THE SIGNATURE AND THE ELLIPTIC GENUS OF EVEN 4-MANIFOLDS WITH S^1 ACTIONS

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ABSTRACT. We prove the vanishing of the signature of oriented smooth 4-manifolds with even intersection form and admitting circle actions. When the manifold is Spin, such a vanishing follows from the Atiyah-Hirzebruch vanishing of $\widehat{A}$-genus regardless of the parity of the intersection form. We prove the vanishing of the signature in the *non-Spin* case by proving the vanishing of the $\widehat{A}$-genus via the rigidity of the elliptic genus under S^1 actions. As a corollary we see that the Enriques surface and its n-fold connected sums admit no smooth S^1 actions.

1. INTRODUCTION

In the study of the topology of oriented smooth 4-manifolds, the intersection form is one of the classical invariants and determining which quadratic forms can represent such an intersection form is still a subject of central interest. In this paper we address this question on 4-manifolds with even intersection form and admitting smooth circle actions. More precisely, let E_8 be the unique irreducible negative definite quadratic form of rank eight and let H be the hyperbolic quadratic form. By the classification of quadratic forms any indefinite even intersection form Q is of the form $aE_8 \oplus bH$. Throughout the paper, we shall assume that the manifolds are oriented, compact, connected and smooth. We prove the following theorem.

Theorem 1.1. *Let M be an even 4-manifold admitting smooth (isometric) circle actions, and let $Q = aE_8 \oplus bH$ denote its intersection form. Then, the signature of M vanishes,* $\mathrm{sign}(M) = 0$, *i.e. the intersection form is $Q = bH$.*

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If the 4-manifold is Spin, the vanishing of the signature follows from the Atiyah-Hirzebruch vanishing of the $\widehat{A}$-genus [3] regardless of the parity of the intersection form. Therefore, the main goal of this paper is to prove the theorem in the non-Spin case. Note that even non-Spin manifolds are not simply-connected [1]. As an example, consider the oriented 4-manifold $S^2 \times S^2/(x,y) \sim (-x,-y)$, which is oriented, non-Spin, has even intersection form H, and admits smooth S^1 actions [1].

Since in dimension 4, the two characteristic numbers satisfy

$$\mathrm{sign}(M) = -8\,\widehat{A}(M),$$

we prove the vanishing of the signature by proving the vanishing of the $\widehat{A}$-genus. This, however, does not follow from index theory for the Dirac operator, since such an operator is not defined on non-Spin manifolds. In a similar fashion to that of [7, 8], we prove the vanishing of the $\widehat{A}$-genus on even non-Spin 4-manifolds admitting circle actions by means of the rigidity of the elliptic genus.

The elliptic genus was introduced as a topological genus by Ochanine [12, 13] and reinterpreted by Witten [15, 16] in a Quantum Field Theoretical context. Witten conjectured the rigidity property under S^1 actions on *Spin* manifolds which was proved by Taubes [14], Bott and Taubes [5] and Liu [11]. The rigidity theorem has been extended to Spin^c manifolds by Dessai [6] and to π_2-finite non-Spin manifolds by the authors [7, 8]. The latter extension allowed the authors to prove the vanishing of the $\widehat{A}$-genus of π_2-finite non-Spin manifolds admitting circle actions [7, 8], therefore extending Atiyah and Hirzebruch's vanishing theorem [3].

The note is organized as follows. In Section 2 we recall the definition of the elliptic genus and prove the rigidity theorem. In Section 3 we briefly recall the argument that renders the vanishing of the $\widehat{A}$-genus and in Section 4 we give some of applications.

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2. Rigidity of the Elliptic Genus

Let $D\colon \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator acting on sections of the vector bundles E and F over the manifold M. The index of D is the virtual vector space $\operatorname{ind}(D) = \ker(D) - \operatorname{coker}(D)$. If M admits a circle action preserving D, i.e. such that S^1 acts on E and F, and commutes with D, $\operatorname{ind}(D)$ admits a Fourier decomposition into complex 1-dimensional irreducible representations of S^1 $\operatorname{ind}(D) = \sum a_m L^m$, where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on $\mathbb{C}$ given by $e^{i\theta} \mapsto e^{im\theta}$. The elliptic operator D is called *rigid* if $a_m = 0$ for all $m \neq 0$, i.e. $\operatorname{ind}(D)$ consists only of the trivial representation with multiplicity a_0. The elliptic operator D is called *universally rigid* if it is rigid under any S^1 action on M by isometries.

Let $\bigwedge_c^{\pm}$ be the even and odd complex differential forms on the oriented, compact, smooth 4-manifold M under the Hodge $*$-operator, respectively. The signature operator

$$d_s^M = d - *d* \ : \ \bigwedge\nolimits_c^+ \longrightarrow \bigwedge\nolimits_c^-$$

is elliptic and the virtual dimension of its index equals the signature of M, $\operatorname{sign}(M)$. If W is a complex vector bundle on M endowed with a connection, we can *twist* the signature operator to forms with values in W

$$d_s^M \otimes W \colon \bigwedge\nolimits_c^+(W) \longrightarrow \bigwedge\nolimits_c^-(W).$$

This operator is also elliptic and the virtual dimension of its index is denoted by $\operatorname{sign}(M, W)$.

Definition 2.1. *Let $T = TM \otimes \mathbb{C}$ denote the complexified tangent bundle of M and let R_i be the sequence of bundles defined by the formal series*

$$R(q,T) = \sum_{i=0}^{\infty} q^i R_i = \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q^i} T \otimes \bigotimes_{j=1}^{\infty} S_{q^j} T,$$

where $S_t T = \sum_{k=0}^{\infty} t^k S^k T$, $\bigwedge_t T = \sum_{k=0}^{\infty} t^k \bigwedge^k T$, and $S^k T$, $\bigwedge^k T$ denote the k-th symmetric and exterior tensor powers of T, respectively. The elliptic genus *of M is defined as*

$$\Phi(M) = \operatorname{ind}(d_s^M \otimes R(q,T)) = \sum_{i=0}^{\infty} q^i \cdot \operatorname{sign}(M, R_i). \tag{1}$$

The first few terms of the sequence $R(q,T)$ are $R_0 = 1$, $R_1 = 2T$, $R_2 = 2(T^{\otimes 2} + T)$, etc. In particular, the constant term of $\Phi(M)$ is $\operatorname{sign}(M)$.

Theorem 2.1. *Let M be an oriented, compact, connected, even, smooth 4-manifold admitting smooth S^1 actions. Then, each of the operators $d_s \otimes R_i$ is universally rigid.*

Proof. Without loss of generality let us assume that the S^1 action is effective.

The proof is carried out along the lines of [5] from which we recall the main arguments and give appropriate modifications. By the Atiyah-Segal G-signature theorem [4]

$$\Phi(M) = \sum_{\{P\}} \mu(P)$$

where P runs over the connected components of the fixed point set of the S^1 action [5, p. 155]. Note that in this dimension the connected components P are oriented, totally geodesic submanifolds of even codimension, i.e. oriented surfaces or isolated fixed points. The contribution $\mu(P)$ of P to $\Phi(M)$ is given by the index of the signature operator on P twisted by an appropriate power series of vector bundles on P; namely,

$$\begin{aligned} \mu(P) &= \frac{(1+\lambda^{m_1})}{(1-\lambda^{m_1})} \prod_{l=1}^{\infty} \frac{(1+q^k\lambda^{m_1})(1+q^k\lambda^{-m_1})}{(1-q^k\lambda^{-m_1})(1-q^k\lambda^{-m_1})} \\ &\times \frac{(1+\lambda^{m_2})}{(1-\lambda^{m_2})} \prod_{l=1}^{\infty} \frac{(1+q^k\lambda^{m_2})(1+q^k\lambda^{-m_2})}{(1-q^k\lambda^{-m_2})(1-q^k\lambda^{-m_2})}, \end{aligned}$$

if P is an isolated fixed point and $TM|_P = E^{\#}_{m_1} \oplus E^{\#}_{m_2}$, where $E^{\#}_{m_i}$ denotes the canonical underlying real bundle of the complex bundle E_{m_i} on which S^1 acts by sending λ to λ^{m_i}; or

$$\mu(P) = \mathrm{ind}\left(d_s^P \otimes \frac{\bigwedge_{\lambda^{m_2}} E_{m_2}}{\bigwedge_{-\lambda^{m_2}} E_{m_2}} \right),$$

when P is 2-dimensional, where $TM|_P = TP \oplus E^{\#}_{m_2}$, i.e. E_{m_1} is now a trivial representation ($m_1 = 0$) and d_s^P is the signature operator on P.

The contributions $\mu(P)$ are meromorphic functions on the 2-dimensional torus $\mathbb{T}_{q^2} = \mathbb{C}^*/q^2$ considered as the quotient of the multiplicative group of non-zero complex numbers $\mathbb{C}^*$ by the subgroup generated by the element $q^2 \neq 0$. The rigidity of the elliptic genus is equivalent to the function $\Phi(M) = \sum_{\{P\}} \mu(P)$ having no poles at all on $\mathbb{T}_{q^2}$.

Define the translation $t_a\Phi(M)$ of $\Phi(M)$ with $a \in \mathbb{C}^*$, by the map at the character level $\lambda \mapsto a\lambda$. The rigidity theorem for $\Phi(M)$ follows from showing that none of the translations $t_a\Phi(M)$, by points $a \in \mathbb{T}_{q^2}$ of finite order, has a pole on the circle $|\lambda| = 1$. Let $k \in \mathbb{N}$ and a be any root of the form

$$a = \alpha^s, \qquad \alpha^k = q,$$

with k and s relatively prime. Define the formal power series

$$\varphi_a(F) = \frac{\bigwedge_a F}{\bigwedge_{-a} F} \otimes \bigotimes_{n=1}^{\infty} \left(\frac{\bigwedge_{aq^n} F}{\bigwedge_{-aq^n} F} \otimes \frac{\bigwedge_{a^{-1}q^n} F^*}{\bigwedge_{-a^{-1}q^n} F^*} \right)$$

where F denotes a complex vector bundle and F^* its complex dual bundle, and define

$$\varphi_{\pm q^{1/2}}(E) = \frac{\bigotimes_{n=1}^{\infty} \bigwedge_{\pm q^{n-1/2}} E}{\bigotimes_{n=1}^{\infty} \bigwedge_{\mp q^{n-1/2}} E}$$

for a real vector bundle E. The translations $t_{\alpha^s}\Phi(M)$ can be expressed as *twists* of Φ on the connected components M_k of the fixed point submanifold of the subgroup $\mathbb{Z}_k = \{e^{2\pi i l/k} | l = 1, ..., k\} \subset S^1$ which *do contain fixed points of the* S^1 *action*, as follows

$$t_{\alpha^s}\Phi(M) = \operatorname{ind}\left(d_s^{M'_k} \otimes R(q, TM'_k \otimes \mathbb{C}) \otimes \varphi_{\alpha^{\omega_{r_1}}}(T_{r_1}) \otimes \varphi_{\alpha^{\omega_{r_2}}}(T_{r_2})\right), \quad (2)$$

where M'_k is the submanifold M_k with a specific orientation and the bundles T_r are defined below.

Given (2), [5, Proposition 6.1] says that the translations $t_{\alpha^s}\Phi(M)$ converge on some annulus containing the unit circle $|\lambda| = 1$ to the Laurent series of a meromorphic function on $\mathbb{T}_{q^2}$ which has no poles on the unit circle. The function $t_{\alpha^s}\tau_q(M)$ on the variable q is regular on an annulus containing the unit circle for all $k \in \mathbb{N}$, so that $t_{\alpha^s}\Phi(M)$ has no poles on the unit circle $|\lambda| = 1$. Hence, $\Phi(M)$ has no poles at all on $\mathbb{T}_{q^2}$, and must be constant, i.e. the rigidity theorem follows.

In order to define the aforementioned twists, the submanifolds M_k must be orientable. The connected components of M_k that we are interested in are only those that contain S^1-fixed points since they are intermediate steps between M and the S^1-fixed point submanifolds P. The orientability of such components is confirmed as follows.

Since the transformations in $\mathbb{Z}_k \subset S^1$ are orientation-preserving, the codimension of the components of M_k is even, so that in our case, they are isolated points or surfaces. The surfaces containing S^1-fixed points are orientable since they either coincide with a S^1-fixed component P, or are S^1-invariant with positive Euler characteristic, i.e. they are 2-dimensional spheres. The subgroup $\mathbb{Z}_k$ acts on the normal bundle of M_k in M

$$\begin{cases} T|_{M_k} = T^{\#}_{r_1} \oplus T^{\#}_{r_2} & \text{if } M_k \text{ is an isolated fixed point, or} \\ T|_{M_k} = TM_k \oplus T^{\#}_{r_2} & \text{if } M_k \text{ is a surface,} \end{cases} \quad (3)$$

where $T^{\#}_r$ is an irreducible real representation of $\mathbb{Z}_k$ with $1 \le r \le [k/2]$, $[k/2]$ is the greatest integer smaller than or equal to $k/2$, and TM_k is a trivial representation. The S^1 action on M induces an S^1 action on M_k, whose differential induces an action on $T|_{M_k}$ preserving the decomposition, and making each $T^{\#}_r$ an S^1 bundle over M_k, for $1 \le r \le [(k-1)/2]$. Each $T^{\#}_r$, with $r \ne k/2$ if k is even, is endowed with a complex structure such that $\lambda \in S^1$ acts by λ^r, for $1 \le r \le [(k-1)/2]$. Hence, $T^{\#}_r$ comes from a complex vector bundle T_r. For k even, the action on $T^{\#}_{k/2} = T_{k/2}$ is multiplication by -1, and it does not necessarily come from a complex vector bundle, while the $T_r^{\#}$ inherit an orientation from the complex structure on T_r, for $r = 1, \ldots, [(k-1)/2]$. Hence, if k is odd, TM_k has an induced orientation. If k is even, however, we only know that $TM_k \oplus T_{k/2}$ is orientable, if $T_{k/2}$ does appear in (3). On the other hand, we know that M_k is also orientable. Let us, therefore, choose an orientation. In this way, $T_{k/2}$ inherits an orientation from M and M_k.

We must choose the orientation of M_k' to be compatible with the orientations of the S^1-fixed point submanifolds P as follows. Recall the decomposition of TM along P

$$\begin{cases} TM|_P = E_{m_1} \oplus E_{m_2} & P \text{ is isolated fixed point,} \\ TM|_P = TP \oplus E_{m_2} & P \text{ is a surface.} \end{cases} \tag{4}$$

When k is odd, the decomposition (3) determines an orientation on TM_k denoted by $+1$. If $P \subset M_k$, choose the exponents along P so that each $m_j \not\equiv 0 \pmod{k}$ is congruent to some $r \in \{1, \ldots, (k-1)/2\}$. Choose the orientation of TP and the sign of those $m_j \equiv 0 \pmod{k}$ so that the induced orientation on $TM|_P$ is the given one. The induced orientation on $TM_k|_P$ will be the $+1$ orientation. For m_1 and m_2 let $(l_1, \omega_1), (l_2, \omega_2) \in \mathbb{Z} \times \{1, \ldots, k-1\}$ be such that

$$s \cdot m_1 = l_1 \cdot k + \omega_1, \qquad s \cdot m_2 = l_2 \cdot k + \omega_2 \tag{5}$$

and define

$$\varepsilon(P) = l_1 + l_2. \tag{6}$$

The orientation for M_k' is now defined as $(+1) \cdot (-1)^{\varepsilon(P)}$, if $M_k \supseteq P$. Lemma 2.1 below ensures that this orientation is well defined. When k is even, $TM|_{M_k}$ decomposes according to (3), so that $TM_k \oplus T_{k/2}$ always inherits an orientation if the summand $T_{k/2}$ does appear in (3). Let us choose an orientation for TM_k and call it $+1$, which induces an orientation on $T_{k/2}$. If $P \subset M_k$, select the exponents at P as follows. If $m_j \not\equiv 0, k/2 \pmod{k}$, make the choice as before so that $(m_j)_{\text{mod } k} \in \{1, \ldots, k/2 - 1\}$. Choose the signs for those $m_j \equiv 0, k/2 \pmod{k}$ and the orientation of TP to make the induced orientation of $(TM_k \oplus T_{k/2})|_P$ correct. This ensures that the induced orientation of $TM|_P$ is correct. The induced orientation of $TM_k|_P$, however, *may not* be the correct one $(+1)$. Let $\varepsilon_0 = 0, 1$, with $\varepsilon_0 = 0$ if the induced orientation on $TM_k|_P$ is correct, and $\varepsilon_0 = +1$ if the induced orientation on $TM_k|_P$ is incorrect. For each m_j, define (l_j, ω_j) by (5) and set

$$\varepsilon(P) = \varepsilon_0 + l_1 + l_2. \tag{7}$$

The orientation of M_k' is given by $(+1) \cdot (-1)^{\varepsilon(P)}$, if $M_k \supset P$. The orientation of M_k' is well defined by the following lemma.

Lemma 2.1. *Let M be an oriented, compact, connected,* even *smooth 4-manifold admitting a smooth S^1 action. Let $k \in \mathbb{N}$ and M_k be a connected component of the fixed point set of $\mathbb{Z}_k$. Let $s \in \mathbb{Z}$ be relatively prime to k, and P, $P' \subset M_k$ be connected components of the S^1-fixed point set. Use the prescription* (6) *or* (7) *above to define the numbers $\varepsilon(P)$ and $\varepsilon(P')$. Then $(-1)^{\varepsilon(P)} = (-1)^{\varepsilon(P')}$.*

Proof. Lemma 2.1 is analogous to [5, Lemma 8.1], so we shall concentrate in the relevant changes to the proof. Let $k \in \mathbb{N}$, P be a connected fixed point submanifolds of S^1 contained in a connected component of M_k, which we shall also denote also by M_k. Just as before, given that we are working in dimension 4, there are three cases: (i) $\dim M_k = \dim P = 2$; (ii) $\dim M_k = 2$, $\dim P = 0$; (iii) $\dim M_k = \dim P = 0$. In (i) and (iii) the components P and M_k coincide so that we do not have to check any compatibility of the exponents $\varepsilon(P)$. In case (ii), we can have two isolated S^1-fixed points p and p'. Consider a path joining p and p' which avoids other S^1-fixed points. Let the path flow with the S^1 action to generate a sphere $S \cong S^2$ with "north" and "south" poles p and p' respectively (which, in fact, coincides with the component of M_k). Let the sets of integers $\{m_1, m_2\}$ and $\{m'_1, m'_2\}$ denote the exponents of the S^1 action on T_pM and $T_{p'}M$ respectively.

By [5, Lemmas 9.1, 9.2], the number

$$\varepsilon(P) - \varepsilon(P') \equiv c \cdot (m_1 + m_2 - m'_1 - m'_2) \pmod 2,$$

where c is a constant, while

$$(m_1 + m_2 - m'_1 - m'_2) = \int_S c_1(TM|_S),$$

so that the lemma is reduced to finding the parity of the integer represented by the last integral. Notice that

$$TM|_S = TS \oplus \nu,$$

where ν is the (real rank 2) normal bundle of S in M. The three bundles on S can be considered as complex vector bundles over S (cf. [5, p. 159]). Hence,

$$c_1(TM|_S) = c_1(TS) + c_1(\nu),$$

and

$$\int_S c_1(TM|_S) = \int_S c_1(TS) + \int_S c_1(\nu) = 2 + \int_S c_1(\nu),$$

so that the parity depends on the last integral only. Given that we are working in dimension 4

$$\int_S c_1(\nu) = S \cdot S \equiv 0 \pmod 2,$$

the self-intersection number of this sphere, which by the hypothesis on the intersection form is an even number. □

3. Vanishing of the signature

Proof of Theorem 1.1. Since in dimension 4

$$\mathrm{sign}(M) = -8\,\widehat{A}(M)$$

we shall prove Theorem 1.1 by proving the vanishing of $\widehat{A}(M)$.

Since we are also considering the case when M may be non-Spin, $\widehat{A}(M)$ may only be defined as a characteristic number and may not represent the index of an elliptic operator. Thus, $\widehat{A}(M)$ may, in principle, be a rational number.

Since S^1 acts on M, the equivariant genus $\Phi(M)_g$ is defined for any $g \in S^1$ as

$$\Phi(M)_g = \sum \mathrm{sign}(M, R_i)_g \cdot q^i,$$

where $\mathrm{sign}(M, R_i)_g = \mathbf{tr}|_g \ker(d_s \otimes R_i) - \mathbf{tr}|_g \mathrm{coker}(d_s \otimes R_i)$. The coefficients of its q-development are now equivariant twisted signatures. Thus, according to Theorem 2.1, the value of $\Phi(M)_g$ does not depend on g. Applying the Atiyah-Bott fixed point theorem [2], $\Phi(M)_g$ can be expressed in terms of the fixed point set M^g of g and the action of g on the normal bundle of $M^g \subset M$. In particular, let g be the orientation preserving involution in $\mathbb{Z}_2 \subset S^1$. We denote the transversal self-intersection of M_2 by $M_2 \circ M_2$. In [10, p. 315], Hirzebruch and Slodowy showed that

$$\Phi(M)_g = \Phi(M_2 \circ M_2).$$

On the other hand, applying Theorem 2.1, $\Phi(M) = \Phi(M)_g$, i.e.

$$\Phi(M) = \Phi(M_2 \circ M_2). \tag{8}$$

The codimension of M_2 is positive and even, so that the elliptic genus $\Phi(M)$ can now be computed from the elliptic genera of submanifolds of M of codimension at least 4, i.e. isolated points.

Let us now recall the expansion of $\Phi(M)$ at the other cusp [9]

$$\tilde{\Phi}(M) = \frac{1}{q^{\dim(M)/8}} \sum_{j=0}^{\infty} \widehat{A}(M, R'_j) \cdot q^j,$$

where R'_j is the sequence of virtual tensor bundles given by

$$R'(q, T) = \bigotimes_{k=2m+1} \bigwedge\nolimits_{-q^k} T \otimes \bigotimes_{k=2m+2} S_{q^k} T,$$

and the $\widehat{A}(M, R'_j) = \langle \widehat{A}(M) \cdot \mathrm{ch}(R'_j), [M] \rangle$ may only defined be as characteristic numbers. The first few terms of the sequence are $R'_0 = 1$, $R'_1 = -T$, $R'_2 = \bigwedge^2 T + T$, etc. This expansion is obtained by considering $q = e^{\pi i t}$ and changing the t coordinate in (1) by $t \to -1/t$, and then by $t \to 2t$ (cf. [9]). This expansion has, a priori, a pole of order $1/2$. On the other hand, by (8) we also have

$$\tilde{\Phi}(M) = \tilde{\Phi}(M_2 \circ M_2), \tag{9}$$

whose right hand side has a pole of order at most $1/2 - 1/2 = 0$, since the dimension of any connected component of $M_2 \circ M_2$ is at most 0. Therefore (9) implies that the first coefficient on the left hand side vanishes,

$$\widehat{A}(M) = 0,$$

i.e.

$$\mathrm{sign}(M) = 0.$$

□

4. Applications

In dimension 4, given the modular nature of the elliptic genera, Φ and $\tilde{\Phi}$ satisfy

$$\begin{aligned} \Phi(M) &= \frac{\delta}{\varepsilon^{1/2}}\,\mathrm{sign}(M) = \frac{-8\,\delta}{\varepsilon^{1/2}}\,\widehat{A}(M), \\ \tilde{\Phi}(M) &= \frac{-8\,\delta}{(\delta^2 - \varepsilon)^{1/2}}\widehat{A}(M) = \frac{\delta}{(\delta^2 - \varepsilon)^{1/2}}\,\mathrm{sign}(M), \end{aligned}$$

where

$$\begin{aligned} \delta &= \frac{1}{4} + 6(q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + \ldots), \\ \varepsilon &= \frac{1}{16} - q + 7q^2 - 28q^3 + 71q^4 - 126q^5 + 196q^6 \pm \ldots, \end{aligned}$$

are the generators of degree 2 and 4 respectively of the space of modular forms for the subgroup

$$\Gamma_0(2) = \left\{ A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod 2 \right\} \subset SL_2(\mathbb{Z}).$$

Hence, we have also proved the following.

Corollary 4.1. *Let M be an oriented, compact, connected even 4-manifold admitting smooth S^1 actions. Then, the elliptic genus vanishes identically on M*

$$\Phi(M) = \mathrm{sign}(M) = 0 \quad \textit{and} \quad \tilde{\Phi}(M) = 0.$$

Furthermore, every genus vanishes on M and, in particular, the Witten genus [9]

$$\varphi_W(M) = \mathrm{ind}\left(d_s^M \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T\right) = 0$$

vanishes on M. □

Corollary 4.2. *The Enriques surface and the n-fold sum of Enriques surfaces admit no smooth S^1-actions.*

Proof. The Enriques surface has intersection form $E_8 \oplus H$, so it is incompatible with Theorem 1.1. Similarly for the n-fold connected sum of Enriques surfaces which have intersection form $nE_8 \oplus nH$. □

Finally, let us remark that the arguments in Sections 2 and 3, and particularly equation (2), prove the rigidity of the elliptic genus and the vanishing of the signature in a wider class of *non-Spin* 4-manifolds, including non-Spin π_2-finite manifolds [7, 8].

Theorem 4.1. *Let M be an oriented, compact, connected, smooth 4-manifold admitting a smooth S^1 action. Furthermore, assume that the normal bundle to every S^1-invariant 2-sphere contained in M is topologically trivial. Then,*

(1) *the elliptic genus $\Phi(M)$ of M is rigid under the S^1 action;*
(2) *the signature and $\widehat{A}$-genus of M vanish*

$$\mathrm{sign}(M) = 0, \quad \widehat{A}(M) = 0;$$

(3) *the elliptic genus $\Phi(M)$ and all other genera vanish on M.*

□

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