A PROBLEM OF VON NEUMANN AND MAHARAM ABOUT ALGEBRAS SUPPORTING CONTINUOUS SUBMEASURES

STEVO TODORCEVIC

ABSTRACT. We show that a σ-algebra $\mathbb{B}$ carries a strictly positive continuous submeasure if and only if $\mathbb{B}$ is weakly distributive and it satisfies the σ-finite chain condition of Horn and Tarski [2; 2.4(ii))]

1. INTRODUCTION

Recall that a *measure algebras* is a complete boolean algebra $\mathbb{B}$ supporting a strictly positive countably additive measure. The problem of characterizing measure algebras in the class of complete boolean algebras was first proposed by von Neumann in the 1930s (see [10]). The first necessary condition isolated in [10] is the *countable chain condition* asserting that every cellular family of members of $\mathbb{B}$ must be countable. Recall, that a *cellular family* is any family of pairwise disjoint members of $\mathbb{B}$. The second necessary condition given in [4] is the *weak (σ-)distributivity* of $\mathbb{B}$ asserting that for every double sequence a_{nk} (indexed by non-negative integers) of elements of $\mathbb{B}$,

$$\bigwedge_n \bigvee_k a_{nk} = \bigvee_F \bigwedge_n a_{nF(n)},$$

where $F = (F(n))$ ranges over all sequences of finite sets of non-negative integers and where $a_{nF(n)}$ is defined to be the supremum of a_{nk} for $k \in F(n)$. So this is a natural weakening of the usual *σ-distributive law* where the $F(n)'$s are assumed to be all singletons. The first major advance on von Neuman's problem is given by Maharam [9] who has correctly identified it as a problem which is in part a metrization problem for the corresponding sequential topology of $\mathbb{B}$. In fact, Maharam [9] shows that the sequential topology of $\mathbb{B}$ is metrizable if and only if $\mathbb{B}$ supports a strictly positive continuous submeasure. Recall that, a *strictly positive continuous submeasure* on a σ-algebra $\mathbb{B}$ is a function $\nu : \mathbb{B} \to [0, \infty)$ such that:

Date: 12 July 2004.

(*a*) $\nu(a) = 0$ if and only if $a = \mathbf{0}$,
(*b*) $a \leq b$ implies $\nu(a) \leq \nu(b)$,
(*c*) $\nu(a \vee b) \leq \nu(a) + \nu(b)$.
(*d*) $\bigwedge_n a_n = \mathbf{0}$ implies $\nu(a_n) \to 0$ for every decreasing sequence a_n.

If (d) is weakenned to

(d^-) If a_n is an increasing sequence then $\nu(a_n) \to \nu(\bigvee_n a_n)$

then one says that $\mathbb{B}$ supports a *strictly positive submeasure*. A submeasure ν on $\mathbb{B}$ is *nontrivial* if for every a in $\mathbb{B}^+ = \mathbb{B} \setminus \{\mathbf{0}\}$ there is $b \leq a$ such that $0 < \nu(b) < \nu(a)$). The work of Maharam [9] gives a natural decomposition of von Neuman's problem into the following two parts:

(I) Does every weakly distributive complete boolean algebra $\mathbb{B}$ satisfying the countable chain condition support a strictly positive continuous submeasure?
(II) Given that $\mathbb{B}$ supports a strictly positive continuous submeasure, does it support also a strictly positive countably additive measure?

The part (II) is the original form of the well-known and well-studied Control Measure Problem which has a strong degree of absoluteness and many reformulations in different areas of Functional Analysis(see [4], [8]). Already at the very start of the analysis of von Neuman's problem it has been realized that this formulation of its Part(I) lacks in absoluteness. To explain this, recall that a *Souslin algebra* is a non-atomic σ-distributive boolean algebra satisfying the countable chain condition. Its existence is equivalent to the negation of Souslin hypothesis and is therefore independent of the standard axioms of set theory. In [9], Maharam shows that no Souslin algebra can support a non-trivial submeasure a result this clearly indicates that the countable chan condition along with weak distributivity may not be sufficient even for the existence of nontrivial submeasure at least if one is not willing to go beyond the standard axioms of set theory. Remarkably, quite recently it has been shown by Barlcar, Jech and Pazák [3] that the positive answer to (I) is consistent with the standard axioms of set theory. However, one still would like to see if supplementing von Neumann's list with some other necessary conditions one arrives at a positive answer on these questions without going beyond the standard axioms of set theory. It turns out that such a condition can be found in a parallel series of investigations that begun with the paper of Horn and Tarski [7] on the following problem of Tarski (see [12], [13]):

(III) Does every boolean algebra satisfying the countable chain condition supports a strictly positive *finitely*-additive measure?

While working on this problem Horn and Tarski [7] have realized that the countable chain condition has to be strengthened even if one wants to have a finitely additive measure on $\mathbb{B}$. For example, it is readily seen that if μ is a strictly positive measure on $\mathbb{B}$ then the sets $\mathbb{B}_n = \{a \in \mathbb{B} : \mu(a) > \frac{1}{n+1}\}$ witness that $\mathbb{B}$ satisfies the following chain condition that is considerably stronger than the countable chain condition:

(σbcc) There is a decomposition $\mathbb{B}^+ = \bigcup_{n=0}^{\infty} \mathbb{B}_n$ such that for every n, the piece $\mathbb{B}_n$ contains no cellular subfamily of size $n+2$.

On the other hand, note that of $\nu : \mathbb{B} \to [0,\infty)$ is a strictly positive continuous submeasure then ν is *exhaustive* in the sense that for every $\epsilon > 0$ the set $\{a \in \mathbb{B} : \nu(a) > \epsilon\}$ contains no infinite cellular family. It follows that every σ-algebra supporting a strictly positive continuous submeasure satisfies the following chain condition which is also considered by Horn and Tarski [7] and which is still considerably stronger from the countable chain condition:

(σfcc) There is a decomposition $\mathbb{B}^+ = \bigcup_{n=0}^{\infty} \mathbb{B}_n$ such that no $\mathbb{B}_n$ contains an infinite cellular subfamily.

Looking at the difference between σbcc and σfcc one may wander if there is a finite upper bound on the sizes of cellular subfamilies of $\{a \in \mathbb{B} : \nu(a) > \epsilon\}$ whenever μ is a strictly positive continuous submeasure on $\mathbb{B}$ and where $\epsilon > 0$. The positive answer to this question is yet another equivalent formulation of the Control Measure Problem (II) discussed above. Regardless on the state of the Control Measure Problem, one can still ask whether the two chain conditions σbcc and σfcc are equivalent requirements on a given boolean algebra $\mathbb{B}$. In [7], Horn and Tarski also ask if any of these two chain conditions is sufficient for the existence of a strictly positive finitely additive measure on $\mathbb{B}$. This was answered into the negative by Gaifman [6]. It turns out however that the σfcc of Horn and Tarski is indeed the right chain condition for answering the first part of von Neumanns question as the following result shows.

Theorem 1. *The following are equivalent for every complete Boolean algebra $\mathbb{B}$:*

(1) $\mathbb{B}$ *carries a strictly positive continuous submeasure.*
(2) (*a*) $\mathbb{B}$ *is weakly distributive and*
    (*b*) $\mathbb{B}$ *satisfies the σ-finite chain condition.*

It follows, in particular, that von Neumann's problem in the class of σ-algebras satisfying the σ-finite chain condition is equivalent to the Control Measure Problem. So for all practical purposes, von Neumann's problem [9] has now been reduced to the Control Measure Problem of Maharam [9].

2. A P-ideal of converging sequences

Recall, the definition of *strong convergence* of sequences a_n of members of some σ-algebra $\mathbb{B}$ as introduced in [9]: $a_n \to a$ if and only if lim sup $(a_n \triangle a) = \mathbf{0}$. As shown in [9], if $\mathbb{B}$ is weakly distributive this notion of convergence satisfies the axioms of the abstract theory of convergence of Fréchet [5] and, in particular, his diagonal sequence condition:

(L4) If $a_n \to a$ and if for each n we are given that $a_{nk} \to a_n$ then there is an increasing sequence k_n of non-negative integers such that $a_{nk_n} \to a$.

In fact, in the context of σ-algebras $\mathbb{B}$ satisfying the countable chain condition, where we are working here, this principle is equivalent to the weak distributivity of $\mathbb{B}$ as well as to the following *diagonal sequence property* (see [9] and [16]):

(DS) Given a double sequence a_{nk} such that for each n the sequence a_{nk} decreases monotonically to $\mathbf{0}$ as $k \to \infty$ then there is an increasing sequence k_n of non-negative integers such that $a_{nk_n} \to \mathbf{0}$.

Hence, defining the closure $\bar{A}$ of a subset A of $\mathbb{B}$ to be the collection of all limits of strongly converging sequences of members of A, we get a closure operator and the corresponding *sequential topology* of $\mathbb{B}$. In [9], Maharam shows that a σ-algebra $\mathbb{B}$ supports a strictly positive continuous submeasure if and only if its sequential topology is metrizable (in which case $\mathbb{B}$ becomes a topological group under the relation of symmetric difference). This shows clearly the upmost importance of the sequential topology in the study of von Neumann's problem. As correctly realized in [9] the metrizability of the sequential topology of $\mathbb{B}$ is captured by the algebraic (or better to say, combinatorial) properties of the ideal of all countable subsets of $\mathbb{B}$ which don't contain $\mathbf{0}$ in their closures. In fact, it turns out that it is more convenient to work with the orthogonal of this ideal. Thus, let $\mathcal{I}_{\mathbb{B}}$ be the collection of all countable subsets A of $\mathbb{B}^+$ for which we can find a maximal cellular family $\mathcal{C}$ of $\mathbb{B}$ such that

$$c \restriction A = \{a \in A : c \cdot a \neq \mathbf{0}\}$$

is finite for all $c \in \mathcal{C}$. Note that if a maximal cellular family $\mathcal{C}$ witnesses the membership of A in $\mathcal{I}_{\mathbb{B}}$ then so does every other maximal cellular family $\mathcal{D}$ which *refines* $\mathcal{C}$, i.e., has the property that every d from $\mathcal{D}$ is included in a (necessarily unique) member of $\mathcal{C}$. It follows that that $\mathcal{I}_{\mathbb{B}}$ is indeed an *ideal* of subsets of $\mathbb{B}^+$, i.e., that it is closed under unions, since for every pair $\mathcal{C}_0$ and $\mathcal{C}_1$ of maximal cellular families of $\mathbb{B}$ there is a maximal cellular family $\mathcal{D}$ of $\mathbb{B}$ which refines them both. The ideal $\mathcal{I}_{\mathbb{B}}$ appears for the first time in [1] for $\mathbb{B}$ a Souslin algebra. The σ-distributivity of the Souslin algebra is used in [1] to show that $\mathcal{I}_{\mathbb{B}}$ is a *P-ideal*, i.e. it has the following weak form of

σ-completeness: For every sequence A_n of members of $\mathcal{I}_{\mathbb{B}}$ there is a member B of $\mathcal{I}_{\mathbb{B}}$ such that $A_n \setminus B$ is finite for all n. To see this, for each n, fix a maximal cellular family $\mathcal{C}_n$ witnessing $A_n \in \mathcal{I}_{\mathbb{B}}$. Using σ-distributivity of $\mathbb{B}$ we can find a maximal cellular family $\mathcal{C}$ of $\mathbb{B}$ which refines $\mathcal{C}_n$ for all n, and therefore witnesses simultaneously $A_n \in \mathcal{I}_{\mathbb{B}}$ for all n. Since $\mathbb{B}$ satisfies the countable chain condition $\mathcal{C}$ is countable so we can enumerate it in a simple sequence c_k. Let

$$B = \bigcup_n A_n \setminus (\bigcup_{k<n} c_k \upharpoonright A_n).$$

Clearly, this B almost includes A_n for every n. Note that the same argument applies if the cellular family $\mathcal{C}$ only *almost refines* each of the families $\mathcal{C}_n$ in the sense that for each n every element of $\mathcal{C}$ intersects only finitely many members of $\mathcal{C}_n$. Note also that the existence of such an $\mathcal{C}$ is guaranteed by the weak distributivity of $\mathbb{B}$. This shows the following useful fact.

Lemma 1. *The ideal $\mathcal{I}_{\mathbb{B}}$ is a P-ideal for every weakly distributive σ-algebra $\mathbb{B}$ satisfying the countable chain condition.*

This fact has been first put out in print by Quickert [11] while analyzing a problem of Prikry that is quite closely related to that of von Neumann [10] (see also [15]).

Back to the sequential topology of $\mathbb{B}$, note that if A is a countably infinite subset of $\mathbb{B}^+$ then A belongs to $\mathcal{I}_{\mathbb{B}}$ if and only if for some (all) one-to-one enumerations a_n of A the sequence a_n converges to $\mathbf{0}$, i.e., lim sup $a_n = \mathbf{0}$. So the orthogonal

$$\mathcal{I}_{\mathbb{B}}^{\perp} = \{X \subseteq \mathbb{B} : X \text{ is countable and } X \cap A \text{ is finite for all } A \in \mathcal{I}_{\mathbb{B}}\}.$$

More generally, we say that an arbitrary subset X of $\mathbb{B}^+$ is *orthogonal* to $\mathcal{I}_{\mathbb{B}}$ and write $X \perp \mathcal{I}_{\mathbb{B}}$ if X has a finite intersection with every member of $\mathcal{I}_{\mathbb{B}}$. Thus, $\mathcal{I}_{\mathbb{B}}^{\perp}$ is simply the collection of all countable sets that are orthogonal to $\mathcal{I}_{\mathbb{B}}$. Put it even more simply, a set is orthogonal to $\mathcal{I}_{\mathbb{B}}$ if its closure misses $\mathbf{0}$. We say that $\mathcal{I}_{\mathbb{B}}^{\perp}$ is *countably generated* if there is a sequence X_n of subsets of $\mathbb{B}^+$ that are orthogonal to $\mathcal{I}_{\mathbb{B}}$ with the property that every member of $\mathcal{I}_{\mathbb{B}}^{\perp}$ is included modulo a finite set in some member of the sequence. The following result of Maharam [9] reveals our interest in these notions.

Lemma 2. *(*[9]*) A σ-algebra $\mathbb{B}$ supports a strictly positive continuous submeasure if and only if the orthogonal of $\mathcal{I}_{\mathbb{B}}$ is countably generated.*

Proof. For the convenience of the reader we sketch the argument of the reverse implication as it appears in [9] though in a slightly different terminology and one unnecessary assumption. First of all note that if $\mathcal{I}_{\mathbb{B}}^{\perp}$ is countably generated then $\mathbb{B}$ satisfies the countable chain condition as well as the diagonal sequence principle (DS). So we can choose a sequence X_n of subsets of $\mathbb{B}^+$ which is increasing in n and consists of sets that are at the same time upwards closed and closed in the sequential topology. As already pointed out, the corresponding sequence V_n of complements consists of neighborhoods of $\mathbf{0}$ that decrease in n and are downwards closed in the natural ordering of $\mathbb{B}$. Note that any subset of $\mathbb{B}$ whose closure does not contain $\mathbf{0}$ must be included in some X_n. It follows that the V_n form a countable neighborhood base of $\mathbf{0}$. Note also that the operations $\vee$ and $\triangle$ are continuous at $(\mathbf{0},\mathbf{0})$ or else we would, say, be able to find a neighborhood U of $\mathbf{0}$ and two sequences a_n and b_n such that $a_n, b_n \in V_n$ but $a_n \vee b_n$ is not in U. It follows that lim sup $a_n = \mathbf{0}$ and lim sup $b_n = \mathbf{0}$ while lim sup $a_n \vee b_n \neq \mathbf{0}$, a contradiction. It follows, that $\mathbb{B}$ with the symmetric difference as the group operation is a metrizable abelian group so it has an invariant metric ϱ. Then $\nu(a) = \varrho(\mathbf{0}, a)$ defines a strictly positive continuous measure on $\mathbb{B}$. □

Note that in the case when $\mathcal{I}_{\mathbb{B}}^{\perp}$ is countably generated one can cover $\mathbb{B}^+$ by countably many sets that are orthogonal to $\mathcal{I}_{\mathbb{B}}$. It is actually this condition that forms a part of a general dichotomy for P-ideals considered in [14]. For general P-ideals $\mathcal{I}$ of countable subsets of some set S the alternative that the set S can be covered by countably many subsets that are orthogonal to $\mathcal{I}$ is unlikely to imply that $\mathcal{I}^{\perp}$ is countably generated. Therefore, the following beautiful discovery of Balcar, Jech, and Pazák [3] (based on a previous work from [2]) came as a surprise.

Lemma 3. [3] *The following are equivalent for a weakly distributive σ-algebra satisfying the countable chain condition:*

(1) *The orthogonal of $\mathcal{I}_{\mathbb{B}}$ is countably generated.*
(2) *$\mathbb{B}^+$ can be decomposed into countably many subsets orthogonal to $\mathcal{I}_{\mathbb{B}}$.*

Proof. For the convenience of the reader again, we sketch the arguments as they appear in [3] and [2] though in a different terminology. Only the implication from (2) to (1) requires an argument. So, let X_n be a sequence of subsets of $\mathbb{B}^+$ which cover $\mathbb{B}^+$ and which are orthogonal to $\mathcal{I}_{\mathbb{B}}$. Again we may assume that X_n increase with n and that they are upwards closed as well as closed in the sequential topology of $\mathbb{B}$. Hence the corresponding sequence V_n of complements is a sequence of open neighborhoods that are downwards closed under the ordering of $\mathbb{B}$. First of all note that $\bigcap_n \overline{V_n} = \{\mathbf{0}\}$. For suppose there is a non-zero member b in that intersection. Then for each n we can find a sequence a_{nk} of members of V_n which strongly converges to b as $k \to \infty$. By (DS) we can find find a diagonal sequence a_{nk_n} such that $a_{nk_n} \to b$. So there must be an m such that $c = \bigwedge_{n>m} a_{nk_n} \neq \mathbf{0}$.

Pick an $n > m$ such that c does not belong to V_n. Then $c \leq a_{nk_n} \in V_n$ contradicting the downward closure of V_n. Now we claim that V_n form a neighborhood base at $\mathbf{0}$ which is of course just another formulation of (1). We first check that V_n refine *closed* neighborhoods of $\mathbf{0}$. Otherwise, fixing a closed neighborhood U of $\mathbf{0}$ not refined by any V_n and using the fact that the boolean operation $\vee$ is separately continuous, we can build a sequences a_n such that $\bigvee_{k=n}^{m} a_k \in V_n \setminus U$ for all $n \leq m$. It follows that $\bigvee_{k \geq n} a_k \in \overline{V_n} \setminus U$ for all n. Since the $\overline{V_n}$ have only the point $\mathbf{0}$ in their intersection, we conclude that lim sup $a_n = \mathbf{0}$. This shows that $a_n \to \mathbf{0}$. So there must be an n such that $a_n \in U$, a contradiction. Similar argument shows that for every open neighborhood W of $\mathbf{0}$ we could find a neighborhood V such that $V \vee V \subseteq W$ giving the continuity of the operations $\vee$ and $\triangle$ and therefore the regularity of the sequential topology. □

It follows that a weakly distributive σ-algebra $\mathbb{B}$ satisfying the countable chain condition supports a strictly positive continuous submeasure if and only if $\mathbb{B}^+$ can be decomposed into countably many subsets orthogonal to $\mathcal{I}_{\mathbb{B}}$ if and only if $\mathbf{0}$ is a G_δ-point in $\mathbb{B}$.

3. Proof of Theorem 1

We prove that (2) implies (1). Thus, we start with a weakly distributive σ-algebra $\mathbb{B}$ satisfying the countable chain condition. Let $\mathcal{I} = \mathcal{I}_{\mathbb{B}}$ be the corresponding P-ideal, i.e., the collection of all countable subsets B of $\mathbb{B}^+$ for which there is a maximal cellular family $\mathcal{A}$ of $\mathbb{B}^+$ such that every member of $\mathcal{A}$ meets only finitely many members of B. We shall show that if $\mathbb{B}^+$ cannot be decomposed into countably many sets each orthogonal to $\mathcal{I}$ then the σ-algebra $\mathbb{B}$ does not satisfy the stronger σ-finite chain condition of Horn and Tarski. Then the result will follow from Lemmas 2 and 3 above. So from now on we assume that $\mathbb{B}^+$ cannot be covered by countably many sets orthogonal to $\mathcal{I}$ and work for showing that $\mathbb{B}$ fails to satisfy the σ-finite chain condition. So, consider a decomposition $\mathbb{B}^+ = \bigcup_{k=0}^{\infty} \mathbb{B}_k$ and work towards showing that some $\mathbb{B}_k$ contains an infinite cellular family.

Following [14], we let $\mathcal{P}$ be the collection of all pairs $p = \langle x_p, \mathfrak{X}_p \rangle$, where

(i) x_p is an element of $\mathcal{I}$.
(ii) $\mathfrak{X}_p$ is a countable collection of cofinal subsets of $\langle \mathcal{I}, \subseteq^* \rangle$.

We order $\mathcal{P}$ by letting $q \leqslant p$ (q extends p) when:

(iii) $x_p \subseteq x_q$
(iv) $\mathfrak{X}_p \subseteq \mathfrak{X}_q$
(v) For every $X \in \mathfrak{X}_p$ the set $\{x \in X : x_q \setminus x_p \subseteq x\}$ is $\subseteq^*$-cofinal in $\mathcal{I}$ and it belongs to $\mathfrak{X}_q$.

Lemma 4. *For every $p \in \mathcal{P}$ and every maximal cellular family $\mathcal{A}$ of $\mathbb{B}^+$ there is $q \leqslant p$ such that some member of $x_q \setminus x_p$ refines a member of $\mathcal{A}$.*

Proof. Let $\mathbb{B}^+ \restriction \mathcal{A}$ be the collection of all $b \in \mathbb{B}^+$ which refine a member of $\mathcal{A}$. Then $\mathbb{B}^+ \restriction \mathcal{A}$ is coinitial in $\mathbb{B}^+$ so by our assumption it cannot be covered by countably many sets orthogonal to $\mathcal{I}$. Suppose the conclusion of the lemma fails. Then for every member $b \in (\mathbb{B}^+ \restriction \mathcal{A}) \setminus x_p$ there is $X \in \mathfrak{X}_p$ such that

$$X(b) = \{x \in X : b \in x\}$$

is not cofinal in $\langle \mathcal{I}, \subseteq^* \rangle$. For $X \in \mathfrak{X}_p$, let

$$\mathcal{B}(X) = \{b \in (\mathbb{B}^+ \restriction \mathcal{A}) \setminus x_p : X(b) \text{ is not cofinal in } \langle \mathcal{I}, \subseteq^* \rangle\}.$$

By our assumption $\{\mathcal{B}(X)\}_{X \in \mathfrak{X}_p}$ is a countable collection which covers $(\mathbb{B}^+ \restriction \mathcal{A}) \setminus x_p$.

So we shall reach a contradiction the moment we show that each $\mathcal{B}(X)$ is orthogonal to $\mathcal{I}$. For suppose that for some $X \in \mathfrak{X}_p$ there is an infinite subset z of $\mathcal{B}(X)$ belonging to $\mathcal{I}$. Since $\mathcal{I}$ is a P-ideal, removing a finite subset of z we may assume that

$$\mathcal{Y} = \{y \in X : z \subseteq y\}$$

is cofinal in $\langle \mathcal{I}, \subseteq^* \rangle$. Pick $b \in z$. Then $b \in \mathcal{B}(X)$ and $\mathcal{Y} \subseteq X(b)$, a contradiction since $X(b)$ is not supposed to be cofinal in $\langle \mathcal{I}, \subseteq^* \rangle$. □

Choose a countable elementary submodel M of some large enough $\langle H_\theta, \in \rangle$ such that M contains all the relevant objects.

Lemma 5. *Suppose $p \in \mathcal{P} \cap M$ and $z_M \in \mathcal{I}$ is such that $z_M \subseteq M \cap \mathbb{B}^+$ and $x \subseteq^* z_M$ for all $x \in \mathcal{I} \cap M$. Then for every $k \in \mathbb{N}$ there is $q \leqslant p$ in $\mathcal{P} \cap M$ such that $x_q \setminus x_p \subseteq z_M$ and such that either*

(*a*) *$x_q \setminus x_p$ contains a member of $\mathbb{B}_k$, or*

(*b*) *there is no $r \leqslant q$ in $\mathcal{P}$ for which $x_r \setminus x_p$ contains a member of $\mathbb{B}_k$.*

Proof. Suppose such a q cannot be found. Let $\mathcal{D}$ be the collection of all $q \in \mathcal{P}$ satisfying (a) or (b). Clearly $\mathcal{D} \in M$ and $\mathcal{D}$ is dense below p. Let

$$Y_0 = \{y \in \mathcal{I} : \exists\, F \in [y]^{<\omega}\ \forall\, q \leqslant p\ (q \in \mathcal{D} \implies x_q \setminus x_p \not\subseteq y \setminus F)\}.$$

Clearly $Y_0 \in M$. Note that every member y of $\mathcal{I} \cap M$ belongs to Y_0 since the finite set $y \setminus z_M$ witnesses this by our assumption that there is no $q \leqslant p$ satisfying the conclusion of the lemma. By elementarity, it follows that Y_0 is actually equal to $\mathcal{I}$. Fix a mapping $A \mapsto z_A$ from $[\mathcal{I}]^\omega$ into $\mathcal{I}$ belonging to M and having the property that

(vi) $z_A \subseteq \bigcup A$ and $x \subseteq^* z_A$ for every $x \in A$.

Then by the fact that $Y_0 = \mathcal{I} \supseteq \{z_A : A \in [\mathcal{I}]^\omega\}$, for each $A \in [\mathcal{I}]^\omega$ we can fix a finite set $F_A \subseteq z_A$ such that

(vii) $\forall\, q \leqslant p\ (q \in \mathcal{D} \implies x_q \setminus x_p \not\subseteq z_A \setminus F_A)$

Moreover we may assume that the mapping $A \mapsto F_A$ belongs to M. Find in M a stationary subset S of $[\mathcal{I}]^\omega$ and a finite set F such that $F_A = F$ for all $A \in S$. Let

$$p_1 = \langle x_p, \mathfrak{X}_p \cup \{z_A \setminus F \colon A \in S\}\rangle.$$

Then $p_1 \in \mathcal{P} \cap M$. So there is $q \in \mathcal{D} \cap M$ such that $q \leqslant p_1$. By the definition of the ordering of $\mathcal{P}$ there exists cofinally many members of the set $\{z_A \setminus F \colon A \in S\}$ which contain the difference $x_q \setminus x_p$. So fix one such $z_A \setminus F$ in this set so that

(viii) $x_q \setminus x_p \subseteq z_A \setminus F$.

Since $F = F_A$, this contradicts (vii). This finishes the proof of Lemma 5. □

Similarly one proves the following

Lemma 6. *Suppose $p \in \mathcal{P} \cap M$ and let $z_M \in \mathcal{I}$ be such that $z_M \subseteq M \cap \mathbb{B}^+$ and $x \subseteq^* z_M$ for all $x \in \mathcal{I} \cap M$. Then for every finite subset F of $\mathbb{B}^+ \cap M$, every $p \in \mathcal{P} \cap M$ and every $k \in \mathbb{N}$ there is $q \leqslant p$ in $M \cap \mathcal{P}$ such that $x_q \setminus x_p \subseteq z_M$ and such that either*

(*a*) *$x_q \setminus x_p$ contains a member of $\mathbb{B}_k$ disjoint from every member of F, or*

(*b*) *there is no $r \leqslant q$ in $\mathcal{P}$ such that $x_r \setminus x_p$ contains a member of $\mathbb{B}_k$ that is disjoint from every member of F.*

Working in M and starting from $p_0 = \langle \emptyset, \{\mathcal{I}\}\rangle$ we build a decreasing sequence $p_n = \langle x_n, \mathfrak{X}_n\rangle$ ($n \in \mathbb{N}$) of members of $\mathcal{P} \cap M$ and a $\subseteq$-decreasing sequence z_n ($n \in \mathbb{N}$) of members of $\mathcal{I}$ such that

(ix) $\forall\, n \in \mathbb{N}\ \ x \subseteq^* z_n$

(x) $\forall\, n \in \mathbb{N}\ \ x_{n+1} \setminus x_n \subseteq z_n$

The set z_0 is of course chosen arbitrarily, while z_{n+1} is obtained from z_n by removing from it a finite set. This is done in the following manner. Via some book-keeping device, we associate to each n and each $X \in \mathfrak{X}_n$ an integer $m \geqslant n$ where we perform the following procedure. We first let

$$X_1 = \{x \in X \colon x_m \setminus x_n \subseteq x\}.$$

Then $X_1 \in M$ and X_1 is cofinal in $\langle \mathcal{I}, \subseteq^*\rangle$. So there is a finite set $F_m \subseteq z_m$ such that

$$X_2 = \{x \in X_1 \colon z_m \setminus F_m \subseteq x\}$$

is still cofinal in $\langle \mathcal{I}, \subseteq^*\rangle$. We let $z_{m+1} = z_m \setminus F_m$ and

$$p_{m+1} = \langle x_m, \mathfrak{X}_m \cup \{x \setminus F_m \colon x \in \mathcal{I}\}\rangle.$$

This together with (x) ensures that

(xi) $\forall\, l > m\ \ (x_l \setminus x_m) \cap F_m = \emptyset$

and therefore that

(xii) $\forall\, x \in X_2 \;\; x_\omega \setminus x_n \subseteq x$

where $x_\omega = \bigcup_{i=0}^{\infty} x_i$. This will ensure that if we let

$$\mathfrak{X}_\omega = \Big(\bigcup_{i=0}^{\infty} \mathfrak{X}_i\Big) \cup \{\{x \in X \colon x_\omega \setminus x_n \subseteq x\} \colon n \in \mathbb{N},\; X \in \mathfrak{X}_n\}$$

then $p_\omega = \langle x_\omega, \mathfrak{X}_\omega\rangle$ is a member of $\mathcal{P}$ extending all p_n's. Using Lemmas 5 and 6 we can arrange our book-keeping device so that the following two conditions are satisfied

(xiii) For every $k, n \in \mathbb{N}$ there is $m \geqslant n$ such that either $(x_{m+1} \setminus x_m) \cap \mathbb{B}_k \neq \emptyset$ or $\forall\, q \leqslant p_{m+1} \;\; (x_q \setminus x_m) \cap \mathbb{B}_k = \emptyset$.

(xiv) For every $k, n \in \mathbb{N}$, for every $F \in [x_n]^{<\omega}$ there is $m \geqslant n$ such that either

$$\exists\, b \in (x_{m+1} \setminus x_m) \cap \mathbb{B}_k \;\forall\, a \in F \;\; (b \cdot a = 0)$$

or

$$\forall\, q \leqslant p_{m+1} \;\forall\, b \in (x_q \setminus x_m) \cap \mathbb{B}_k \;\exists\, a \in F \;\; (b \cdot a \neq 0).$$

This finishes our description of the recursive construction. Note that in particular x_ω belongs to $\mathcal{I}$. So we can choose a maximal cellular family $\mathcal{A}$ of $\mathbb{B}^+$ such that every member of $\mathcal{A}$ meets only finitely many members of x_ω. Applying Lemma 4 to p_ω and $\mathcal{A}$ we find an extension $q \leqslant p_\omega$ such that $x_q \setminus x_\omega$ contains a member b_0 which refines some member of the cellular family $\mathcal{A}$. Fix $k_0 \in \mathbb{N}$ such that $b_0 \in \mathbb{B}_{k_0}$. From (xiii) we infer that

(xv) $\exists^\infty m \;\; (x_{m+1} \setminus x_m) \cap \mathbb{B}_{k_0} \neq \emptyset$.

Pick $\overline{m}$ such that

(xvi) $\forall\, m > \overline{m} \;\forall\, a \in (x_m \setminus x_{\overline{m}}) \;\; (b_0 \cdot a = 0)$.

Pick $m_0 > \overline{m}$ such that we can find $a_0 \in x_{m_0+1} \setminus x_{m_0}$ in $\mathbb{B}_{k_0}$ (see (xv)). Applying (xiv) to $k = k_0$, $n = m_0 + 1$ and $F = \{a_0\}$ and noticing (see (xv)) that the second alternative fails we get $m_1 \geqslant m_0 + 1$ and $a_1 \in (x_{m_1+1} \setminus x_{m_1}) \cap \mathbb{B}_{k_0}$ such that $a_0 \cdot a_1 = 0$. Applying (xvi) we get

(xvii) $\forall\, a \in \{a_0, a_1\} \;\; (b_0 \cdot a = 0)$.

So applying (xiv) for $k = k_0$, $n = m_1 + 1$ and $F = \{a_0, a_1\}$ we get an $m_2 > m_1$ satisfying one of its two alternatives of which the second one is in contradiction to (xvii). This gives us an $a_2 \in (x_{m_2+1} \setminus x_{m_2}) \cap \mathbb{B}_{k_0}$ such that

$$a_2 \cdot a_0 = 0 = a_2 \cdot a_1,$$

and so on. It is clear that continuing this procedure into the infinity we produce an infinite cellular family of elements of $\mathbb{B}_{k_0}$. This finishes the proof of Theorem 1.

References

[1] U. Abraham and S. Todorcevic, *Partition properties of ω_1 compatible with CH*, Fund. Math. 152, (1997), 165–181

[2] B. Balcar, W. Glowczynski and T. Jech, *The sequential topology on complete Boolean algebras*, Fund. Math. 155, (1998), 27–41

[3] B. Balcar, T. Jech and T. Pazák, *Complete ccc Boolean algebras, the order sequential topology and a problem of von Neumann*, preprint, ArXiv 28/12/2003.

[4] D.H. Fremlin, *Chapter 39: Measurable algebras*, Measure Theory, Vol 3, Torres Fremlin, 2002.

[5] M. Fréchet, *Sur la notion de voisinage dans les ensembles abstraits*, Bull. Sci. Math., 42, (1918) 138-156.

[6] H. Gaifman, *Concerning measures on Boolean algebras*, Pacific J. Math. 14, (1964) 61–73.

[7] A. Horn, A. Tarski, *Measures in Boolean algebras*, Trans. Amer. Math. Soc. 64, (1948) 467–497.

[8] N. Kalton, *The Maharam problem.* Séminaire d'Initiation à l'Analyse, Exp. No. 18, 13 pp., Publ. Math. Univ. Pierre et Marie Curie, 94, Univ. Paris VI, Paris, 1989

[9] D. Maharam, *An algebraic characterization of measure algebras*, Ann. of Math. (2) 48, (1947) 154–167.

[10] J. von Neumann, *Problem 163*, The Scottish Book 1937.

[11] S. Quickert, *CH and the Sacks property*, Fund. Math. 171 (2002), no. 1, 93–100.

[12] A. Tarski, *Ideale in vollständigen Mengenkörpern. I*, Fund. Math. 32, (1939). 45–63.

[13] A. Tarski, *Ideale in vollständigen Mengenkörpern. II*, Fund. Math. 33, (1945). 51–65.

[14] S. Todorcevic, *A dichotomy for P-ideals of countable sets*, Fund. Math. 166 (2000), no. 3, 251–267.

[15] B. Velickovic, *CCC forcing and splitting reals*, CRM preprint no 569, 02/2004.

[16] D. A. Vladimirov, *Boolean algebras in analysis*, Kluwer 2004.

Univ. Paris VII-C.N.R.S., UMR 7056, 2 Place Jussieu, 75251 Paris, France.
stevo@math.jussieu.fr

Department of Mathematics, Univ. of Toronto, Toronto, Canada, M5S 3G3.
stevo@math.toronto.edu

Centre de Recerca Matèmatica, Apartat 50, E-08193, Bellaterra, Spain.
stodorcevic@crm.es