ABSOLUTE EXTENSORS AND ABSOLUTE NEIGHBORHOOD EXTENSORS IN ASYMPTOTIC CATEGORIES

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Abstract. This paper deals with extending maps in asymptotic categories, i.e. in categories consisting of metric spaces and asymptotically Lipschitz coarsely proper maps. We demonstrate certain examples of absolute extensors and absolute neighborhood extensors. We give some conditions under which a version of Borsuk's homotopy extension theorem holds in these categories, and in answer to a problem posed by A. Dranishnikov in [D] we show the failure of a general homotopy extension theorem. Finally, we show that a pair of a Hadamard space and its convex subspace has the homotopy extension property.

Introduction

The large scale geometry focuses on properties of unbounded metric spaces. The method consists in neglecting details of bounded size and focusing on phenomena occurring while approaching the infinity (i.e. while moving away from the base point). In a way, this approach is dual to that of the classical topology of metric spaces, where only infinitesimal distances matter.

A direct motivation for this paper was an extensive paper by A. N. Dranishnikov [D], in which he develops, among other things, a theory of large-scale absolute extensors and absolute neighborhood extensors.

In this paper we consider four slightly different categories: $\bar{\mathcal{A}}$, $\mathcal{A}$, $\bar{\mathcal{A}}_l$ and $\mathcal{A}_l$, each consisting of metric spaces and maps that satisfy the asymptotic Lipschitz condition and that additionally keep the preimages of bounded sets bounded. (The last condition can be seen as continuity of these maps at infinity.) We prove that the Euclidean half-spaces $\mathbb{R}^n_+$ are absolute extensors in these categories. With the notion of homotopy appropriately defined, the homotopy extension theorem (HET) holds true under some additional assumptions. We give an example which shows that the additional assumptions are necessary, i.e., that in the asymptotic categories HET does not

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hold in full generality. This answers the problem 13 of [D], §9, p. 1127 (and, according to [D], it also implies a negative answer to Problem 12 in [D]). In the context of Hadamard spaces, however, we show that the addititonal assumptions can be somehow weakened.

In several definitions and some of the results we follow the paper [D]. However, Dranishnikov only studied three out of four categories considered here, and out of those only two in more detail. We give reasons for which the other two categories could be considered more suitable (see section 5). Thus also results which are close to those of [D] need verification in our extended context. Among them, theorems 2.5 and 5.1 are especially important. The proofs of these two theorems are new and avoid a flaw that persisted in [D].

1. CATEGORIES

Metrics will be denoted by letters ρ and σ (sometimes ρ_X etc.) and a closed ball of radius r and center x_0 in a metric space (X, ρ) will be denoted $B_X(x_0, r)$.

Definition 1.1. a) A function $f : (X, \rho) \longrightarrow (Y, \sigma)$ is an *asymptotically Lipschitz map* if there exist constants $\lambda, s \geq 0$ such that

$$\forall x, x' \in X \quad \sigma(f(x), f(x')) \leq \lambda\, \rho(x, x') + s$$

Though an asymptotically Lipschitz function might be noncontinuous, we still call it “map” to emphasize its regularity. We call λ and s the *Lipschitz constants* of the map f. We also say that such f is (λ, s)-Lipschitz.
b) A map f of metric spaces is *proper* if the preimage of every compact set is compact.
c) A map f is *coarsely proper* if the preimage of every bounded set is bounded.
d) A map f is *linearly proper* if there exist $x_0 \in X$ and constants $\alpha, t > 0$ such that $\sigma(f(x), f(x_0)) \geq \alpha\rho(x, x_0) - t$.
e) A metric space is *proper* if every closed ball in this space is compact.

For example, a function $f(t) = \sqrt{|t|}$ is asymptotically Lipschitz and proper, but neither Lipschitz nor linearly proper.

Definition 1.2. a) The *coarse asymptotic category* $\bar{\mathcal{A}}$ is a category in which objects are metric spaces and morphisms are coarsely proper asymptotically Lipschitz maps.
b) The *asymptotic category* $\mathcal{A}$ is a subcategory of the category $\bar{\mathcal{A}}$ restricted to proper metric spaces and proper asymptotically Lipschitz maps.
c) The subcategory of $\bar{\mathcal{A}}$ (or $\mathcal{A}$), consisting only of the linearly proper maps, will be denoted by $\bar{\mathcal{A}}_l$ (or $\mathcal{A}_l$, respectively).

It is easily seen that all these classes of objects and morphisms indeed form categories. The category $\mathcal{A}_l$ is denoted as $\tilde{\mathcal{A}}$ in [D].

Proposition 1.3. *Morphisms of $\mathcal{A}$ are continuous.*

Proof. Let $f : X \longrightarrow Y$ be such a morphism. It suffices to show that if $A \subseteq Y$ is closed, then $f^{-1}(A)$ intersects each ball along a closed set.

Let $x_0 \in X$ and let $y_0 = f(x_0) \in Y$. Note that for $x \in B_X(x_0, r)$ we have $\sigma(y_0, f(x)) \leq \lambda r + s$. In particular, if $x \in f^{-1}(A)$, then $x \in f^{-1}(A \cap B_Y(y_0, \lambda r + s))$. Consequently

$$f^{-1}(A) \cap B_X(x_0, r) = f^{-1}(A \cap B_Y(y_0, \lambda r + s)) \cap B_X(x_0, r)$$

But A is closed and $B_Y(y_0, \lambda r + s)$ is compact, so by the properness of f the right hand side above is a compact set. ■

In general, a proper map between metric spaces might not be continuous (unlike stated in [D], at the beginning of §1). As an example, take an injective function from any countable, proper, non-discrete metric space into a discrete space.

Proposition 1.4. *A continuous map of proper metric spaces is coarsely proper if and only if it is proper.*

An easy proof is omitted.

The above two propositions show that all proper asymptotically Lipschitz maps of proper metric spaces are morphisms of $\bar{\mathcal{A}}$, and that $\mathcal{A}$ contains all continous $\bar{\mathcal{A}}$-morphisms between its objects.

2. Absolute extensors

In this section, we focus on the case of the categories $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$, though we do mention it if certain definitions or results also apply to $\mathcal{A}$ or $\mathcal{A}_l$. However, we defer the detailed study of the latter two categories till section 5.

Definition 2.1. In the categories $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$, a *subobject* of an object (X, ρ) is any subset A of X with the induced metric $\rho|_{A\times A}$ and the inclusion morphism $i : A \hookrightarrow X$, $i(x) = x$. In the categories $\mathcal{A}$ and $\mathcal{A}_l$, a *subobject* is a closed subset with the induced metric and the inclusion.

Definition 2.2. An object Y of a category $\mathcal{K}$ is its *absolute extensor*, $Y \in AE(\mathcal{K})$, if for any object X in $\mathcal{K}$, its subobject (A, i) and morphism $f : A \longrightarrow Y$ there exists a morphism $\bar{f} : X \longrightarrow Y$ such that $\bar{f} \circ i = f$.

Unless otherwise stated, a Cartesian product of two metric spaces is equipped with an l_1 metric, that is

$$\rho_{X\times Y}\big((x',y'),(x'',y'')\big)=\rho_X(x',x'')+\rho_Y(y',y'')$$

If we took any other standard product metric, e.g. $\sup(\rho_X,\rho_Y)$ or $\sqrt{\rho_X^2+\rho_Y^2}$, we would obtain spaces isomorphic in $\bar{\mathcal{A}}$ (or $\mathcal{A}$, if our spaces were proper).

For $n>0$, let $\mathbb{R}^n$ denote the Euclidean space with l_1 metric and let $\mathbb{R}^n_+$ denote the closed half-space $\{x\in\mathbb{R}^n : \ x_1\geq 0\}$. We identify $\mathbb{R}^n$ with $\{0\}\times\mathbb{R}^n\subseteq\mathbb{R}^{n+1}_+$

Example 2.3. *Neither in $\bar{\mathcal{A}}$, nor in $\mathcal{A}$ there exists a retraction $r:\mathbb{R}^{n+1}_+\longrightarrow\mathbb{R}^n$.*

Proof. It suffices to show that there is no such retraction in $\bar{\mathcal{A}}$. On the contrary, assume that $r:\mathbb{R}^{n+1}_+\longrightarrow\mathbb{R}^n$ is a coarsely proper retraction with Lipschitz constants λ and s. Represent $\mathbb{R}^{n+1}$ as a locally finite complex K of simplices of diameter less than 1. Let V be the set of vertices of K and extend $r|_V$ affinely onto each of the simplices of K to get a function $R:K\to\mathbb{R}^n$.

It can be easily shown that R is a proper retraction of $\mathbb{R}^{n+1}_+$ onto $\mathbb{R}^n$. But such a retraction would extend to a map $\mathbb{R}^{n+1}_+\cup\{\infty\}\to\mathbb{R}^n\cup\{\infty\}$ between the Alexandroff one-point compactifications, yielding a retraction of the $n+1$-disk onto its boundary — a contradiction. ■

Corollary 2.4. *$\mathbb{R}^n\notin AE(\mathcal{A})$ and $\mathbb{R}^n\notin AE(\bar{\mathcal{A}})$, for each integer $n>0$.* ■

Thus extending maps might be a non-trivial task even if we do not require continuity. The next theorem states a positive result in that direction.

Theorem 2.5. *Let A be a subset of a metric space (X,ρ) and $f:A\longrightarrow\mathbb{R}_+$ be an asymptotically Lipschitz map with Lipschitz constants λ,s. Then:*
a) There exists an extension $\bar f:X\longrightarrow\mathbb{R}_+$ of f with the same constants,
b) If f is a coarsely proper map, then $\bar f$ may be chosen to be coarsely proper as well.
c) The same holds for linearly proper maps.

Proof. a) We let $\bar f(x):=f(x)$ for $x\in A$ and

$$\bar f(x):=\inf_{a\in A}\{f(a)+\lambda\,\rho(a,x)\}\quad\text{for } x\in X\setminus A. \tag{1}$$

We shall check that $\bar f$ is asymptotically Lipschitz as desired.

First, let $x,y\in X\setminus A$. For any number $\varepsilon>0$, let $a\in A$ be such that

$$f(a)\leq\bar f(x)-\lambda\,\rho(a,x)+\varepsilon \tag{2}$$

Together with the definition of $\bar f(y)$ and the triangle inequality this yields

$\bar f(y) \le f(a)+\lambda\,\rho(a,y) \le \bar f(x)-\lambda\,\rho(a,x)+\varepsilon+\lambda\,\rho(a,y) \le \bar f(x)+\lambda\,\rho(x,y)+\varepsilon$

By the arbitrariness of ε and the symmetry between x and y, we obtain

$$\bigl|\bar f(x)-\bar f(y)\bigr| \le \lambda\,\rho(x,y)$$

Hence $\bar f$ restricted to $X\setminus A$ is a Lipschitz map with constant λ.

Now let $x\in X\setminus A,\ y\in A$. Again, let $a\in A$ satisfy (2). Then

$$\bar f(y)=f(y)\le f(a)+\lambda\,\rho(a,y)+s$$

since f was (λ,s)-Lipschitz. From this and (2) it follows that

$$\bar f(y)\le \bar f(x)-\lambda\,\rho(a,x)+\varepsilon+\lambda\,\rho(a,y)+s\le \bar f(x)+\lambda\,\rho(y,x)+\varepsilon+s$$

Since ε was chosen arbitrarily, $\bar f(y)\le \bar f(x)+\lambda\,\rho(y,x)+s$.
On the other hand, $\bar f(x)\le \bar f(y)+\lambda\,\rho(y,x)$ by the definition of $\bar f(x)$. Hence

$$\bigl|\bar f(x)-\bar f(y)\bigr|\le \lambda\,\rho(y,x)+s \qquad \text{for } x\in X\setminus A \text{ and } y\in A$$

Thus $\bar f$ is indeed λ,s-Lipschitz on X.

b) We shall prove that $\bar f^{-1}([0,M))$ is bounded for any $M>0$. Let $x_0\in A,\ R>0$ be such that

$$f^{-1}([0,M))\subseteq B_A(x_0,R)\subseteq B_X(x_0,R)$$

Now if $\rho(x_0,x)>R+\frac{M}{\lambda}$ then for $a\in A$ such that $\rho(a,x_0)\le R$

$f(a)+\lambda\,\rho(a,x)\ge\lambda\,\rho(a,x)\ge\lambda(\rho(x,x_0)-\rho(a,x_0))>\lambda\bigl(R+\frac{M}{\lambda}-R\bigr)=M$

and for $a\in A$ with $\rho(a,x_0)>R$ we have

$$f(a)+\lambda\,\rho(a,x)\ge f(a)\ge M$$

Thus if $\rho(x_0,x)>R+\frac{M}{\lambda}$ then $f(a)+\lambda\,\rho(a,x)\ge M$ for all $a\in A$, yielding $\bar f(x)\ge M$. This means that $\bar f^{-1}([0,M))\subseteq B_X\bigl(x_0,R+\frac{M}{\lambda}\bigr)$, as desired.

c) If f is linearly proper, then we can take $R=O(M)$. Hence $R+\frac{M}{\lambda}=O(M)$ as well and therefore $\bar f^{-1}([0,M))\subseteq B_X\bigl(x_0,O(M)\bigr)$, so $\bar f$ is also linearly proper. ■

Corollary 2.6. $\mathbb{R}_+\in AE(\bar{\mathcal{A}})$ *and* $\mathbb{R}_+\in AE(\bar{\mathcal{A}}_l)$. ■

The following lemma leads to further examples of absolute extensors in $\bar{\mathcal{A}}$ and in $\bar{\mathcal{A}}_l$.

Lemma 2.7. *If a metric space space* (Y,σ) *is an absolute extensor in the category of all asymptotically Lipschitz maps (not necessarily coarsely proper), then* $Y\times\mathbb{R}_+\in AE(\bar{\mathcal{A}})\cap AE(\bar{\mathcal{A}}_l)$.

Proof. Consider the case of $\bar{\mathcal{A}}$ first. Let (X,ρ) be a metric space, $A \subseteq X$ be any subset and let $F : A \longrightarrow Y \times \mathbb{R}_+$ be an asymptotically Lipschitz coarsely proper map to be extended onto X. Pick any $x_0 \in A$. Let $g = \pi_Y \circ F$, $f = \pi_{\mathbb{R}_+} \circ F$, and define $h : A \longrightarrow \mathbb{R}_+$ as follows:

$$h(a) := f(a) + \sigma(g(a), g(x_0))$$

Since F is coarsely proper, then of course h is coarsely proper as well. Hence in $\bar{\mathcal{A}}$ we can find its extension $\bar{h} : X \longrightarrow \mathbb{R}_+$. Let $\bar{g}$ denote any asymptotically Lipschitz extension of g onto X (neither g nor $\bar{g}$ needs to be coarsely proper).

Let us define $\bar{F} : X \longrightarrow Y \times \mathbb{R}_+$ as follows:

$$\bar{F}(x) := \big(\bar{g}(x), \max\big(\bar{h}(x) - \sigma\big(\bar{g}(x), \bar{g}(x_0)\big), 0\big)\big)$$

We check that $\bar{F}$ extends F and that $\bar{F} \in \bar{\mathcal{A}}$. First, for $a \in A$ we have

$$\begin{aligned}\bar{F}(a) &= \big(g(a), \max\big(f(a) + \sigma(g(a), g(x_0)) - \sigma(g(a), g(x_0)), 0\big)\big)\\ &= (g(a), \max(f(a), 0)) = F(a)\end{aligned}$$

so indeed $\bar{F}|_A = F$. Next, $\bar{F}$ is asymptotically Lipschitz as a composition of asymptotically Lipschitz functions. To verify that $\bar{F}$ is coarsely proper, take any $M > 0$ and let $R > 0$ be such that $\bar{h}^{-1}([0, M + f(x_0)]) \subseteq B(x_0, R)$. Since $\bar{F}(x_0) = (g(x_0), f(x_0))$, we have

$$\begin{aligned}\sigma_{Y\times\mathbb{R}_+}\big(\bar{F}(x), \bar{F}(x_0)\big) &= \sigma\big(\bar{g}(x), g(x_0)\big) + \big|\max\big(\bar{h}(x) - \bar{h}(x_0), 0\big) - f(x_0)\big|\\ &\geq \bar{h}(x) - f(x_0) \geq M\end{aligned}$$

In the case of $\bar{\mathcal{A}}_l$, the proof is fully analogous. ■

We already know that $\mathbb{R}_+$ is an absolute extensor in the category of metric spaces and asymptotically Lipschitz maps (see theorem 2.5a). In this category (unlike in $\bar{\mathcal{A}}$), the Cartesian product $\times$ is a categorical product. Hence $(\mathbb{R}_+)^n \in AE(\bar{\mathcal{A}}) \cap AE(\bar{\mathcal{A}}_l)$ for all $n \geq 1$.

One can extend theorem 2.5a to state that $\mathbb{R}$ is an absolute extensor for asymptotically Lipschitz maps. The only argument that needs to be added is that if $f : A \longrightarrow \mathbb{R}$ is (λ, s)-Lipschitz, then for $x \in X$ fixed and $a \in A$ varying, the value of $f(a) + \lambda\,\rho(a,x)$ is bounded from below. Indeed, pick any $a_0 \in A$, and since f is (λ, s)-Lipschitz, it easily follows that $f(a) + \lambda\,\rho(a,x) \geq f(a_0) - \lambda\,\rho(a_0,x) - s = \mathit{const}$.

Corollary 2.8. *$\mathbb{R}^n_+ \in AE(\bar{\mathcal{A}}) \cap AE(\bar{\mathcal{A}}_l)$, for all $n \geq 1$.* ■

The analogous fact for $\mathcal{A}$ and $\mathcal{A}_l$ was proved as a theorem 3.3 in [D].

To obtain further nontrivial examples of absolute extensors (at least in $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$), let us first recall the notion of δ-hyperbolic spaces. They have been first defined by Gromov in [G]. One of the equivalent definitions says that a complete geodesic metric space is *δ-hyperbolic*, if for every triangle, each of its sides belongs to the δ-neighborhood of the union of the other two sides. For further reference, see [BBI] or [BH].

Now let us refer to a theorem by Urs Lang (theorem 1.2 in [L]), which says that any complete δ-hyperbolic geodesic metric space Y is an absolute extensor in the category of all metric spaces and asymptotically Lipschitz maps. This, together with our lemma 2.7, leads directly to the following:

Corollary 2.9. *For every complete δ-hyperbolic geodesic metric space Y, the space $Y \times \mathbb{R}_+$ is an absolute extensor in the categories $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$.* ■

3. Absolute neighborhood extensors

Throughout this section, let $\mathcal{K}$ denote any of the 4 categories considered: $\bar{\mathcal{A}}$, $\mathcal{A}$, $\bar{\mathcal{A}}_l$ or $\mathcal{A}_l$, and let $\mathcal{K}_l$ denote either $\bar{\mathcal{A}}_l$ or $\mathcal{A}_l$.

Definition 3.1. Let (X,ρ) be an object of $\mathcal{K}$, let $A \subseteq X$ be its subobject and let $W \subseteq X$ be a set containing A.
a) $W \subset X$ is an *asymptotic neighborhood* [1] of A if $A \subseteq W$ and

$$\lim_{A \ni a \to \infty} \rho(a, X \setminus W) = \infty$$

where $a \to \infty$ means that $\rho(a, a_0) \to \infty$ for some $a_0 \in A$.
b) W is a *linear neighborhood* of A if there exist $x_0 \in X$ and a constant $\kappa > 0$ such that

$$\kappa\, \rho(a, X \setminus W) \geq \rho(a, x_0) \quad \text{for all } a \in A$$

c) For a given map $f : A \longrightarrow Y$ of A into a metric space (Y, σ), we say that W is an *(f, y_0, t)-neighborhood* of A, where $t > 0$ is a constant and $y_0 \in Y$ is a base point, if

$$\kappa\, \rho(a, X \setminus W) \geq \sigma(f(a), y_0) + t \quad \text{for all } a \in A$$

Definition 3.2. Let Y be an object of $\mathcal{K}$.
a) We say Y is an *absolute neighborhood extensor* in $\mathcal{K}$, $Y \in ANE(\mathcal{K})$, if for any object X, its subobject A and a morphism $f : A \longrightarrow Y$, there exist an asymptotic neighborhood W of A and a morphism $\bar{f} : W \longrightarrow Y$, which is an extension of f.
b) Y is $ANE'(\mathcal{K})$ if for every $y_0 \in Y,\ t > 0$ there exists an extension of f onto a (f, y_0, t)-neighborhood.

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[1] This definition is not equivalent to that of an asymptotically open set, given in [D], §4. However any asymptotic neighborhood is an asymptotically open set in the sense of [D].

Since every morphism f is coarsely proper, an (f, y_0, t)-neighborhood is an asymptotic neighborhood and $ANE' \subseteq ANE$.

Proposition 3.3. *For a linearly proper map $f : A \longrightarrow Y$, there exist $y_0 \in Y$ and $t > 0$ such that any (f, y_0, t)-neighborhood is a linear neighborhood.*

Proof. Let f be linearly proper, hence there exist $\alpha, t > 0$, $x_0 \in X$, $y_0 \in Y$ such that for all $a \in A$ we have $\sigma(f(a), y_0) \geq \alpha\, \rho(a, x_0) - t$. Let W be a (f, y_0, t)-neighborhood of A. Then we have

$$\kappa\, \rho(a, X \setminus W) \geq \sigma(f(a), y_0) + t \geq \alpha\, \rho(a, x_0)$$

which means that

$$\tfrac{\kappa}{\alpha}\, \rho(a, X \setminus W) \geq \rho(a, x_0)$$

i.e. W is a linear neighborhood of A (with base point x_0 and constant $\frac{\kappa}{\alpha}$). ■

Corollary 3.4. *A space Y is an $ANE'(\mathcal{K}_l)$ if and only if every morphism $f : A \longrightarrow Y$ in $\mathcal{K}_l$ has an extension to some linear neighborhood.* ■

The following theorem was stated without a proof as proposition 4.1 in [D] for $\mathcal{K} = \mathcal{A}$ and $\mathcal{K} = \mathcal{A}_l$.

Theorem 3.5. (A. N. Dranishnikov) *If $\mathbb{R}_+ \times Y \in AE(\mathcal{K})$, then $Y \in ANE'(\mathcal{K})$.*

Proof. Let (X, ρ) be an arbitrary (proper) metric space, A be a subobject of X and let $f : A \longrightarrow Y$ be a morphism. Let σ denote the l_1 metric in the AE space $\mathbb{R}_+ \times Y$. Let $\bar{f} : X \longrightarrow \mathbb{R}_+ \times Y$ be an extension of $(0, f) : A \longrightarrow \mathbb{R}_+ \times Y$. Let λ, s be the Lipschitz constants of $\bar{f}$. Let us also fix $y_0 \in Y$ and $t > 0$.

Let $U = \{(\tau, y) \in \mathbb{R}_+ \times Y : \ \tau \leq \rho(y_0, y) + s + t\}$. This means that

$$\sigma\big((0, y), (\mathbb{R}_+ \times Y) \setminus U\big) \geq \sigma(y, y_0) + s + t$$

Let $W = \bar{f}^{-1}(U)$. We check that $W \subseteq X$ is a (f, y_0, t)-neighborhood of A. For any $a \in A$ if $x \in X \setminus W$, then $\bar{f}(x) \in (\mathbb{R}_+ \times Y) \setminus U$ and $\sigma\big(\bar{f}(a), \bar{f}(x)\big) \geq \sigma(f(a), y_0) + s + t$. On the other hand, $\lambda\, \rho(a, x) + s \geq \sigma\big(\bar{f}(a), \bar{f}(x)\big)$, therefore $\lambda\, \rho(a, x) \geq \sigma(f(a), y_0) + t$. But since $x \notin W$ was chosen arbitrarily, we obtain

$$\lambda\, \rho(a, X \setminus W) \geq \sigma(f(a), y_0) + t$$

The required extension of the map $f : A \longrightarrow Y$ onto U is given as $\pi_Y \circ (\bar{f}|_W)$, where π_Y denotes the projection of $\mathbb{R}_+ \times Y$ onto Y (note that $\pi_Y|_U$ is a morphism in $\mathcal{K}$). ■

The next corollary follows from theorem 3.3 in [D], corollary 2.8 and theorem 3.5:

Corollary 3.6. $\mathbb{R}^n \in ANE'(\mathcal{K})$. ■

We can find many more interesting examples of $ANE'(\bar{\mathcal{A}})$ if we combine theorem 3.5 with corollary 2.9. We get:

Corollary 3.7. *Every complete δ-hyperbolic geodesic metric space belongs to $ANE'(\bar{\mathcal{A}})$.* ■

Let us turn back to the case of $\mathbb{R}$. By corollary 3.6, $\mathbb{R} \in ANE'(\mathcal{A})$ and $\mathbb{R} \in ANE'(\bar{\mathcal{A}})$. However, a stronger assertion holds:

Theorem 3.8. *If A is a subobject of X in either $\bar{\mathcal{A}}$ or $\mathcal{A}$, then any morphism $f : A \longrightarrow \mathbb{R}$ extends to a morphism $\bar{f}$ of a linear neighborhood W of A.*

Proof. Let f be (λ, s)-Lipschitz. In $\bar{\mathcal{A}}$, for $x \notin A$ the extension is given by the formula

$$\bar{f}(x) := \inf_{a \in A \setminus B(x_0, \frac{s}{\lambda})} \left\{ f(a) + \frac{2\left|f(a) - f(x_0)\right|}{\rho(a, x_0)} \rho(a, x) \right\} \tag{3}$$

where $x_0 \in A$ is an arbitrarily chosen base point. Note that

$$\left|f(a) - f(x_0)\right| \leq \lambda\, \rho(a, x_0) + s \leq 2\lambda\, \rho(a, x_0)$$

hence the coefficient $\frac{2|f(a)-f(x_0)|}{\rho(a,x_0)}$ is less than or equal to 4λ, and it plays a role analogous to λ in (1). Thus the proof of theorem 2.5 can easily be adapted to show that $\bar{f}$ is everywhere greater than $-\infty$ and is asymptotically Lipschitz.

It is slightly harder to show that on the linear neighborhood $W = \bigcup_{a \in A} B_X(a, \frac{1}{4}\rho(a, x_0))$ the map $\bar{f}$ is coarsely proper. We skip the details, as well as the proof in the case of $\mathcal{A}$. ■

In section 6, we shall see that in the last theorem one cannot replace $\mathbb{R}$ by $\mathbb{R}^2$.

4. Homotopy

The following section does not contain original ideas; it rather reorganizes slightly some of the concepts found in [D]. Our goal here is to clarify the definitions, and to verify that for maps extendable onto linear neighborhoods, Homotopy Extension Theorem holds in $\bar{\mathcal{A}}$ as well as in $\mathcal{A}$ (in particular, that in $\bar{\mathcal{A}}_l$ it holds for all maps).

Definition 4.1. Given pointed metric spaces (X, x_0, ρ) and (Y, y_0, σ), we define their *asymptotic product* $X \tilde{\times} Y$ as a set

$$X \tilde{\times} Y = \{(x, y) \in X \times Y : \ \rho(x, x_0) = \sigma(y, y_0)\}$$

with the metric induced by the l_1 metric of $X \times Y$

The projections π_X and π_Y restricted to the asymptotic product are linearly proper, and hence are morphisms, unlike the projections taken on the entire (metric) product.

Consider the pointed space $\mathbf{R}^2_+ = (\mathbb{R}_+ \times \mathbb{R},\, (0,0))$ and its two pointed subspaces $\mathbf{R}_+ = (\{0\} \times \mathbb{R}_+,\, (0,0))$ and $\mathbf{R}_- = (\{0\} \times \mathbb{R}_-,\, (0,0))$. Please note that in each of the four categories considered here, for any object X, there exist natural isomorphisms: $i_+ : X \to X\tilde{\times}\mathbf{R}_+$ and $i_- : X \to X\tilde{\times}\mathbf{R}_-$. As a notational convention, we shall further assume the basepoints to be implicit.

Definition 4.2. A *homotopy* between two maps $f, g : (X, x_0, \rho) \longrightarrow (Y, y_0, \sigma)$ is a morphism $H : X\tilde{\times}\mathbf{R}^2_+ \longrightarrow Y$ such that $H \circ i_+ = f$ and $H \circ i_- = g$. In this context we call $X\tilde{\times}\mathbf{R}^2_+$ the *homotopy space.*

In each of the categories considered, the homotopy space is isomorphic to the space

$$\{(x,t) \in X \times \mathbb{R}_+ :\ 0 \leq t \leq \rho(x, x_0)\}$$

equipped with l_1 metric, which we also denote ρ. The isomorphism maps the subspaces $X\tilde{\times}\mathbf{R}_+$ and $X\tilde{\times}\mathbf{R}_-$ onto $X \times \{0\}$ and $\{(x, \rho(x,x_0)) : x \in X\}$ respectively. Further on we shall find it convenient to identify these two realizations of the homotopy space.

The following theorem was proved in the case of $\mathcal{A}$ by A. N. Dranishnikov in [D] (theorem 4.3). The proof given below works both for $\mathcal{A}$ and $\bar{\mathcal{A}}$.

Theorem 4.3. *(Homotopy extension theorem for $\mathcal{A}$ and $\bar{\mathcal{A}}$, HET) If A is a subobject of X and $f : X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+ \longrightarrow Y$ has an extension g onto a linear neighborhood W of $X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+$ in $X\tilde{\times}\mathbf{R}^2_+$, then f also has an extension $\bar{f} : X\tilde{\times}\mathbf{R}^2_+ \longrightarrow Y$.*

Proof. Let us remind the proof in the classical (topological) case: we construct a morphism (e.g. a continuous function) $F : X \times I \longrightarrow W$ such that $F|_{X\times 0 \cup A \times I} = id$ (using the compactness of the interval and the Urysohn lemma) and define $\bar{f} = g \circ F$. The proof in our case is similar.

Let W be the linear neighborhood of $X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+$ and let $\kappa > 0$ be such that for any $\xi \in A\tilde{\times}\mathbf{R}^2_+$ there holds $\kappa\, \rho(\xi, X \setminus W) \geq \rho\big(\xi, (x_0, 0)\big)$. Let $V = \bigcup_{a\in A} B_X\big(a, \frac{1}{\kappa}\, \rho(a, x_0)\big)$. We see that $V\tilde{\times}\mathbf{R}^2_+ \subseteq W$.

For brevity, let us denote $||x|| := \rho(x, x_0)$. Consider a map $\phi : A \cup (X \setminus V) \longrightarrow \mathbb{R}_+$ defined as $\phi(x) = ||x_0||$ for $x \in A$ and $\phi(x) = 0$ for $x \in X \setminus V$. Of course, this map is not proper, but it is Lipschitz, since for $a \in A,\ x \in X \setminus V$ we have $|\phi(a) - \phi(x)| = \phi(a) = ||a|| \leq \kappa\, \rho(a, X \setminus V) \leq \kappa\, \rho(a, x)$, and on A it is a short map (i.e. Lipschitz with $\lambda = 1$). The theorem 2.5a) implies that

ϕ has a Lipschitz extension $\bar{\phi}$. If we take $\psi(x) = \min\big(\bar{\phi}(x), ||x||\big)$, we get a Lipschitz extension ψ of ϕ, satisfying $\psi(x) \leq ||x||$.

Now define $F : X\tilde{\times}\mathbf{R}^2_+ \longrightarrow W$ as follows:

$$F(x,t) = \Big(x, \frac{\psi(x)}{||x||}t\Big) \quad \text{for } x \neq x_0 \qquad \text{and} \qquad F(x_0,0) = (x_0,0)$$

Clearly, F restricted to $X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+$ is an identity and the image of F takes values in $V\tilde{\times}\mathbf{R}^2_+ \cup X\tilde{\times}\mathbf{R}_+ \subseteq W$. Also, F is (coarsely) proper: we have

$$\rho\big(F(x,t),(x_0,0)\big) \geq ||x|| \geq \tfrac{1}{2}\rho\big((x,t),(x_0,0)\big)$$

since $0 \leq t \leq ||x||$. To show that F is Lipschitz, first let us note that for $||x'|| \leq ||x''||$ we have

$$\begin{aligned}
&\left|\frac{\psi(x')}{||x'||}t' - \frac{\psi(x'')}{||x''||}t''\right| \leq \\
&\leq \quad \psi(x')t'\left|\frac{1}{||x'||} - \frac{1}{||x''||}\right| + \frac{\psi(x')}{||x''||}|t'-t''| + \frac{t''}{||x''||}|\psi(x') - \psi(x'')| \leq \\
&\leq \quad \frac{\psi(x')}{||x'||}\,\frac{t'}{||x'||}\,\big|\,||x''|| - ||x'||\,\big| + \frac{\psi(x')}{||x'||}|t'-t''| + \frac{t''}{||x''||}|\psi(x') - \psi(x'')| \leq \\
&\leq \quad \lambda\cdot 1\cdot\rho(x',x'') + \lambda\,|t'-t''| + 1\cdot\lambda\cdot\rho(x',x'') \leq 2\lambda\big(\rho(x',x'') + |t'-t''|\big)
\end{aligned}$$

where λ denotes the Lipschitz constant of ψ. By symmetry, the inequality holds as well for $||x'|| \geq ||x''||$. Consequently

$$\begin{aligned}
&\rho\big(F(x',t'),F(x'',t'')\big) \leq \\
&\leq \quad \rho(x',x'') + 2\lambda\left(\rho(x',x'') + |t'-t''|\right) \leq (2\lambda+1)\big(\rho(x',x'') + |t'-t''|\big) = \\
&= \quad (2\lambda+1)\,\rho\big((x',t'),(x'',t'')\big)
\end{aligned}$$

Now we conclude, just as in the classical case, that $\bar{f} := g \circ F$ is the required extension. ■

Corollary 4.4. *In the categories $\bar{\mathcal{A}}_l$ and $\mathcal{A}_l$, HET holds in full generality.* ■

The proof of theorem 4.3 in the case of $\bar{\mathcal{A}}$ is further discussed in remark 5.4.

5. $\mathcal{A}$ VERSUS $\bar{\mathcal{A}}$

In this section we prove and discuss a theorem analogous to theorem 2.5, but in $\mathcal{A}$. Further, we discuss certain simplifications that we could have made if we wanted to prove theorems 3.5 and 4.3 only in $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$, disregarding the other two categories considered here. We conclude that for studying asymptotic topology, $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$ seem to be a better choice than $\mathcal{A}$ or $\mathcal{A}_l$.

Theorem 5.1. *Let A be a closed subset of a metric space (X, ρ). In the category $\mathcal{A}$, each morphism $f : A \longrightarrow \mathbb{R}_+$ with Lipschitz constants λ and s has an extension $\bar{f} : X \longrightarrow \mathbb{R}_+$ with Lipschitz constants λ and $13s$.*

Proof. Put $d = \frac{s}{3\lambda}$. Let us denote $X_0 := \{x \in X : \ \rho(x, A) \geq d\}$. First we extend f to a map $\bar{f}_0 : A \cup X_0 \longrightarrow \mathbb{R}_+$, using for $x \in X_0$ the same formula (1) that we have used in the proof of theorem 2.5:

$$\bar{f}_0(x) := \inf_{a \in A} \{f(a) + \lambda\, \rho(a, x)\} \quad \text{for } x \in X_0$$

We have already proven there that this formula gives a classical λ-Lipschitz map on $X \setminus A$, so in particular $\bar{f}_0\big|_{X_0}$ is continuous. Since X_0 and A are both open-closed in $X_0 \cup A$, then $\bar{f}_0$ is continuous. We also know that $\bar{f}_0$ is (λ, s)-Lipschitz.

Now we shall find an extension over the remaining part of X. For $i = 0, 1, 2, \ldots$ let

$$A_i := f^{-1}([2si; 2s(i+1)]) \qquad B_i := \{x \in X : \rho(x, A_i) \leq d\}$$

The sets B_i cover the set $\{x \in X : 0 < \rho(x, A) < d\}$ on which $\bar{f}$ needs to be defined yet. We shall first extend $\bar{f}_0$ onto the sets B_{2i}, and then onto the sets B_{2i+1}.

First observe that for $|i - j| \geq 2$ we have $B_i \cap B_j = \emptyset$. Indeed, for $a \in A_i$, $a' \in A_j$ we have $2s \leq |f(a) - f(a')| \leq \lambda\, \rho(a, a') + s$, hence $\rho(a, a') \geq \frac{s}{\lambda} = 3d$. Therefore, $\rho(A_i, A_j) \geq 3d$ and $\rho(B_i, B_j) \geq d > 0$ since B_i is a d-neighborhood of A_i.

Now as $\bar{f}_0$ is (λ, s)-Lipschitz, we deduce that on the intersection of the set B_i and its domain, $\bar{f}_0$ takes values in the set

$$I_i := [2si - \lambda d - s, 2s(i+1) + \lambda d + s] \cap \mathbb{R}_+ = \big[2s\big(i - \tfrac{2}{3}\big), 2s\big(i + \tfrac{5}{3}\big)\big] \cap \mathbb{R}_+$$

By Tietze theorem, for each $i \geq 1$ there exists a continuous extension $\bar{f}^{2i} : B_{2i} \longrightarrow I_{2i}$ of $\bar{f}_0\big|_{B_{2i} \cap (X_0 \cup A)}$. We take

$$\bar{f}_1 := \bar{f}_0 \cup \bigcup_{i \in \mathbf{N}} \bar{f}^{2i}$$

Since f was a proper map, each bounded set intersects only finitely many of the sets B_i (because it does intersect finitely many sets A_i). Hence, the family of domains of the maps $\bar{f}_0$ and $\bar{f}^{2i}$ is locally finite. Since the sets B_{2i} are pairwise disjoint, the only nonempty intersections between domains of maps $\bar{f}^{2i}$ for $i \geq 0$ are of the form $B_{2i} \cap (X_0 \cup A)$, where the respective functions do agree. Finally, all the domains are closed subsets of their union. Therefore $\bar{f}_1$ is well-defined and continuous on the set

$$X_1 := X_0 \cup A \cup \bigcup_{i \in \mathbf{N}} B_{2i}$$

Now we only have to fill in the "holes" in X_1 corresponding to the sets B_{2i+1}. The set $D_i := B_{2i+1} \cap X_1$ is a subset of $(B_{2i+1} \cap X_0) \cup B_{2i} \cup B_{2i+2}$. Hence on D_i the function $\bar{f}_1$ takes values in the interval $I_{2i} \cup I_{2i+1} \cup I_{2i+2}$. Just as before, we take a family of maps $\bar{f}^{2i+1} : B_{2i+1} \longrightarrow I_{2i} \cup I_{2i+1} \cup I_{2i+2}$ extending the maps $\bar{f}_1\big|_{B_{2i+1} \cap X_1}$ respectively. Again, their union is a well-defined continuous map and we finally set

$$\bar{f} := \bar{f}_1 \cup \bigcup_{i \in \mathbf{N}} \bar{f}^{2i+1}$$

Now we have to check that $\bar{f}$ is a morphism in $\mathcal{A}$. First, $\bar{f}$ is coarsely proper. Indeed, take any $M > 0$. The map $\bar{f}_0$ is proper, hence the preimage $\bar{f}_0^{-1}([0; M))$ is bounded. On the other hand, only finitely many intervals I_i have nonempty intersections with $[0; M)$. Since each of the sets B_i is bounded, so is the set $\big(\bar{f}^{-1}([0; M))\big) \setminus (X_0 \cap A)$.

Second, we have $|\bar{f}(x) - \bar{f}(x')| \le \lambda\rho(x, x') + 13s$ for each $x, x' \in X$. To prove it, note that if $x \in B_i$, then

$$\bar{f}(x) \in I_{i-1} \cup I_i \cup I_{i+1} = \big[2s\big(i - \tfrac{5}{3}\big), 2s\big(i + \tfrac{8}{3}\big)\big] \cap \mathbb{R}_+$$

Moreover for each $x \in B_i$ there exits $a \in A_i$ such that $\rho(x, a) \le d$ and $f(a) \in [2si, 2s(i + 1)]$. Therefore for these x and a we have $\big|\bar{f}(x) - f(a)\big| \le \frac{16}{3}s$.

Thus if $x \in B_i$ and $x' \in B_j$, then for the apropriate $a \in A_i$ and $a' \in A_j$ we have

$$\begin{aligned}\big|\bar{f}(x) - \bar{f}(x')\big| &\le \big|\bar{f}(x) - f(a)\big| + |f(a) - f(a')| + \big|\bar{f}(x') - f(a')\big| \le \\ &\le \tfrac{32}{3}s + |f(a) - f(a')|\end{aligned}$$

However, since f was (λ, s)-Lipschitz, the triangle inequality yields

$$\begin{aligned}|f(a) - f(a')| &\le \lambda\,\rho(a, a') + s \le \lambda(\rho(a, x) + \rho(x, x') + \rho(a', x')) + s \le \\ &\le \lambda\,\rho(x, x') + 2\lambda d + s = \lambda\,\rho(x, x') + \tfrac{5}{3}s\end{aligned}$$

Put together, we get

$$\big|\bar{f}(x) - \bar{f}(x')\big| \le \lambda\,\rho(x, x') + \tfrac{37}{3}s \le \lambda\,\rho(x, x') + 13s$$

Please note that $A_i \subseteq B_i$, so we have also covered the case when either x or x' belongs to A. The last case to be checked is when $x \in B_i$ and $x' \in X_0$.

Again, let $a \in A_i$ and $\rho(x,a) \le d$. By the definition of $\bar{f}(x')$, there holds

$$\begin{aligned}\bar{f}(x') &\le f(a) + \lambda\,\rho(a,x') \le \\ &\le \left(\bar{f}(x) + \left|\bar{f}(x) - f(a)\right|\right) + (\lambda\,\rho(a,x) + \lambda\,\rho(x,x')) \le \\ &\le \bar{f}(x) + \lambda\,\rho(x,x') + \tfrac{17}{3}s\end{aligned}$$

On the other hand, by the formula for f_0 there is $a' \in A$ such that $\bar{f}(x') \ge f(a') + \lambda\,\rho(a',x') - s$. This gives

$$\begin{aligned}\bar{f}(x) &\le f(a) + \tfrac{16}{3}s \le f(a') + \lambda\,\rho(a,a') + \tfrac{19}{3}s \le \\ &\le \left(\bar{f}(x') - \lambda\,\rho(a',x') + s\right) + \lambda\,\rho(a,a') + \tfrac{19}{3}s \le \\ &\le \bar{f}(x') + \lambda\,\rho(a,x') + \tfrac{22}{3}s \le \bar{f}(x') + \lambda\,\rho(a,x) + \lambda\,\rho(x,x') + \tfrac{22}{3} \le \\ &\le \bar{f}(x') + \lambda\,\rho(x,x') + \tfrac{25}{3}s\end{aligned}$$

The last two inequalities yield

$$\left|\bar{f}(x) - \bar{f}(x')\right| \le \lambda\,\rho(x,x') + \tfrac{25}{3}s \le \lambda\,\rho(x,x') + 13s$$

This concludes the proof. ■

Corollary 5.2. $\mathbb{R}_+ \in AE(\mathcal{A})$. ■

A theorem very similar to the last one has been proven first in [D] by A. N. Dranishnikov as Theorem 3.1. However, the proof presented there had some flaws, the most important being that it only worked for geodesic metric spaces. Indeed, Dranishnikov extended the map inductively onto consecutive sets C_i such that for some $\mu, m > 0$, the μ-neighborhood of C_i was a subset of C_{i+1} and the extension restricted to $C_{i+1} \setminus C_i$ had values in $[mi, m(i+1)]$. Then he concluded that for $j = i+k$, $x \in C_i$, $y \in C_j \setminus C_{j-1}$ there held $\rho(x,y) \ge \mu(k-1)$, which would only be true in geodesic spaces. In fact, in a non-geodesic space X the sets C_i do not even need to cover X. Moreover in his proof Dranishnikov had to increase the Lipschitz constant λ by a factor $\alpha > 1$.

Remark 5.3. *Comparison of theorems 2.5 and 5.1.*

As we have just seen, the theorem is not much harder to prove in $\mathcal{A}$ than in $\bar{\mathcal{A}}$. However, there is a lot of technical details concerning continuity, which we have to struggle with in $\mathcal{A}$, not being rewarded much more. In fact, the topology of metric spaces is induced only by small values of metric (less than any given constant), while we tend to concentrate on large scale problems.

Remark 5.4. *Simplifications in case of $\bar{\mathcal{A}}$ - discussion of theorems 4.3 and 3.5.*

If we forgot about continuity and only worked in $\bar{\mathcal{A}}$, we could have changed some of the definitions from section 3. First, it would suffice to define a linear neighborhood by the condition

$$\exists \kappa, t > 0 \quad \kappa\, \rho(a, X \setminus W) \geq \rho(a, x_0) - t$$

In such case, the maps ϕ, ψ and F from the proof of theorem 4.3 were not necessarily continuous, but they were still asymptotically Lipschitz and the entire proof would keep working. Second, we might have consequently changed the definition 3.1c as follows: W is an f-neighborhood if

$$\exists \kappa, t > 0,\ y_0 \in Y \quad \kappa\, \rho(a, X \setminus W) \geq \sigma(f(a), y_0) - t$$

Thanks to the new definition of linear neighborhood, we would still be able to prove proposition 3.3. Finally, the definition of ANE' would of course say that every f needs to have an extension onto some f-neighborhood. Now the proof of the theorem 3.5 would become even simpler.

In section 3 we decided to give the more complicated versions of definitions to take care of phenomena that take place only on the boundaries of asymptotic neighborhoods (like continuity in the case of $\mathcal{A}$). If we define the *width* of a set as a supremum of distance from its points to the points of its complement, then the boundaries mentioned above have finite width. However, the behavior of objects of finite width seems to be irrelevant for us. For example, in $\bar{\mathcal{A}}$ every map can be trivially extended over the k-boundary of its domain (i.e. the k-neighborhood minus the domain itself), by setting in each point x a value, taken by the map in any point of its domain, lying within the distance of $2k$ from x.

The observations presented in this section should convince us that the categories $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_l$ are more efficient tools for studying asymptotic topology than $\mathcal{A}$ and $\mathcal{A}_l$. In particular, if we need to extend homotopies, $\bar{\mathcal{A}}_l$ seems to be a best choice. However, in this paper we examine all four categories in order to display the differences between them and to stay closer to the paper [D], where $\mathcal{A}$ is the main object of study.

6. General HET — a counterexample

In [D], Dranishnikov posed a natural question: in asymptotic categories, does HET hold true in full generality, i.e. does it hold for any map f with its values in $Y \in ANE$? The following example shows that it does not even hold for $Y = \mathbb{R}^2$, no matter how we pick the definition of asymptotic neighborhood (please note it does hold for $Y = \mathbb{R}$ by virtue of theorem 3.8).

First, we need to introduce some auxiliary terminology.

Definition 6.1. If J is an oriented arc, with initial point p and terminal point q, then the *index* of a continuous map $h : J \longrightarrow \mathbb{R}^2 \setminus \{0\}$ is defined as

$$\operatorname{ind}(h) := \tfrac{1}{2\pi}(a(q) - a(p))$$

where $a : J \longrightarrow \mathbb{R}$ is any continuous map satisfying $a(t) \equiv \arg h(t) \pmod{2\pi}$.

Clearly, if $h(p) = h(q)$, then $\operatorname{ind}(h) \in \mathbf{Z}$. Also, if we partition J into smaller arcs $J_1, \dots J_n$, then $\operatorname{ind} h = \operatorname{ind} h|_{J_1} + \dots + \inf h|_{J_n}$.

Example 6.2. *Let* $X = \{(x, y) \in \mathbb{R}^2 : \ y = x^2\}$*, let* $\xi_0 = (0, 0)$ *be the base point and let* $A = \{(x, y) \in X : x \leq 0\}$*. There exists an asymptotically Lipschitz proper map* $f : X \tilde{\times} \mathbf{R}_+ \cup A \tilde{\times} \mathbf{R}^2_+ \longrightarrow \mathbb{R}^2$ *which in the category* $\bar{\mathcal{A}}$ *does not extend over* $X \tilde{\times} \mathbf{R}^2_+$.

Proof. Let σ denote the l_1 metric on $\mathbb{R}^2$ and let ρ be the induced metric on X. Below we consider the homotopy space as a subset of $X \times \mathbb{R}_+$. Precisely, we take $X \tilde{\times} \mathbf{R}^2_+ = \{(x, y, t) \in \mathbb{R}^3 : y = x^2 \text{ and } 0 \leq t \leq x^2\}$. This is slightly different from what we had in the proof of HET, since x^2 is not exactly equal to $\rho(\xi_0, (x, x^2))$. But it is still isomorphic to the homotopy space defined before. We extend ρ onto $X \tilde{\times} \mathbf{R}^2_+$ by the formula $\rho\big((x', y', t'), (x'', y'', t'')\big) = |x' - x''| + |y' - y''| + |t' - t''|$ The asymptotic product $\tilde{\times}$ of a subset of X by a subset of $\mathbf{R}^2_+$ should be understood as a respective subspace.

Also, let σ denote the l_1 metric on $\mathbb{R}^2$. Let $B = \{(x, y) \in X : x \geq 0\}$. On $A \tilde{\times} \mathbf{R}^2_+$, let

$$f(x, y, t) := (x, t)$$

The values of f grow slowly along the left branch $A \times \{0\}$ of the parabola, but they grow rapidly (linearly) as a function of time t above the set A.

Next, for $(x, y) \in B$ let

$$f(x, y, 0) := x(\cos(2\pi x), \sin(2\pi x))$$

The right branch of the parabola X is being wound onto an arithmetic spiral with approximately constant linear velocity.

Let us check that f is an asymptotically Lipschitz proper map on its domain, $A \tilde{\times} \mathbf{R}^2_+ \cup B \tilde{\times} \mathbf{R}_+ = A \tilde{\times} \mathbf{R}^2_+ \cup X \tilde{\times} \mathbf{R}_+$. It is asymptotically Lipschitz on each of the two branches $A \times \{0\}$, $B \times \{0\}$ of the parabola $X \times \{0\}$ as well as along the time axis above the set $A \times \{0\}$. And for any two symmetrical points lying on the opposite branches of parabola, say $\xi_1 = (x, x^2, 0)$ and $\xi_2 = (-x, x^2, 0)$, we have $\sigma(f(\xi_i), f(\xi_0)) = O(|x|)$ for $i = 1, 2$, hence $\sigma(f(\xi_1), f(\xi_2)) = O(|x|) = O(\rho(\xi_1, \xi_2))$.

Now we only need to prove that f has no extension onto $X \times \mathbb{R}_+$. On the contrary, suppose that $\bar{f}$ is such an extension. We might assume that $\bar{f}$ is continuous, because we can always "improve" $\bar{f}$ to be so, like in example 2.3. We only need some care to keep the original values of f unchanged.

(Continuity has nothing to do with the nature of this example, but here allows to use familiar topological notions without the need of reformulating the definitions for the noncontinuous asymptotically Lipschitz case.)

Hence $\bar{f}$ is an asymptotically Lipschitz proper map with Lipschitz constants λ, s. Pick a number $R_0 > 0$ such that for $|x| \geq R_0$ we have $\bar{f}(x, x^2, t) \neq 0$. Such a number exists, since $\bar{f}$ is proper.

For $x > R_0$, let there be curves given by the parameterizations:

$$\begin{aligned} j_x &: [R_0^2, x^2] \longrightarrow \mathbb{R}^2 \setminus \{0\} & \qquad j_x(\tau) &= \bar{f}(\sqrt{\tau}, \tau, 0) \\ j'_x &: [0, x^2] \longrightarrow \mathbb{R}^2 \setminus \{0\} & \qquad j'_x(\tau) &= \bar{f}(x, x^2, \tau) \\ j''_x &: [R_0^2, x^2] \longrightarrow \mathbb{R}^2 \setminus \{0\} & \qquad j''_x(\tau) &= \bar{f}(\sqrt{\tau}, \tau, \tau) \\ j_0 &: [0, R_0^2] \longrightarrow \mathbb{R}^2 \setminus \{0\} & \qquad j_0(\tau) &= \bar{f}(R_0, R_0^2, \tau) \end{aligned}$$

Note that $j_x \cdot j'_x \cdot (-j''_x) \cdot (-j_0)$ forms a loop (here "$\cdot$" means concatenation of arcs and "$-$" means reversing the orientation) and that for any two x such loops are homotopic in $\mathbb{R}^2 \setminus \{0\}$ (in the topological sense). Consequently, for $x > R_0$ there should be

$$\operatorname{ind} j_x + \operatorname{ind} j'_x - \operatorname{ind} j''_x - \operatorname{ind} j_0 = \mathit{const}$$

It is easy to calculate $\operatorname{ind} j_x$: since $j_x(\tau) = f(\sqrt{\tau}, \tau, 0) = \sqrt{\tau}\, \exp 2\pi i \sqrt{\tau}$, we have $\operatorname{ind} j_x = x - R_0 = x + O(1)$. Next, we have j'_x. Let $\mu = 2\lambda + \frac{s}{R_0}$. By the Lipschitz condition for $\bar{f}$, we get

$$\begin{aligned} \sigma\big(j'_x(\tau), (x, \tau)\big) &= \sigma\big(\bar{f}(x, x^2, \tau), f(-x, x^2, \tau)\big) = \sigma\big(\bar{f}(x, x^2, \tau), \bar{f}(-x, x^2, \tau)\big) \\ &\leq \lambda(2x) + s \leq \mu x \end{aligned}$$

since $x > R_0$. In particular, for $\tau > \mu x$ if $j'_x(\tau) = (x', y')$, then $\sigma\big((x', y'), (x, \tau)\big) \leq \mu x$ which implies $y' > 0$. Therefore $\operatorname{ind} j'_x|_{[\mu x, x^2]} \leq \frac{1}{2}$. And what is the value of $\operatorname{ind} j'_x|_{[0, \mu x]}$? Let

$$\beta(R) = \inf\big\{\sigma\big(\bar{f}((x, x^2, t), 0\big) : \ 0 \leq t \leq x^2, \ |x| \geq R\big\}$$

Since $\bar{f}$ is proper, we have $\lim_{R \to \infty} \beta(R) = \infty$. Now we see that j'_x maps the interval $[0, \mu x]$ into $\mathbb{R}^2 \setminus B(0, \beta(x))$. But j'_x is asymptotically Lipschitz with exactly the same constants λ, s as $\bar{f}$. Hence if only $\beta(x) > s$, then the number of leaps j'_x takes around 0 on $[0, \mu x]$ is not greater than $\frac{\lambda \mu x}{\beta(x)}$. That is,

$$\Big|\operatorname{ind} j'_x|_{[0, \mu x]}\Big| \leq \frac{\lambda \mu x}{\beta(x)} + 1 = o(x)$$

which means that $\operatorname{ind} j'_x = o(x)$.

Now we only need a similar calculation for j''_x. By the Lipschitz inequality for $\bar{f}$ we have

$$\sigma\big(j''_x(\tau), (-\sqrt{\tau}, \tau)\big) = \sigma\big(\bar{f}(\sqrt{\tau}, \tau, \tau), \bar{f}(-\sqrt{\tau}, \tau, \tau)\big) \leq \lambda(2\sqrt{\tau}) + s \leq \mu\sqrt{\tau}$$

(as we take $\tau \geq R_0^2$). For $\sqrt{\tau} > \mu$ and $j''_x(\tau) = (x'', y'')$ this gives $\sigma\big((x'', y''), (-\sqrt{\tau}, \tau)\big) \leq \mu\sqrt{\tau}$ which implies $|y'' - \tau| \leq \mu\sqrt{\tau} < \tau$, which in turn gives $y'' > 0$. We conclude that

$$\big|\operatorname{ind} j''_x\big| \leq \Big|\operatorname{ind} j''_x|_{[R_0^2, \mu^2]}\Big| + 1 = O(1)$$

This yields

$$\operatorname{ind} j_x + \operatorname{ind} j'_x - \operatorname{ind} j''_x - \operatorname{ind} j_0 = x + O(1) + o(x) + O(1) + O(1) = x + o(x) \neq const$$

A contradiction. ■

7. Extending maps on Hadamard spaces

Studying the counterexample from the previous section carefully, one may ask if it is possible to give a nicer example with spaces X and A geodesic. This section gives a partial negative answer to this question, under the additional assumption of nonpositive curvature.

Recall that a *Hadamard space* is a complete simply connected metric space of nonpositive curvature in the sense of CAT(0)-condition. In such space every two points x, x' are connected by a unique geodesic segment, whose length is equal to the distance from x to x' (Cartan–Hadamard Theorem). We denote the segment by $[xx']$ and its length by $|xx'|$. As a reference for Hadamard spaces, see [BBI], [B] or [BH].

We say that a subset A of Hadamard space is *convex*, if for every two points $a, a' \in A$, the geodesic segment $[aa']$ is also contained in A.

If X is a Hadamard space, $A \subseteq X$ is its closed convex subset and $x \in X$ is a point, then there exists a unique point $\pi(x) \in A$ such that $|x\pi(x)|$ is minimal. The retraction $\pi : X \longrightarrow A$ is called the *nearest point projection* or simply *projection* onto the subset A. It turns out to be 1-Lipschitz. For a proof, see [BH], Proposition 2.4, pp. 176–177.

There is one more fact important for us. If A is a convex closed subset of a Hadamard space X, then for $t > 0$ the closed t-neighborhood of A:

$$A_t := \{x \in X : |x\pi(x)| \leq t\}$$

is also a closed convex subset. This is due to convexity of the distance function to A. See [BH], Corollary 2.5, p. 178.

Theorem 7.1. *Let A be a closed convex subset of a Hadamard space X. Let $x_0 \in X$ be a base point and let*

$$W := \bigcup_{a \in A} B_X(a, \tfrac{1}{4}|ax_0|)$$

Then W is a linear neighborhood of A and the restricted projection $\pi|_W$ is a coarsely proper 1-Lipschitz retraction onto A.

Proof. Clearly W satisfies the definition of a linear neighborhood with a constant $\kappa = 4$.

Let $x \in W$ and let $a \in A$ be such that $x \in B_X(a, \frac{1}{4}|ax_0|)$. Let $a' = \pi(x) \in A$. Therefore $|xa'| \leq |xa|$, hence $|aa'| \leq |xa| + |xa'| \leq 2|xa| \leq \frac{1}{2}|ax_0|$. We use the triangle inequality once again and get $|a'x_0| \geq |ax_0| - |aa'| \geq |ax_0| - \frac{1}{2}|ax_0| = \frac{1}{2}|ax_0|$. Since $|xx_0| \leq |ax_0| + |ax| \leq \frac{5}{4}|ax_0|$, we obtain $|a'x_0| = |\pi(x)x_0| \geq \frac{2}{5}|xx_0|$. Hence $\pi|_W$ is coarsely proper (even linearly proper). ■

Note that a Hadamard space needs not be proper or even locally compact, so we cannot claim properness of $\pi|_W$.

Corollary 7.2. *Every asymptotically Lipschitz coarsely proper map defined on a closed convex subset A of a Hadamard space X can be extended to a coarsely proper map $\bar{f}$ having the same Lipschitz constants as f and defined on a linear neighborhood of A.* ■

We cannot use the above corollary to prove the homotopy extension theorem in case when A is a convex susbet of a Hadamard space X, because there is no reasonable metric on $X \tilde{\times} \mathbf{R}^2_+$ in which the subset $X \tilde{\times} \mathbf{R}_+ \cup A \tilde{\times} \mathbf{R}^2_+$ is convex. Instead, in theorem 7.5 below, we shall build a coarsely proper Lipschitz retraction $R : X \tilde{\times} \mathbf{R}^2_+ \longrightarrow X \tilde{\times} \mathbf{R}_+ \cup A \tilde{\times} \mathbf{R}^2_+$ directly.

To this end, denote $A_0 = A$ and for $t > 0$ let A_t be the closed t-neighborhood of A in X and $\pi_t : X \longrightarrow A_t$ be the projection onto A_t.

Lemma 7.3. *If $x \notin A_s$ and $x' = \pi_s(x)$, then $|x'\pi(x')| = s$*

Proof. Since $x' \in A_s$, we have $|x'\pi(x')| \leq s$. Assume that the equality does not hold. Let then $x'' \in [x'x] \cap A_s$ be a point such that $s > |x''\pi(x')| > |x'\pi(x')|$. We see that $|xx''| = |xx'| - |x'x''| < |xx'|$, but x' was to be a point of A_s closest to x, a contradiction. ■

Lemma 7.4. *If $x \notin A_s$, then $\pi_s(x) \in [x\pi(x)]$ and $\pi(\pi_s(x)) = \pi(x)$ and also $|\pi_s(x)\pi(x)| = s$.*

Proof. Let $x' = \pi_s(x)$ and let $x_s \in [x\pi(x)]$ be such a point that $|x_s\pi(x)| = s$. Hence $x_s \in A_s$ and we have $|xx_s| \geq |xx'|$. We already know that

$|x'\pi(x')| = s$. Putting it together, we have

$$\begin{aligned}|xx_s| + s \geq |xx'| + s = |xx'| + |x'\pi(x')| \geq |x\pi(x')| \geq |x\pi(x)| = \\ = |xx_s| + |x_s\ pi(x)| = |xx_s| + s\end{aligned}$$

Therefore all of the above inequalities are actually equalities. In particular, $|xx_s| = |xx'|$, so $x_s = x'$ (as $x_s \in A_s$ and x' is the unique point of A_s closest to x). Analogously, $|x\pi(x)| = |x\pi(x')|$ implies that $\pi(x) = \pi(x') = \pi(\pi_s(x))$. Now lemma 7.3 yields $|\pi_s(x)\pi(x)| = |x'\pi(x')| = s$. ■.

Theorem 7.5. *Let A be a closed convex subset of a Hadamard space X. Then there exists a Lipschitz retraction*

$$R : X \times \mathbb{R}_+ \longrightarrow X \times \{0\} \cup A \times \mathbb{R}_+$$

Proof. Define

$$R(x,t) = \begin{cases} (\pi_0(x), t-d) & \text{for } t \geq d \\ (\pi_{d-t}(x), 0) & \text{for } t < d \end{cases} \qquad \text{where } d = |x\pi_0(x)|$$

Fix the usual l_1 metric ρ on $X \times \mathbb{R}_+$.

We need to check that R is Lipschitz. Let then $(x_i, t_i) \in X \times \mathbb{R}_+$, let $a_i = \pi_0(x_i)$ and $d_i = |x_i a_i|$ for $i = 1, 2$.

First note that $d_1 - d_2 = |x_1a_1| - |x_2a_2| \leq |x_1a_2| - |x_2a_2| \leq |x_1x_2|$. By symmetry, we get $|d_1 - d_2| \leq |x_1x_2|$.

Second, we see that $|\pi_s(x)\pi_t(x)| = |s - t|$ if only $|x\pi_0(x)| \geq \max(s,t)$. (This follows from lemma 7.4).

Now if both $t_1 \geq d_1$ and $t_2 \geq d_2$, then

$$\begin{aligned}\rho\big(R(x_1,t_1), R(x_2,t_2)\big) &= |\pi_0(x_1)\pi_0(x_2)| + |(t_1 - d_1) - (t_2 - d_2)| \leq \\ &\leq |\pi_0(x_1)\pi_0(x_2)| + |t_1 - t_2| + |d_1 - d_2| \leq \\ &\leq |x_1x_2| + |t_1 - t_2| + |d_1d_2| \leq 2|x_1x_2| + |t_1 - t_2| \leq \\ &\leq 2\rho\big((x_1,t_1),(x_2,t_2)\big)\end{aligned}$$

Next, if $t_1 < d_1$ and $t_2 < d_2$, then

$$\begin{aligned}&\rho\big(R(x_1,t_1), R(x_2,t_2)\big) = \\ &= |\pi_{d_1-t_1}(x_1)\pi_{d_2-t_2}(x_2)| \leq \\ &\leq |\pi_{d_1-t_1}(x_1)\pi_{d_1-t_1}(x_2)| + |\pi_{d_1-t_1}(x_2)\pi_{d_2-t_2}(x_2)| \leq \\ &\leq |x_1x_2| + |(d_1 - t_1) - (d_2 - t_2)| \leq |x_1x_2| + |t_1 - t_2| + |d_1 - d_2| \leq \\ &\leq 2\rho\big((x_1,t_1),(x_2,t_2)\big)\end{aligned}$$

Finally, if $t_1 < d_1$ but $t_2 \geq d_2$, then

$$\begin{aligned}
&\rho\big(R(x_1,t_1),R(x_2,t_2)\big) = \\
&= \quad |\pi_{d_1-t_1}(x_1)\pi_0(x_2)| + t_2 - d_2 \leq \\
&\leq \quad |\pi_{d_1-t_1}(x_1)\pi_0(x_1)| + |\pi_0(x_1)\pi_0(x_2)| + t_2 - d_2 \leq \\
&\leq \quad (d_1 - t_1) + (t_2 - d_2) + |x_1x_2| \leq |t_1 - t_2| + |d_1 - d_2| + |x_1x_2| \leq \\
&\leq \quad 2\rho\big((x_1,t_1),(x_2,t_2)\big)
\end{aligned}$$

The last case when $t_1 \geq d_1$ and $t_2 < d_2$ is analogous to the previous one. Therefore R is the required 2-Lipschitz retraction. ■

Theorem 7.6. *Let A be a closed convex subset of a Hadamard space X. Then there exists a linearly proper Lipschitz retraction*

$$R' : X\tilde{\times}\mathbf{R}^2_+ \longrightarrow X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+$$

Proof. This time it is convenient to represent the homotopy space as

$$X_H = \{(x,t) \in X \times \mathbb{R}_+ : \ 0 \leq 2t \leq |xx_0|\}$$

equipped with the usual l_1 metric, denoted by ρ_{X_H}, where $x_0 \in X$ is a chosen base point. We let $R' = R|_{X_H}$, where R is the map defined in the previous theorem. We first check that R' takes its values in the set $X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+ = X \times \{0\} \cup \{(a,t) \in A \times \mathbb{R}_+ : \ 0 \leq t \leq |ax_0|\}$.

Let then $(x,t) \in X_H$, hence $t \leq 2t \leq |xx_0|$. Let $d = |x\pi_0(x)|$. If $t < d$, then clearly $R'(x,t) = (\pi_{d-t}(x), 0) \in X \times \{0\}$. Otherwise we have $R'(x,t) = (\pi_0(x), t-d)$, where $\pi_0(x) \in A$ and $0 \leq t-d \leq |xx_0| - d = |xx_0| - |x\pi_0(x)| \leq |\pi_0(x)x_0|$, so $R'(x,t) \in A\tilde{\times}\mathbf{R}^2_+$. In both cases it turns out that $R'(x,t) \in X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+$, as desired.

To show that R' is linearly proper, we prove that for $(x,t) \in X_H$ we have $||R'(x,t)|| \geq \frac{1}{3}||(x,t)||$, where $||(x,t)|| := \rho_{X_H}\big((x,t),(x_0,0)\big) = |xx_0| + t$. Again, there are two cases. If $t < d$ then observe that $x = \pi_d(x)$, hence $|x\pi_{d-t}(x)| = t$. Because of that, we have $||R'(x,t)|| = |x_0\pi_{d-t}(x)| \geq |xx_0| - |x\pi_{d-t(x)}| = |xx_0| - t \geq \frac{1}{2}|xx_0| \geq \frac{1}{3}(|xx_0| + t) = \frac{1}{3}||(x,t)||$ (the last two inequalities follow from the fact that in X_H we have $2t \leq |xx_0|$). And if $t \geq d$ then $||R'(x,t)|| = |x_0\pi_0(x)| + t - d \geq |x_0\pi_0(x)| \geq |xx_0| - |x\pi_0(x)| = |xx_0| - d \geq |xx_0| - t \geq \frac{1}{3}||(x,t)||$. ■

The last part of the proof shows why this time have we put the factor 2 in the definition of X_H. Clearly this factor does not affect the space X_H up to isomorphism, but if it was not present, the restriction $R|_{X_H}$ might not be proper.

Corollary 7.7. *If A is a convex subset of a Hadamard space, Y is any space and $f : X\tilde{\times}\mathbf{R}_+ \cup A\tilde{\times}\mathbf{R}^2_+ \longrightarrow Y$ is a morphism in either of the categories $\bar{\mathcal{A}}$, $\mathcal{A}$, $\bar{\mathcal{A}}_l$, $\mathcal{A}_l$, then it can be extended in the respective category to a homotopy $\bar{f} : X\tilde{\times}\mathbf{R}^2_+ \longrightarrow Y$, defined as $\bar{f} := f \circ R'$.* ■

References

[B] W. Ballmann, *Lectures on Spaces of Nonpositive Curvature*, DMV Seminar 25, Birkhäuser (1995), viii+112

[BH] M. R. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, A Series of Comprehensive Studies in Mathematics 319, Springer (1999), XXI+643

[BBI] D. Burago, Y. Burago, S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics 33, AMS (2001), xiv+415

[D] A. N. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys, 55 (2000), vol. 6, 1085-1129

[G] M. Gromov, *Hyperbolic groups*, in S. M. Gersten (ed), Essays in group theory, MSRI Publ. no. 8, New York: Springer (1987), 75–263

[L] U. Lang, *Extendability of Large-Scale Lipschitz Maps*, Transactions of the American Mathematical Society, 10 (1999), vol. 351, 3975–3988

[Y] T. Yagasaki, *A short survey on coarse topology*, Research in general and geometric topology, Sūrikaisekikenkyūsho Kōkyūroku, 1126 (2000), 66–78