(1)})$, it follows that it is a bounded sequence in $\mathbf{L}^2$ and then the sequence $(\mathbf{v}_n, w_n, \mathbf{\Psi}_n, u_n, T_n)$ is bounded in $\mathbf{W}_0^{1,2}$. The Rellich-Kondrasov theorem implies the existence of a subsequence which is convergent in $\mathbf{L}^2$. In a similar way the sequence $(\mathbf{z}_{n_j}, y_{n_j}, \mathbf{\Psi}_{n_j}, s_{n_j})$, where $\mathbf{z}_{n_j} = \mathbf{u}_{n_j} - \hat{\mathbf{v}}_{n_j}, y_{n_j} = w_{n_j} - \hat{w}_{n_j}, \mathbf{\Phi}_{n_j} = \mathbf{\Psi}_{n_j} - \hat{\mathbf{\Psi}}_{n_j}, s_{n_j} = u_{n_j} - \hat{u}_{n_j}$, has a subsequence which is convergent in $\mathbf{L}^2$. It follows that the sequence

$$(\mathbf{v}_{n_j}, \mathbf{z}_{n_j}, w_{n_j}, y_{n_j}, \mathbf{\Psi}_{n_j}, \mathbf{\Phi}_{n_j}, u_{n_j}, s_{n_j}, T_{n_j}),$$

is convergent in $\mathcal{Z}$.□

This lemma implies that the orbits starting in $\mathcal{D}$ are precompact (see [11]). From the inequality (58) and the positivity of the thermal conductivity constant it follows that $\mathcal{A}^{-1}\mathbf{0} = \mathbf{0}$. Now, we can state a result on the asymptotic behavior of solutions.

Let us consider the following problem characterized by the equations

(70)

$$\begin{aligned}
&I\left[(\mu+\kappa)\Delta v_\alpha + (\lambda+\mu)v_{\rho,\rho\alpha}\right] - \\
&2h(\mu+\kappa)v_\alpha - 2h\mu w_{,\alpha} + 2h\kappa\varepsilon_{3\alpha\beta}\psi_\beta + I\lambda_0 u_{,\alpha} + \rho I\xi^2 v_\alpha = 0, \\
&(\mu+\kappa)\Delta w + \mu v_{\nu,\nu} + \kappa\varepsilon_{3\alpha\beta}\psi_{\beta,\alpha} + \rho\xi^2 w = 0, \\
&\kappa\varepsilon_{3\rho\alpha}v_\rho - \kappa\varepsilon_{3\rho\alpha}w_{,\rho} + \gamma\Delta\psi_\alpha + (\alpha+\beta)\psi_{\rho,\rho\alpha} - 2\kappa\psi_\alpha + b_0\varepsilon_{3\alpha\beta}u_{,\beta} + J\xi^2\psi_\alpha = 0, \\
&-I\lambda_0 v_{\rho,\rho} - 2hb_0\varepsilon_{3\alpha\beta}\psi_{\alpha,\beta} + Ia_0\Delta u - (2ha_0 + I\lambda_1)u + \zeta\xi^2 u = 0, \\
&\beta_0 v_{\rho,\rho} + \beta_1 u = 0, \text{ on } \Sigma
\end{aligned}$$

and the boundary conditions

$$\mathbf{v} = \mathbf{0},\ \ w = 0,\ \ \mathbf{\Psi} = \mathbf{0},\ \ u = 0,\ \text{on } \Gamma, \tag{71}$$

Theorem 6. *Let* $\mathbf{U}_0 = (\mathbf{v}_0, \mathbf{z}_0, w_0, y_0, \mathbf{\Psi}_0, \mathbf{\Phi}_0, u_0, s_0, T_0) \in \mathcal{D}$ *and let* $\mathbf{U}(t)$ *be the solution to the initial boundary value problem (54). Then*

$$T(t) \to 0 \text{ as } t \to \infty \ \ in \ \ L^2. \tag{72}$$

Moreover

$$\begin{aligned}
&v_\alpha(t), w(t), \psi_\alpha(t), u(t) \to 0 \ \ as \ \ t \to \infty \ \ in \ \ W_0^{1,2}, \\
&z_\alpha(t), y(t), \phi_\alpha(t), s(t) \to 0 \ \ as \ \ t \to \infty \ \ in \ \ L^2,
\end{aligned} \tag{73}$$

whenever the problem (70), (71) has only the null solution

Proof. We have to study the structure of the ω-limit set. Thus, we consider the equation

$$\hat{\mathcal{A}}\mathbf{U} = i\xi\mathbf{U}, \tag{74}$$

for some real number ξ, where $\mathbf{U} \in \mathcal{D}(\hat{\mathcal{A}})$ and $\hat{\mathcal{A}} = \mathcal{A}_{\mathcal{L}}$ is the generator of a group on $\mathcal{L}$. As $\mathbf{U} \in \mathcal{L}$, then $< \mathcal{A}\mathbf{U}, \mathbf{U} >= 0$. As I, k, h and T_0 are positive, it follows that $T = 0$ and the asymptotic behavior of the temperature is proved. The fact that the equation (74) has nontrivial solutions is equivalent to the fact that the system

$$\begin{aligned}
&\mathbf{A}\mathbf{v} + \mathbf{B}w + \mathbf{C}\mathbf{\Psi} + \mathbf{D}u + \xi^2\mathbf{v} = \mathbf{0}, \\
&F\mathbf{v} + Gw + H_1\mathbf{\Psi} + \xi^2 w = 0, \\
&\mathbf{Z}\mathbf{v} + \mathbf{M}w + \mathbf{K}\mathbf{\Psi} + \mathbf{N}u + \xi^2\mathbf{\Psi} = \mathbf{0}, \\
&R_1\mathbf{v} + Q\mathbf{\Psi} + P_1 u + S_1 T + \xi^2 u = 0, \\
&V_1\mathbf{v} + W_1 u = 0,
\end{aligned}$$

has only the trivial solution. This system is equivalent to system (70) and the result is proved.□

System (70) contains one equation more than the number of unknowns. Thus, generically we expect that the null solution is the unique solution and we expect asymptotic stability. However, it could happen that for certain geometries and choice of the parameters there exists nontrivial solutions. Qualitatively, the situation is similar to the asymptotic behavior in classical thermoelasticity. In this theory there exists some examples of undamped solutions [10].

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