

ON UNISERIAL MODULES THAT ARE NOT QUASI-SMALL

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ABSTRACT. We prove \aleph_0 -th root property for uniserial modules and we show that the Krull-Schmidt theorem holds for direct sums of uniserial modules that are not quasi-small.

1. INTRODUCTION

In this note we give some results about uniserial modules that are not quasi-small, especially we are interested in direct summands of serial modules that are a (possibly infinite) direct sum of these uniserial modules.

Let us briefly recall some notions, for details see [3] and [1]. Throughout the paper we consider unitary right modules over an associative ring R with unit. We say that a module U is *uniserial* if its lattice of submodules is a chain. Let $S = \text{End}_R(U)$ be an endomorphism ring of a non-zero uniserial module U . It is known (see [3]) that S is a semilocal ring and that two cases may occur. If S is local, we say that U is of type 1 in the other case the right S -module $S/J(S)$ is a direct sum of two non-isomorphic simple S -modules and we say that S is of type 2. Uniserial modules of type 2 can be also characterized by existence of $f, g \in S$ such that f is a monomorphism, g is an epimorphism but neither f nor g is an automorphism of U . A module M is said to be *serial* if it is a (possibly infinite) direct sum of uniserial modules. Two following problems are of particular interest

- (i) Let M be serial. Is every direct summand of M serial?
- (ii) Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be families of non-zero uniserial modules such that $\bigoplus_{i \in I} U_i \simeq \bigoplus_{j \in J} V_j$. Is there a bijection $\sigma: I \rightarrow J$ such that $U_i \simeq V_{\sigma(i)}$ for any $i \in I$?

In fact, solving the first problem, we can consider only countable direct sums of uniserial modules because any direct summand of a serial module can be decomposed as a direct sum of direct summands of countable direct sums of uniserial modules ([3, Corollary 2.49]). We can do also similar restriction for the second problem using [3, Theorem 2.50] but we will not need this.

Answer to the first question is no in general. In [5] Puninski gives an example of pure projective module over exceptional nearly simple chain ring that has no decomposition into a direct sum of indecomposable modules. Since any finitely presented module over a chain ring is serial, this result serves as a counter-example to (i). However, if we consider M of finite Goldie dimension (i.e. M is finite direct sum of uniserial modules), then any direct summand of M is serial (see [7]).

Answer to the second problem is no even if we consider only finite families of uniserial modules ([3, Example 9.20]). If we consider $|I|$ or $|J|$ finite, then we have a weaker assertion called the weak Krull-Schmidt theorem (see [3, Theorem 9.13]) that roughly says that two indecomposable decompositions of serial module of finite Goldie dimension are unique up to two permutations. Without the assumption $|I|$ or $|J|$ finite, the weak Krull-Schmidt theorem does not hold. We give more details in next section.

We can also study these problems only for some kinds of serial modules. For example, if M is a direct sum of uniserial modules of type 1, then any two direct sum decompositions of M have isomorphic refinements ([3, Corollary 2.54]) and thus any direct summand of M is serial. From Krull-Schmidt-Remak-Azumaya theorem we see that if one of families in (ii) contains only uniserial modules of type 1, then we can find a bijection σ of desired property. In this paper we give similar results for direct sums of uniserial modules that are not quasi-small (definition is recalled in next section). We prove that if M is a direct sum of uniserial modules that are not quasi-small, then any direct summand of M is serial and if $\{U_i \mid i \in I\}$, $\{V_j \mid j \in J\}$ are families of uniserial modules that are not quasi-small such that $\bigoplus_{i \in I} U_i \simeq \bigoplus_{j \in J} V_j$, then there is a bijection $\sigma: I \rightarrow J$ such that $U_i \simeq V_{\sigma(i)}$.

Last section of this note is devoted to uniserial modules over right chain rings. We describe right chain rings possessing uniserial modules that are not quasi-small. As an application we also give a non-model theoretical proof of Puninski's classification of pure projective modules over nearly simple chain domains from [6].

2. TWO IMPORTANT SUBMODULES OF A UNISERIAL MODULE

Sometimes without previous acknowledge, we shall use following easy but useful observation.

Remark 2.1. *Let U, V be uniserial right modules over a ring R . Then $U \simeq V$ if and only if there is a monomorphism $f: U \rightarrow V$ and an epimorphism $g: U \rightarrow V$. In particular, if U is a submodule of V and there is an epimorphism $g: U \rightarrow V$, then $U \simeq V$.*

Proof. This is the particular case of [3, Lemma 9.2(a)]. To prove the second statement observe that the inclusion $U \subseteq V$ is a monomorphism. \square

Let U, V be uniserial modules. Recall that U and V have the same monogeny (epigeny) class if and only if there are monomorphisms (epimorphisms) $f: U \rightarrow V$ and $g: V \rightarrow U$. In this case we write $[U]_m = [V]_m$ ($[U]_e = [V]_e$). Let U be a uniserial right module over a ring R . Let U_m be the intersection of all submodules of U that are isomorphic to U . Observe that if $S \subseteq \text{End}_R(U)$ is the set containing all monic endomorphisms of U , then $U_m = \bigcap_{f \in S} \text{Im } f$. Next Lemma gives the basic properties of U_m .

Lemma 2.2. *Let U be a uniserial right module such that there is a monomorphism $f: U \rightarrow U$ that is not an epimorphism. Then*

- (i) $U_m \subsetneq U$
- (ii) *Let V be a submodule of U . Then $[V]_m = [U]_m$ if and only if $U_m \subsetneq V$.*
- (iii) U_m *is the largest submodule of U that does not have the same monogeny class as U .*
- (iv) *Let V be a uniserial module, and let $f: U \rightarrow V$ be a monomorphism. Then $f(U_m) \subseteq V_m$. Moreover, the equality holds if and only if $[U]_m = [V]_m$.*
- (v) U_m *is an invariant submodule of U .*
- (vi) U/U_m *contains no minimal submodule and $(U/U_m)_m = 0$.*

Proof. (i) This is clear.

(ii) If $U_m \subsetneq V$, then there is a monomorphism from U to V . Clearly, the inclusion $V \subseteq U$ is also a monomorphism, thus we have $[U]_m = [V]_m$. On the other hand if $f: U \rightarrow V$ and $g: U \rightarrow U$ are monomorphisms such that $g(U) \neq U$, then $\text{Im } f \circ g \subsetneq V$ and therefore $U_m \subsetneq V$.

(iii) This is an immediate consequence of (ii).

(iv) If $f(U) \subseteq V_m$, there is nothing to prove. If $V_m \subsetneq f(U)$, then using (ii), we see that $[U]_m = [V]_m$ and the result follows from (iii), since $f(U_m)$ is the greatest submodule of $f(U)$ having the monogeny class different from V .

(v) To see that U_m is an invariant submodule of U it is enough to use (iv) and the fact that any endomorphism of a uniform module can be written as a sum of at most two monic endomorphisms (if $f \in \text{End}_R(U)$ is not monic, then $1 + f$ is monic because $1 = (1 + f) - f$ has to have zero kernel, so $f = (1 + f) - 1$).

(vi) We have already seen above that any submodule containing U_m as a proper submodule contains also an isomorphic copy of U as a proper submodule. Let $f: U \rightarrow U$ be a monomorphism. Then $f(U_m) = U_m$ and f

induces a monomorphism $f': U/U_m \rightarrow U/U_m$ such that $\text{Im } f' = f(U)/U_m$. Therefore $(U/U_m)_m = 0$. \square

Now, let us introduce the dual notion. Let U be a uniserial module. Let $T \subseteq \text{End}_R(U)$ be the set of all epic endomorphisms. Define $U_e = \sum_{f \in T} \text{Ker } f$. Observe that $u \in U_e$ if and only if $f(u) = 0$ for some $f \in T$.

Lemma 2.3. *Let U be a uniserial right module such that there is an epimorphism $g: U \rightarrow U$ that is not a monomorphism. Then*

- (i) $U_e \neq 0$
- (ii) For any submodule $V \subseteq U$ we have $[U/V]_e = [U]_e$ if and only if $V \not\subseteq U_e$.
- (iii) U_e is the smallest submodule of U such that U/U_e does not have the same epigeny class as U .
- (iv) Let V be a uniserial module and $f: V \rightarrow U$ an epimorphism. Then $V_e \subseteq f^{-1}(U_e)$ (and also $f(V_e) \subseteq U_e$). Moreover, if $[U]_e = [V]_e$, then $V_e = f^{-1}(U_e)$ and $U_e = f(V_e)$.
- (v) U_e is an invariant submodule of U .
- (vi) U_e is not cyclic and $(U_e)_e = U_e$.

Proof. (i) This is clear.

(ii) Let $V \subseteq U_e$. Then there is an epimorphism $f: U \rightarrow U$ such that $f(V) = 0$. Then f induces an epimorphism $h: U/V \rightarrow U$. Of course, there is also an epimorphism $\pi: U \rightarrow U/V$ and thus $[U]_e = [U/V]_e$.

On the other side suppose that V is a non-zero submodule of U such that there is an epimorphism $h: U/V \rightarrow U$. Let $f = h \circ \pi$, where $\pi: U \rightarrow U/V$ denotes the canonical projection. Then V is a proper submodule of $\text{Ker } f^2$.

(iii) Follows easily from (ii).

(iv) Suppose $f^{-1}(U_e) \not\subseteq V_e$. Then $\text{Ker } f \not\subseteq V_e$, therefore $[V]_e = [U]_e$. But $[U/U_e]_e = [V]_e$ because $U/U_e \simeq V/f^{-1}(U_e)$. Thus we have a contradiction. Now suppose $[U]_e = [V]_e$ and $V_e \not\subseteq f^{-1}(U_e)$. Then $f(V_e) \not\subseteq U_e$ because $\text{Ker } f \not\subseteq V_e$, and $[U]_e = [U/f(V_e)]_e$. But $U/f(V_e)$ is an epimorphic image of V/V_e and it cannot have the same epigeny class as V , a contradiction.

(v) Follows easily from (iv) and the fact that any endomorphism of a couniform module can be written as a sum of at most two epimorphisms (if $f: U \rightarrow U$ is not an epimorphism, then $1 + f$ is an epimorphism since $U = \text{Im } f + \text{Im } 1 + f$, so $f = (1 + f) - 1$).

(vi) If $0 \neq u \in U_e$, then there is an epimorphism $f: U \rightarrow U$ such that $f(u) = 0$. Let $v \in U$ be such that $f(v) = u$. Then $f^2(v) = 0$ and thus $v \in U_e$. Since $u \neq 0$, $uR \not\subseteq vR$. From (iv) we have that any epimorphism $f: U \rightarrow U$ induces an epimorphism on U_e . Thus $(U_e)_e = U_e$. \square

Let U be a non-zero uniserial module. Then $U_e = 0$ if and only if any epimorphism $f: U \rightarrow U$ is an automorphism. In this case U is of type 1. Similarly $U_m = U$ if and only if any monomorphism $g: U \rightarrow U$ is an automorphism. In this case U is again of type 1. Thus U is of type 2 if and only if $U_e \neq 0$ and $U_m \neq U$. In particular, if $U_m \subsetneq U_e$, then U is of type 2.

Lemma 2.4. *Let U be a uniserial right module of type 2. Then*

- (i) *There is a cyclic uniserial module of type 2 having the same monogeny class as U .*
- (ii) *There is a non-zero cyclic submodule $V \subseteq U$ such that U/V is a module of type 2 having the same epigeny class as U .*

Proof. (i) Let $u \in U \setminus U_m$. Then, by Lemma 2.2, $[uR]_m = [U]_m$ and there is always a monomorphism $f: uR \rightarrow uR$ that is not an epimorphism. Thus we need to find some $v \in U \setminus U_m$ such that there is a non-monic epimorphism $h: vR \rightarrow vR$. Let $f: U \rightarrow U$ be a non-monic epimorphism. Take some $v \in U$ such that $f(v) \notin U_m$ (note that $U_m \subsetneq f(U)$). Since U_m is fully invariant by Lemma 2.2 (v), $v \notin U_m$, and we have $[vR]_m = [f(v)R]_m = [U]_m$. Since $f(v)R$ is a homomorphic image of vR , there is an isomorphism $g: f(v)R \rightarrow vR$ by Remark 2.1. We are done if we put $h = g \circ f|_{vR}$.

- (ii) The proof is dual to (i). \square

Recall that a R -module M is said to be *small* if for every family $\{N_i \mid i \in I\}$ of R -modules and any homomorphism $\varphi: M \rightarrow \bigoplus_{i \in I} N_i$ there is a finite subset $I_0 \subseteq I$ such that $\pi_i \varphi = 0$ for every $i \in I \setminus I_0$. Similarly a R -module M is said to be *quasi-small* if the following condition holds: Let $\{N_i \mid i \in I\}$ be a family of R -modules such that M is isomorphic to a direct summand of $\bigoplus_{i \in I} N_i$. Then there is a finite subset $I' \subseteq I$ such that M is isomorphic to a direct summand of $\bigoplus_{i \in I'} N_i$. Clearly any finitely generated module is small and any small module is quasi-small. Fuchs and Salce proved that any non-countably generated uniserial module is small (see [3, Proposition 2.45]).

Let us recall some characterization of uniserial modules that are not quasi-small from [1].

Lemma 2.5. *Let U be a countably generated uniserial right module. Then U is not quasi-small if and only if for any $u \in U$ there is a monomorphism $f: U \rightarrow U$ such that f is not an epimorphism and $f(u) = u$. Thus if U is not quasi-small, then for any $u \in U$ there is an epimorphism $g: U \rightarrow U$ such that $g(u) = 0$.*

Proof. See [1, Lemma 4.5]. \square

Proposition 2.6. *Let U be a uniserial right module over a ring R . Then U is not quasi-small if and only if $U_m \subsetneq U = U_e$ and U is a countably generated module.*

Proof. Let U be a uniserial module that is not quasi-small. Since it is not small, it has to be countably generated. Any module with local endomorphism ring is quasi-small by [3, Theorem 9.29], so U has to be of type 2 and thus $U_m \neq U$. Finally, let $u \in U$ and let $f: U \rightarrow U$ be a monomorphism such that it is not an automorphism and such that $f(u) = u$. Then $1 - f$ is an epimorphism having u in its kernel, so $U_e = U$.

Now, let U be a countably generated uniserial module of type 2 satisfying $U_e = U$. Observe that by Lemma 2.3 U is not a finitely generated module. Let $g: U \rightarrow U$ be any monomorphism that is not an automorphism. Let $0 \neq u \in U$. Then there is an epimorphism $f: U \rightarrow U$ such that $f(u) = 0$. Then $f + g$ is an automorphism and $(f + g)(u) = g(u)$. Now $(f + g)^{-1} \circ g: U \rightarrow U$ is a monomorphism that is not an automorphism and $(f + g)^{-1}g(u) = u$. We conclude by Lemma 2.5. \square

Lemma 2.7. *Let U be a uniserial right module that is not quasi-small. Then no non-zero factor of U is quasi-small.*

Proof. See [1, Lemma 4.5]. \square

Lemma 2.8. *Let U be a uniserial right module satisfying $U_m \subsetneq U_e$. Then for any uniserial module V such that $[U]_m = [V]_m$ we have $V_m \subsetneq V_e$. Moreover, U_e is a union of its proper submodules that are isomorphic to V .*

Proof. We can suppose that V is a submodule of U such that $U_m \subsetneq V \subsetneq U_e$ because $[V]_m = [U_e]_m$. Now, there is an epimorphism $f: U \rightarrow U$ such that $f(V) = 0$. Submodule $W = f^{-1}(V)$ is isomorphic to V (use Remark 2.1), let $g: V \rightarrow W$ be some isomorphism. Then $W_m = U_m$ and $gf|_W: W \rightarrow W$ is an epimorphism having V (and thus also W_m) in its kernel. Hence $W_m \subsetneq W_e$.

Let X be a submodule of U_e which is a union of proper submodules of U_e that are isomorphic to V . Suppose $X \neq U_e$. Then there is an epimorphism $f: U \rightarrow U$ such that $f(X) = 0$. Now $f^{-1}(V)$ is a proper submodule of U_e isomorphic to V . Since $X \subsetneq f^{-1}(V)$, we have a contradiction and $X = U_e$. \square

By the statement "weak Krull-Schmidt theorem for uniserial modules over a ring R holds" we mean the following assertion: Let $\{U_i \mid i \in I\}$ and $\{V_j \mid j \in J\}$ be families of non-zero uniserial right modules over R . Then $\bigoplus_{i \in I} U_i \simeq \bigoplus_{j \in J} V_j$ if and only if $|I| = |J|$ and there are bijections $\sigma, \tau: I \rightarrow J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$.

In [1, Theorem 4.8, Theorem 4.9] it was noticed that the weak Krull-Schmidt theorem for uniserial modules over a ring R holds if and only if any uniserial module over R is quasi-small. Next lemma gives a criterion for existence of uniserial R -modules that are not quasi-small. We shall use this lemma to give a characterization of right chain rings possessing uniserial modules that are not quasi-small.

Lemma 2.9. *Let R be a ring. Then the following conditions are equivalent:*

- (i) *There is a uniserial right module over R that is not quasi-small.*
- (ii) *There is a uniserial right module over R such that $U_m \subsetneq U_e$.*
- (iii) *There is a finitely generated uniserial right module U satisfying $U_m \subsetneq U_e$.*

Proof. The implication (i) \Rightarrow (iii) is an easy consequence of Lemma 2.8 and the implication (iii) \Rightarrow (ii) is clear. It remains to prove (ii) \Rightarrow (i). Let U be a uniserial module satisfying $U_m \subsetneq U_e$. Let $u \in U_e \setminus U_m$. Then there is an epimorphism $f: U \rightarrow U$ such that $f(u) = 0$. We define a sequence $u_0, u_1, \dots \in U$ such that $u_0 = u$ and $f(u_i) = u_{i-1}$ for any $i \in \mathbb{N}$. Since for any $i \in \mathbb{N}$ we have $f^i(u_{i-1}) = 0$ and $f^i(u_i) = u \neq 0$, $u_0R \subsetneq u_1R \subsetneq \dots$. We set $U' := \cup_{i \in \mathbb{N}} u_iR$. Then f induces an epic endomorphism on U' . Moreover, $U'_m \subsetneq U'_e = U'$ because for any $v \in U'$ there is $i \in \mathbb{N}$ such that $f^i(v) = 0$. Thus U' is not quasi-small by Proposition 2.6. \square

Lemma 2.10. *Let U be a uniserial right module such that $U_m \neq U$ and $U_e = U$. Given any $u \in U$ and $v \in U \setminus U_m$, there is an isomorphism $f: U \rightarrow U$ such that $u \in f(v)R$.*

Proof. There is a monomorphism $g: U \rightarrow U$ and an epimorphism $g': U \rightarrow U$ such that $\text{Im } g \subsetneq vR$ and $uR \subsetneq \text{Ker } g'$. Put $f = (g + g')^{-1}$. \square

Lemma 2.11. *Let U be a uniserial right module, and let V be a submodule of U satisfying $U_m \subsetneq V$. Then $V_e \subseteq U_e$.*

Proof. Let E be an injective envelope of U , $\nu: V \rightarrow U$ be the inclusion map and $\mu: V \rightarrow E$ be some monomorphism. Let $f: V \rightarrow V$ be an epimorphism. Since E is injective, there is $g: U \rightarrow E$ such that $g \circ \nu = \mu \circ f$. Now $g(U) \subseteq E$ is an epimorphic image of U containing an isomorphic copy of V . Thus there is a monomorphism $h: U \rightarrow g(U)$ and $g(U) \simeq U$ by Remark 2.1. Now we conclude $\text{Ker } f \subseteq U_e$. \square

3. DIRECT SUMS OF UNISERIAL MODULES THAT ARE NOT QUASI-SMALL

In [7, Theorem 7] it was shown that any direct summand of a serial module of finite Goldie dimension is serial. In this section we will use next

(probably well-known) Proposition to prove that direct summands of some serial modules of infinite Goldie dimension are serial.

Proposition 3.1. *Let A be a right module over a ring R and let \mathcal{C} be a class of R -modules. Suppose that there is a countable family $\{A_i \mid i \in \mathbb{N}\}$ of submodules of A such that the following conditions hold:*

- (i) $\sum_{i \in \mathbb{N}} A_i = A$
- (ii) *If $A = B \oplus B'$ is a direct sum decomposition of A and $\pi_B: A \rightarrow B$ is the canonical projection with respect to this decomposition, then for any $i \in \mathbb{N}$ there is some $C \in \mathcal{C}$ such that C is a direct summand of B containing $\pi_B(A_i)$.*

Then A is a countable direct sum of modules from \mathcal{C} .

Proof. By induction we prove following: For any $i \in \mathbb{N}_0$ there are direct summands X_i, Y_i of A satisfying $A = \bigoplus_{j \leq i} X_j \oplus Y_i$, $X_i \in \mathcal{C}$ and $\sum_{j \leq i} A_j \subseteq \bigoplus_{j \leq i} X_j$, where we consider $A_0 = 0$. If $i = 0$, we put $X_0 = 0, Y_0 = A$. Let $i \geq 1$ and suppose X_j, Y_j have been defined for any $j < i$. By assumption $A = \bigoplus_{j < i} X_j \oplus Y_{i-1}$. Let $\pi: A \rightarrow Y_{i-1}$ be the canonical projection with respect to this decomposition. Then there is a decomposition $Y_{i-1} = X_i \oplus Y_i$ such that $X_i \in \mathcal{C}$ and $\pi(A_i) \subseteq X_i$. By hypothesis, $A_i \subseteq \bigoplus_{j \leq i} X_j$. From condition (i), $A = \bigoplus_{i \in \mathbb{N}} X_i$. \square

Let M, N be right modules over R and let $\{f_i \mid i \in I\}$ be a family of homomorphisms from M to N . We say that this family is summable, if for any $m \in M$ there is a finite set $I_0 \subseteq I$ such that for any $i \in I \setminus I_0$ is $f_i(m) = 0$. In this case the family defines a homomorphism $h: M \rightarrow N$ given by $h = \sum_{i \in I} f_i$.

Proposition 3.2. *Let U, V be uniserial right modules over a ring R that are not quasi-small. Then $[U]_m = [V]_m$ if and only if $U \simeq V$.*

Proof. Suppose that $[U]_m = [V]_m$. All we have to do is to find an epimorphism $f: U \rightarrow V$. By Proposition 2.6, we can apply Lemma 2.8, U is a union of submodules isomorphic to V . As U is countably generated, there is a chain $X_1 \subsetneq X_2 \subsetneq \dots \subsetneq U$ such that for any $i \in \mathbb{N}$ there is an epimorphism $f_i: V \rightarrow X_i$. The sum of these epimorphisms induces an epimorphism $\varphi: \bigoplus_{i \in \mathbb{N}} V_i \rightarrow U$. Here all V_i are equal to V and $\varphi(V_i) = X_i$. Since $V = V_e$ and V is countably generated, it is possible to construct by induction elements $v_1, v_2, \dots \in V$ and homomorphisms h_1, h_2, \dots such that the following conditions are satisfied:

- (i) v_1, v_2, \dots generate V .
- (ii) For any $i \in \mathbb{N}$, the homomorphism $h_i: V \rightarrow V_i$ is an epimorphism and $h_{i+1}(v_i) = 0$.

(iii) For any $i \geq 2$, $\varphi(h_i(v_i)) \notin X_{i-1}$.

The family $\{h_i\}_{i \in \mathbb{N}}$ is a summable family of homomorphisms $V \rightarrow \bigoplus_{i \in \mathbb{N}} V_i$, since $h_j(v_i) = 0$ whenever $j > i$. Let $f = \varphi \circ h$, where $h = \sum_{i \in \mathbb{N}} h_i$. By properties (ii) and (iii), $f(v_i) \notin X_{i-1}$ for $i \geq 2$. Thus f is an epimorphism and we are done. \square

Corollary 3.3. *Let $\{U_i \mid i \in I\}$, $\{V_j \mid j \in J\}$ be families of uniserial modules such that none of them is quasi-small. Then $\bigoplus_{i \in I} U_i \simeq \bigoplus_{j \in J} V_j$ if and only if there is a bijection $\sigma: I \rightarrow J$ such that $U_i \simeq V_{\sigma(i)}$ for any $i \in I$.*

Proof. Suppose $\bigoplus_{i \in I} U_i \simeq \bigoplus_{j \in J} V_j$. By [3, Theorem 9.12], there is a bijection $\sigma: I \rightarrow J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ for any $i \in I$. Now the result follows from Proposition 3.2. \square

Corollary 3.4. *Let U be a uniserial right module that is not quasi-small. Then there is an epimorphism $f: U \rightarrow U$ such that $U = \bigcup_{i \in \mathbb{N}} \text{Ker } f^i$.*

Proof. Since $U_e = U \supset U_m$, there is an epimorphism $g: U \rightarrow U$ such that $g(u_0) = 0$ for some $u_0 \in U \setminus U_m$. By induction we define $u_1, u_2, \dots \in U$ such that $g(u_i) = u_{i-1}$. Let $V = \bigcup_{i \in \mathbb{N}} u_i R$ and let $h: V \rightarrow V$ be given by $h = g|_V$. Then $V_m = U_m$, V is countably generated, and $V = \bigcup_{i \in \mathbb{N}} \text{Ker } h^i$, hence $V = V_e$. By Proposition 2.6, V is not quasi-small and $U \simeq V$ by Proposition 3.2. \square

Corollary 3.5. *Let U be a uniserial right module over an arbitrary ring R that is not quasi-small. Then there is $r \in R$ and elements u_1, u_2, \dots generating U such that for any $i \in \mathbb{N}$ is $u_{i+1}r = u_i$.*

Proof. By Proposition 2.6, there is a monomorphism $g: U \rightarrow U$ and an epimorphism $f: U \rightarrow U$ such that $\text{Im } g \subsetneq \text{Ker } f$. Let $u_0 \in \text{Ker } f \setminus \text{Im } g$. By Remark 2.1, $h = f + g: U \rightarrow U$ is an isomorphism. There are $u_1, u_2, \dots \in U$ such that for any $i \in \mathbb{N}$ is $h(u_i) = u_{i-1}$. By induction we can also find $u'_1, u'_2, \dots \in U$ such that for any $i \in \mathbb{N}$ is $f(u'_i) = u'_{i-1}$, where $u'_0 = u_0$. It is easy to prove the following claim: For any $u \in U$ and for any $n \in \mathbb{N}$ is $h^n(u) \in f^n(u) + \text{Im } g$. Using this claim we get $f^i(u_i - u'_i) \in \text{Im } g$ for any $i \in \mathbb{N}$. Since $f^{i-1}(\text{Im } g)$ is a proper submodule of $u'_i R$, $u_i R = u'_i R$ for any $i \in \mathbb{N}$. Let $V = \bigcup_{i \in \mathbb{N}} u'_i R$. Using the same argumentation as in the previous Corollary, V is not quasi-small and $U \simeq V$ by Proposition 3.2. Since $u_0 R \subsetneq u_1 R$, there is $r \in R$ such that $u_1 r = u_0$. Now $h^{i-1}(u_i r) = u_1 r = u_0 = h^{i-1}(u_{i-1})$ for any $i \in \mathbb{N}$. Since h is an isomorphism, $u_i r = u_{i-1}$. \square

Next Theorem gives a positive answer to Problem 16 from [3].

Theorem 3.6. *Let U, V be uniserial right modules and let X be a non-empty set. Then $U^{(X)} \simeq V^{(X)}$ implies $U \simeq V$.*

Proof. Situation is clear if U or V is zero. First suppose that U is a quasi-small non-zero module, say. Then U is isomorphic to a direct summand of $V^{(n)}$ for some $n \in \mathbb{N}$ and, by [2, Theorem 2.7], $U \simeq V$.

Now, let neither U nor V be quasi-small and $U^{(X)} \simeq V^{(X)}$. By Corollary 3.3, $U \simeq V$. \square

Corollary 3.3 gives rise to the question whether the Krull-Schmidt theorem holds for a class of direct sums of uniserial modules that are not quasi-small (see the Introduction of [4]). Equivalently we ask whether every direct summand of such direct sums is serial. We shall try to prove that the answer is yes.

Lemma 3.7. *Let U, V be uniserial right modules that are not quasi-small such that $[U]_e = [V]_e$. If $f: U \rightarrow V$ is a non-zero homomorphism, then $V_m \not\subseteq f(U)$.*

Proof. Since $f(U)$ is a submodule of V and $[V]_e = [U]_e = [f(U)]_e$, there is a monomorphism $g: f(U) \rightarrow V$ and an epimorphism $h: f(U) \rightarrow V$. Therefore, by Remark 2.1, $f(U) \simeq V$. \square

Proposition 3.8. *Let $U_i, i \in \mathbb{N}$ be uniserial right modules that are not quasi-small and of the same epigeny class. Suppose $A \oplus B = \bigoplus_{i \in \mathbb{N}} U_i$. Then for any $i \in \mathbb{N}$ and for any $u \in U_i$ the element $\pi_A(u)$ is contained in a uniserial direct summand of A .*

Proof. Let $u \in U_i$ be such that $0 \neq \pi_A(u)$. Let $X = \pi_A(U_i)$. Since X is a uniform module, there is $j \in \mathbb{N}$ such that $\pi_j|_X$ is a monomorphism. Observe that X is not a quasi-small module by Lemma 2.7 and it has the same epigeny class as U_i because it is a non-zero factor of U_i . We want to find a uniserial module $V \subseteq A$ such that $\pi_A(u) \in V$, $\pi_j(V) = \pi_j(A)$. We distinguish three cases.

If $\pi_j(X) = \pi_j(A)$, we simply put $V = X$.

If there is $k \in \mathbb{N}, k \neq j$ such that $\pi_j \pi_A(U_k) = \pi_j(A)$, then, since $(U_k)_e = U_k$ and $[X]_e = [U_k]_e$, we find an epimorphism $g: X \rightarrow U_k$ such that $g(\pi_A(u)) = 0$ by Lemma 2.5. Let us consider submodule $V' = \{x + g(x) \mid x \in X\} \subseteq \bigoplus_{l \in \mathbb{N}} U_l$. Let $V = \pi_A(V')$. Observe that V is a non-zero homomorphic image of X containing $\pi_A(u)$ and if $\pi_j(X) \neq \pi_j(A)$, then $\pi_j(A) = \pi_j \pi_A(U_k) = \pi_j(V)$ because $\pi_j \pi_A(v)R = (\pi_j \pi_A(v) + \pi_j(x))R$ for any $x \in X$ and $v \in U_k$ such that $\pi_j \pi_A(v) \notin \pi_j(X)$.

Finally, suppose we have a sequence $i_1, i_2, \dots \in \mathbb{N}$ of pairwise different elements such that $\pi_j(\pi_A(U_{i_1})) \subsetneq \pi_j(\pi_A(U_{i_2})) \dots$ and $\cup_{k \in \mathbb{N}} \pi_j \pi_A(U_{i_k}) = \pi_j(A)$. We can construct sequences $u_1, u_2, \dots \in X$ and homomorphisms f_1, f_2, \dots such that the following conditions are satisfied:

- (i) u_1, u_2, \dots are generators of X , $\pi_A(u) \in u_1 R$, and $u_1 R \subset u_2 R \subset \dots$
- (ii) For any $l \in \mathbb{N}$ homomorphism $f_l: X \rightarrow U_{i_{l+1}}$ is an epimorphism such that $f_l(u_l) = 0$ and $\pi_j \pi_A(f_l(u_{l+1})) \notin \pi_j \pi_A(U_{i_l})$.

To construct these sequences we proceed by induction. Observe that since $[X]_e = [U_{i_j}]_e$ and $(U_{i_j})_e = U_{i_j}$ for any $j \in \mathbb{N}$, for any $k \in \mathbb{N}$, and for any $x \in X$ there is an epimorphism $h: X \rightarrow U_{i_k}$ such that $h(x) = 0$. Let $\{v_1, v_2, \dots\}$ be some countable set of generators of X . Let u_1 be any element of $X \setminus (\pi_A(u)R + v_1 R)$ and take $f_1: X \rightarrow U_{i_2}$ any epimorphism such that $f_1(u_1) = 0$. Suppose f_l, u_l have been defined for $l \in \mathbb{N}$. Then there is $u_{l+1} \in X$ such that $\pi_j \pi_A(f_l(u_{l+1})) \notin \pi_j \pi_A(U_{i_l})$ and $v_{l+1} R \subseteq u_{l+1} R$. Since $X_e = X$ and $[X]_e = [U_{i_{l+2}}]_e$, there is an epimorphism $f_{l+1}: X \rightarrow U_{i_{l+2}}$ such that $f_{l+1}(u_{l+1}) = 0$.

Now, let $V' = \text{Im } h$, where $h: X \rightarrow \oplus_{i \in \mathbb{N}} U_i$ is given by $h = 1_X + \sum_{i \in \mathbb{N}} f_i$ and $V = \pi_A(V')$. Again V is an epimorphic image of X containing $\pi_A(u)$. By the construction $\pi_j(V) = \pi_j \pi_A(V') = \pi_j(A)$. To show the last equality, note that $\pi_j \pi_A h(u_l) \notin \pi_j \pi_A(U_{i_{l-1}})$ if $\pi_j(X) \subseteq \pi_j \pi_A(U_{i_{l-1}})$.

Now, $\pi_j|_A: A \rightarrow \pi_j(A)$ gives a homomorphism inducing an isomorphism $V \rightarrow \pi_j(A)$ because $\pi_A(u) \in V$ and $\pi_j|_{\pi_A(u)R}$ is a monomorphism. Therefore $A = V \oplus \text{Ker } \pi_j|_A$. \square

Corollary 3.9. *Let $|I| \leq \aleph_0$ and $U_i, i \in I$ be uniserial right modules of the same epigeny class and suppose that none of them is quasi-small. Then any direct summand of $\oplus_{i \in I} U_i$ is serial.*

Proof. If I is a finite set, then we use [7, Theorem 7]. So we can suppose $I = \mathbb{N}$. Let $\oplus_{i \in \mathbb{N}} U_i = A \oplus B$ and let π_A be a canonical projection with respect to this decomposition. Since all U_i are countably generated, there is a countable set of generators of A of the form $\pi_A(u), u \in U_i$. If $A = C \oplus D$ is a decomposition of A and $\pi_C: A \rightarrow C$ is a canonical projection with respect to this decomposition, then $\pi_C \pi_A(u)$ is a projection of u onto C with respect to decomposition $C \oplus D \oplus B = \oplus_{i \in \mathbb{N}} U_i$. By Proposition 3.8, $\pi_C \pi_A(u)$ is contained in a uniserial direct summand of C . Now we can use Proposition 3.1 taking for \mathcal{C} the class of uniserial modules. \square

Let E be the set of epigeny classes of all uniserial right modules over an arbitrary ring R that are not quasi-small. We say that $[U_1]_e \leq [U_2]_e$ if there

is a submodule $V_2 \subseteq U_2$ and an epimorphism $f: V_2 \rightarrow U_1$. Observe that this definition does not depend on the representatives of $[U_1]_e, [U_2]_e$ chosen.

Lemma 3.10. *The relation \leq is a partial order on the set E . If U, V are uniserial right modules such that neither U nor V is quasi-small and there is a non-zero homomorphism $f: U \rightarrow V$, then $[U]_e \leq [V]_e$.*

Proof. Let us suppose that there are submodules $U_0 \subseteq U$ and $V_0 \subseteq V$ and epimorphisms $f: U_0 \rightarrow V$ and $g: V_0 \rightarrow U$. Let $U' = f^{-1}(V_0) \subseteq U_0$. Then the composition gf induces an epimorphism of U' onto U and thus $U' \simeq U$ and $[U_0]_m = [U]_m$. In the similar way we define $V' = g^{-1}(U_0)$ and show that $V' \simeq V$, $[V_0]_m = [V]_m$. Since U_0, V_0 are images of modules U', V' that are not quasi-small, they are not quasi-small by Lemma 2.7, and $U_0 \simeq U$ and $V_0 \simeq V$ by Proposition 3.2. Then $[U]_e = [V]_e$.

Suppose that $f: U \rightarrow V$ is non-zero. Then $f(U)$ is a submodule of V of the same epigeny class as U and thus $[U]_e \leq [V]_e$. \square

Proposition 3.11. *Let $U_i, i \in \mathbb{N}$ be uniserial right modules that are not quasi-small. Suppose $A \oplus B = \bigoplus_{i \in \mathbb{N}} U_i$. Then for any $a \in A$ there is a serial direct summand of A containing a .*

Proof. Let $X = \{[U_i]_e \mid i \in \mathbb{N}\}$ and for any $x \in X$ let $M_x = \bigoplus_{[U_i]_e = x} U_i$. For a subset $Y \subseteq X$ we put $M_Y = \bigoplus_{[U_i]_e \in Y} U_i$, $A_Y = \pi_A(M_Y)$, $B_Y = \pi_B(M_Y)$. If $Y \subseteq X$ is a non-empty set satisfying $y \leq x \Rightarrow x \in Y$ whenever $x \in X, y \in Y$ (we shall say that Y has the property (*)), then we have $M_Y = A_Y \oplus B_Y$ (observe that by Lemma 3.10 M_Y is an invariant submodule of M). In particular, A_Y is a direct summand of A and B_Y is a direct summand of B . Precisely, $A = A_Y \oplus A'$ and $B = B_Y \oplus B'$, where $A' = A \cap (B_Y \oplus M_{X \setminus Y})$ and $B' = B \cap (A_Y \oplus M_{X \setminus Y})$. If $Y \subsetneq X$ is a non-empty set satisfying (*), then $A' \oplus B'$ and $M_{X \setminus Y}$ are both complements of $A_Y \oplus B_Y = M_Y$ in M . Therefore the canonical projection $\pi_{X \setminus Y}: M \rightarrow M_{X \setminus Y}$ induces an isomorphism of $A' \oplus B'$ onto $M_{X \setminus Y}$.

Now we are ready to prove the Proposition. Let a be a non-zero element of A . There is a unique decomposition $a = a_{e_1} + \dots + a_{e_k}$, where $0 \neq a_{e_i} \in M_{e_i}$ and e_1, \dots, e_k are pair-wise different elements of X . We proceed by induction on k . Without loss of generality we can suppose that e_k is a maximal element of $\{e_1, \dots, e_k\}$. If e_k is not a maximal element of X , we make the following modification: Let $Y = \{x \in X \mid x > e_k\}$. Then Y is a proper non-empty subset of X satisfying (*). Thus $A = A_Y \oplus A'$, where $A' = A \cap (B_Y \oplus M_{X \setminus Y})$. Similarly $B = B_Y \oplus B'$ and the restriction of $\pi_{X \setminus Y}$ on $A' \oplus B'$ gives an isomorphism of $A' \oplus B'$ and $M_{X \setminus Y}$. Observe that $a \in A'$ and $a = \pi_{X \setminus Y}(a)$. Now it is enough to find serial direct summand

of $\pi_{X \setminus Y}(A')$ containing a , where $M_{X \setminus Y} = \pi_{X \setminus Y}(A') \oplus \pi_{X \setminus Y}(B')$ and e_k is a maximal element of $X \setminus Y$.

Thus we can suppose that e_k is a maximal element of X . Using this assumption, $Y' = \{e_k\}$ has the property (*). Therefore $A_{Y'} \oplus B_{Y'} = M_{e_k}$ and by Corollary 3.9 $A_{Y'}$ is serial. If $k = 1$, we are done since $a \in A_{Y'}$. Also there are $A'' \subseteq A, B'' \subseteq B$ such that $A = A'' \oplus A_{Y'}$ and $B = B'' \oplus B_{Y'}$. Let $a = a'' + a_{Y'}$, where $a'' \in A''$ and $a_{Y'} \in A_{Y'}$. Observe that $\pi_{X \setminus Y'}(a) = \pi_{X \setminus Y'}(a'')$. Projection $\pi_{X \setminus Y'}$ induces an isomorphism of $A'' \oplus B''$ and $M_{X \setminus Y'}$. Observe that $\pi_{X \setminus Y'}(a'') = a_{e_1} + \cdots + a_{e_{k-1}}$ so by the induction step on decomposition $M_{X \setminus Y'} = \pi_{X \setminus Y'}(A'') \oplus \pi_{X \setminus Y'}(B'')$ and element $\pi_{X \setminus Y'}(a'')$, we get that there is a serial direct summand S of A'' containing a'' . Then $A_{Y'} \oplus S$ is a serial direct summand of A containing a . \square

Theorem 3.12. *Let $M = \bigoplus_{i \in I} U_i$ be a direct sum of uniserial right modules that are not quasi-small. Let $M = A \oplus B$ be a direct sum decomposition of M . Then there is a partition $I = I_0 \cup I_1$ such that $A \simeq \bigoplus_{i \in I_0} U_i$ and $B \simeq \bigoplus_{i \in I_1} U_i$. Moreover, any two direct sum decompositions of A into indecomposable right modules are isomorphic.*

Proof. First of all let us prove that there is no non-zero uniserial direct summand of M that is quasi-small. If U is a non-zero quasi-small uniserial direct summand of M , then U is isomorphic to a direct summand of $\bigoplus_F U_i$, where $F \subseteq I$ is finite. By [3, Proposition 9.7], $[U]_e = [U_i]_e$ for some $i \in F$, a contradiction with Lemma 2.7.

Now it remains to prove that A is serial. The rest follows easily from Corollary 3.3. If I is a finite set, then A is serial by [7, Theorem 7]. If I is infinite, then by [3, Corollary 2.49] (and its proof) A can be decomposed as a direct sum of direct summands of countable direct sum of uniserial modules that are not quasi-small. Thus we can suppose that I is countable. Then A is a countably generated module and, by Proposition 3.11 and Proposition 3.1, A is serial. \square

4. UNISERIAL MODULES THAT ARE NOT QUASI-SMALL OVER RIGHT CHAIN RINGS

In this section we illustrate our theory in the particular case of right chain rings. Recall that the ring is called a *right chain ring* if the module R_R is uniserial. Thus for any right ideal $I \subseteq R$ we have a cyclic uniserial module R/I and, of course, all cyclic right modules are uniserial. First we give an extension of Puninski's result (see [3, Theorem 9.19]).

Lemma 4.1. *Let R be a right chain ring and U, V be cyclic right modules. If $u \in U = uR$, $v \in V = vR$, then there is an epimorphism $f: U \rightarrow V$ such that $f(u) = v$ or there is an epimorphism $g: V \rightarrow U$ such that $g(v) = u$.*

Proof. There are right ideals $I, J \subseteq R$ and isomorphisms $\alpha: U \rightarrow R/I$, $\beta: V \rightarrow R/J$ such that $\alpha(u) = 1 + I$ and $\beta(v) = 1 + J$. If $I \subseteq J$, then there is an epimorphism $\pi: R/I \rightarrow R/J$ sending $1 + I$ to $1 + J$. Then we put $f = \beta^{-1}\pi\alpha$. If $J \subseteq I$, we define the map g in the same way. \square

Proposition 4.2. *Let $U_i, V_j, i \in I, j \in J$ be non-zero cyclic right modules over a right chain ring R . Then $\bigoplus_{i \in I} U_i \simeq \bigoplus_{j \in J} V_j$ if and only if there is a bijection $\sigma: I \rightarrow J$ such that $U_i \simeq V_{\sigma(i)}$ for every $i \in I$.*

Proof. By [3, Theorem 9.12], there is a bijection $\sigma: I \rightarrow J$ such that $[U_i]_m = [V_{\sigma(i)}]_m$. By Lemma 4.1, there is always an epimorphism from U_i to $V_{\sigma(i)}$ or from $V_{\sigma(i)}$ to U_i , so $U_i \simeq V_{\sigma(i)}$ by Remark 2.1. \square

By [5] there is a non-serial direct summand of a direct sum of cyclic modules over an exceptional nearly simple chain ring. Thus we cannot hope to give the Krull-Schmidt Theorem for the class of direct sums of cyclic uniserial modules over a right chain ring.

We want to describe right chain rings possessing uniserial modules that are not quasi-small. For this reason we want to compute U_m, U_e of a cyclic uniserial module.

Lemma 4.3. *Let R be a right chain ring and let I be a proper right ideal of R . Then $(R/I)_e = RI/I$.*

Proof. Since R is projective, the situation is clear when $I = 0$. Let $x \in R \setminus J(R)$ and suppose $xI \not\subseteq I$. Then the left multiplication by x^{-1} induces an epimorphism $f: R/I \rightarrow R/I$. For any $i \in I$ we have $xi + I \in \text{Ker } f$ and thus $RI/I \subseteq (R/I)_e$ (note that any element of R can be written as a sum of at most two invertible elements). On the other hand any epimorphism $g: R/I \rightarrow R/I$ is induced by a left multiplication on R by some invertible element. Thus if $r + I \in (R/I)_e$, then $xr \in I$ for some $x \in R \setminus J(R)$ and $r \in RI$. \square

Lemma 4.4. *Let R be a right chain ring and let I be a non-zero proper right ideal. Then any monomorphism $f: R/I \rightarrow R/I$ is induced by left multiplication of $r \in R$ such that $rI = I$. Hence, $(R/I)_m = P/I$, where $P = \bigcap_{rI=I} rR$.*

Proof. Any endomorphism of R/I is induced by left multiplication of some element $r \in R$ satisfying $rI \subseteq I$. Suppose $rI = I$, then $r \cdot \text{ann}(r) \not\subseteq I$ since

we suppose I non-zero. If $rs \in I$, then $r(s - i) = 0$ for some $i \in I$ and $s \in I$ follows. On the contrary suppose that left multiplication by r induces a monomorphism $f: R/I \rightarrow R/I$ and that $rI \not\subseteq I$, then $rI \not\subseteq I \not\subseteq rR$ and therefore $rs \in I$ for some $s \in R \setminus I$. But $s + I \in \text{Ker } f$, a contradiction. Now $(R/I)_m = P/I$ by the definitions. \square

Lemma 4.5. *Let R be a right chain ring and let $x \in J(R)$ be such that $I = RxR = I^2$. Then there are $s \in I$ and $r \in R^*$ such that $sx = xr$.*

Proof. By assumptions there are $r_1, r_2, t \in R$ such that $x = r_1xr_2xt$. If $t \in R^*$ we just put $s = r_1xr_2$ and $r = t^{-1}$. If $t \in J(R)$, then $1 - t \in R^*$ and $r_1xr_2x(1 - t) = r_1xr_2x - x = (r_1xr_2 - 1)x$. Now $r_3 = r_1xr_2 - 1$ is invertible and we have $r_3^{-1}r_1xr_2x = x(1 - t)^{-1}$. Then we set $s = r_3^{-1}r_1xr_2$ and $r = (1 - t)^{-1}$. \square

Proposition 4.6. *Let R be a right chain ring. Then there is a uniserial right module over R that is not quasi-small if and only if there is $0 \neq x \in J(R)$ such that the ideal RxR is idempotent.*

Proof. First suppose that the ideal $I = RxR$ is idempotent for some $0 \neq x \in J(R)$. By Lemma 4.5, $s \in I$ and $r \in R^*$ such that $sx = xr$ and consequently $sxR = xR$. Thus $(R/xR)_m \subseteq sR/xR$. But $sR \not\subseteq I$ because $I = II \subseteq IJ(R) \subseteq I$ and I cannot be finitely generated as a right ideal. Thus we have $(R/xR)_m \subseteq sR/xR \not\subseteq I/xR = (R/xR)_e$. Now, we can use Lemma 2.9.

To prove the converse let us suppose the existence of a uniserial module over R that is not quasi-small. By Lemma 2.9, there is a right ideal I such that $(R/I)_m \not\subseteq (R/I)_e = RI/I$. Clearly, I is not a two-sided ideal. Let $y \in RI \setminus I$ and consider canonical epimorphism $\pi: R/I \rightarrow R/yR$. Since $y \in RI$, $[R/I]_e = [R/yR]_e$ and $\pi^{-1}((R/yR)_e) = (R/I)_e$ by Lemma 2.3(iv). Thus $RyR = RI$ for any $y \in RI \setminus I$. Now, there is some $x \in RI \setminus I$ such that $xS \in I$ if and only if $x \in I$. Since $x^2 \notin I$, we have $RI \supseteq RxRxR \supseteq Rx^2R = RI$ and therefore $(RxR)^2 = RxR$. \square

In [6] a classification of pure projective modules over a nearly simple chain domain was given. Later, Puninski asked for a proof of this classification inside the ring theory. Next example brings such a proof (there is one hidden result by Bass, but there is no using of the model theory).

First, let us recall some facts about nearly simple chain rings. We say that R is a *nearly simple chain ring* if the right R -module R_R and the left R -module ${}_R R$ are uniserial, there are exactly three two-sided ideals $0, J(R), R$, and $J(R)^2 = J(R)$. The following Proposition collects some facts about

pure projective modules over nearly simple chain rings proved in [6]. Recall that a module is called *pure projective* if it is a direct summand of a direct sum of finitely presented modules (more usual definition is that M is pure projective if and only if $\text{Hom}_R(M, -)$ is exact on pure exact sequences).

Proposition 4.7. *Let R be a nearly simple chain ring.*

- (i) *If x, y are non-zero elements of $J(R)$, then $R/xR \simeq R/yR$.*
- (ii) *Any pure projective module over R is a direct summand of $R^{(X)} \oplus (R/xR)^{(Y)}$, where $0 \neq x \in J(R)$.*
- (iii) *Let $0 \neq x \in J(R)$. Then $(R/xR)_e = J(R)/xR$ and $(R/xR)_m = 0$.*

Proof. (i) By Lemma 4.3, $(R/xR)_e = (J(R)/xR)$ and $(R/yR)_e = (J(R)/yR)$, hence $[R/xR]_e = [R/yR]_e$. If $x = ty$ for some $t \in R$, then the left multiplication by t induces a monomorphism from R/yR to R/xR , if $y = tx$ for some $t \in R$, then there is a monomorphism from R/xR to R/yR . In all circumstances $R/xR \simeq R/yR$. See also [6, Corollary 4.3].

(ii) By [3, Theorem 3.29], any finitely presented module over a chain ring R is isomorphic to a finite direct sum of modules of type R/rR , $r \in R$. Now the result follows from (i).

(iii) It remains to prove that any non-zero cyclic submodule of R/xR is isomorphic to R/xR . Observe that R/I can be embedded into R/xR if and only if $sI = xR$ for some $s \in R$. If I is not cyclic, then $xR = sI = sIJ(R) = xJ(R)$ that is impossible for $x \neq 0$. Thus any non-zero cyclic submodule of R/xR has to be isomorphic to R/xR by (i). \square

By Proposition 4.6 and Proposition 3.2, there is unique uniserial module V such that V is not quasi-small and $[V]_m = [R/rR]_m$ for some (any) $0 \neq r \in J(R)$. By [1, Theorem 4.9], V is a direct summand of $R/rR^{(\omega)}$, hence it is pure projective. Now we are ready to give the promised example:

Example 4.8 Let R be a nearly simple chain domain. Then any pure projective module over R is isomorphic to $F \oplus U^{(I)} \oplus V^{(J)}$, where F is a free module, U is the (unique) non-zero cyclic finitely presented module that is not free and V is the (unique) uniserial module over R that is not quasi-small such that $[U]_m = [V]_m$.

Proof. By [3, Corollary 2.49], that any pure projective module over R can be decomposed into a direct sum of direct summands of $R^{(\omega)} \oplus U^{(\omega)}$. Thus it is enough to classify direct summands of this module.

Now let $M = A \oplus B = U^{(\omega)} \oplus R^{(\omega)}$. Since R is a domain and every element of U has torsion, we see that there is no non-zero homomorphism from U to R . Thus $U^{(\omega)}$ is an invariant submodule of M . Let $A_1 = \pi_A(U^{(\omega)})$, $B_1 = \pi_B(U^{(\omega)})$, then $U^{(\omega)} = A_1 \oplus B_1$ and there are $A_2 \subseteq A$, $B_2 \subseteq B$

such that $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$. Since $A_2 \oplus B_2 \simeq R^{(\omega)}$ and R is a local ring, A_2 has to be free. It remains to check the decomposition $A_1 \oplus B_1 \simeq U^{(\omega)}$. By [2, Proposition 2.9], $A_1 = A'_1 \oplus A'_2$, where A'_1 is a direct sum of copies of U and A'_2 has no maximal submodule. Thus it remains to prove the following claim: Any non-zero direct summand of $U^{(\omega)}$ that has no maximal submodule is isomorphic to a direct sum of copies of V .

Let $X \oplus Y = U^{(\omega)} = \bigoplus_{i \in \mathbb{N}} U_i$ and let X has no maximal submodule. Let $V_j = \pi_j(X) \subseteq U_j$. Suppose $V_j \neq 0$. Since V_j cannot be cyclic and $\pi_j(X) = \sum_{k \in \mathbb{N}} \pi_j \pi_X(U_k)$, V_j is a countable union of its cyclic submodules. By Lemma 2.11, $(V_j)_e = V_j$ (all non-zero submodules of V_j have the same monogeny class because $U_m = 0$) and we conclude that V_j is not quasi-small and it has the same monogeny class as U , thus $V \simeq V_j$. After all X is isomorphic to a direct summand of $V^{(\omega)}$ and we conclude by Corollary 3.9 and Proposition 3.2. \square

Acknowledgements: Main part of this work was done during the stay of the author in CRM, Bellaterra at Marie Curie Training Site. The author thanks for warm hospitality. Author is also indebted to Dolors Herbera for her help.

REFERENCES

- [1] N. V. Dung, A. Facchini, *Weak Krull-Schmidt for infinite direct sums of uniserial modules*, J. Algebra **193** (1997), 93-106.
- [2] N. V. Dung, A. Facchini, *Direct summands of serial modules*, J. Pure Appl. Algebra **133**(1998), 93-106.
- [3] A. Facchini, *Module Theory. Endomorphism rings and direct sum decompositions in some classes of modules*, Birkhauser, 1998
- [4] A. Facchini, *The Krull-Schmidt Theorem* preprint.
- [5] G. Puninski, *Some model theory over an exceptional uniserial ring and decompositions of serial modules*, J. London Math. Soc. (2) **64**(2001), No. 2, 311–326.
- [6] G. Puninski, *Some model theory over nearly simple uniserial domain and decompositions of serial modules*, J. Pure Appl. Algebra, 2001
- [7] P. Příhoda, *Weak Krull-Schmidt Theorem and direct sum decompositions of serial modules of finite Goldie dimension*, maybe it will be published someday in Journal of Algebra