ON INTEGRAL MODELS OF ALGEBRAIC TORI AND AFFINE TORIC VARIETIES

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To the memory of Albrecht Fröhlich

§1. Introduction

In a joint work with V.E. Voskresenskiĭ [14], an explicit construction of a natural integral model of an algebraic torus, defined over a number field, has been described; in [13], we have constructed a few integral models of the affine toric varieties associated to such a torus. Inspired by a paper of A. Fröhlich's [7], we have conjectured [14] that for any number field k there is a sufficiently big finite normal extension $L \mid k$ having an integral basis over k, and pointed out that our constructions of integral models could be considerably simplified under this conjecture [14]. In the meantime M. V. Bondarko [2] has proved the conjecture. One of the goals of this paper is to describe the arising simplifications in some detail. Our second goal is to restore a few details of our argument and notation, left to the reader in [13] and [14]. We do not dwell on the applications of integral models here; some of the applications have been discussed in [13], [14] (cf. also [5]).

In the next section we shall collect a few results and definitions, relating to the theory of algebraic tori and affine toric varieties defined over an arbitrary field of characteristic 0. Although both our definition of an affine T - toric variety and our Theorem 1 may be known to some authors, we could not find a proof of that theorem in the literature. After the proper terminology has been established, it is a relatively straightforward matter to construct our "standard" model of an algebraic torus T defined over a number field, and the corresponding models of the T-toric varieties. This is done in Section 3.

Notation and conventions. As usual, $\mathbb{Q}, \mathbb{Z}$, $\mathbb{N}$, and $\mathbf{F}_p$ stand for the field of rational numbers, the ring of rational integers, the monoid of non-negative rational integers, and the finite field of p elements respectively. The algebraic closure of a field k and the degree of a finite extension of fields $L \mid k$ are denoted, respectively, by $\bar{k}$ and $[L : k]$. Given a commutative ring A, let A^* stand for the group of invertible elements in A, and let $G_{m,A}$

and $G_{a,A}$ denote the multiplicative and the additive groups defined over A respectively. Assuming the group G acts on a set S, let

$$S^G := \{a \mid \ a \in S,\ g \cdot a = a \text{ for } g \in G\}$$

stand for the subset of the fixed points under that action. Let

$$\mathfrak{M}_{lm} = \{c \ \mid \ c = (c_{ij}),\ c_{ij} \in \mathbb{Z},\ 1 \le i \le l,\ 1 \le j \le m\}$$

stand for the (additive) group of the integral matrices with l rows and m columns, and let I_d denote the unit matrix in $\mathfrak{M}_{dd}$. The elements of an Abelian group $H = H_1 \times \cdots \times H_n$ are sometimes denoted by $\mathbf{a}$ or by $\vec{\alpha}$; we write then

$$\mathbf{a} = (a_1, \ldots, a_n),\ \vec{\alpha} = (\alpha_1, \ldots, \alpha_n),\ a_i \in H_i,\ \alpha_i \in H_i,\ 1 \le i \le n.$$

For $b \in \mathbb{Q}$, let

$$b^+ := \frac{1}{2}\,(|b| + b),\ b^- := \frac{1}{2}\,(|b| - b).$$

As usual, "h.c.f." stands for "the highest common factor"; a *number field* is a finite extension of $\mathbb{Q}$. In this paper, we tend to identify isomorphic objects, whenever possible.

§2. Affine toric varieties

1. Let T be an algebraic torus of dimension d defined over a field k of characteristic zero. The torus T *splits* over $\bar{k}$, so that

$$T \times_k \bar{k} \cong G^d_{m,\bar{k}}\,.$$

The projections

$$\chi_i \colon T \times_k \bar{k} \to G_{m,\bar{k}},\ 1 \le i \le d, \tag{1}$$

are defined over a finite normal extension $L \mid k$; we call any such field L a *splitting field* of the torus T. Projections (1) generate a free Abelian group

$$\hat{T} := \mathrm{Hom}(T \times_k \bar{k}, G_{m,\bar{k}})$$

of rank d, the group of $\bar{k}$-rational characters of T. The absolute Galois group $\mathrm{Gal}(\bar{k} \mid k)$ acts on $\hat{T}$ in a natural way; let

$$\bar{\rho}\, :\ \mathrm{Gal}(\bar{k} \mid k) \to \mathrm{GL}(d, \mathbb{Z})$$

be the integral representation defined by that action. Let L be a splitting field of T, let $[L : k] = n$, and let $\Gamma = \mathrm{Gal}(L \mid k)$ be the Galois group of the extension $L \mid k$. Let $T_L := T \times_k L$; clearly,

$$T_L \cong G^d_{m,L}.$$

The representation $\bar{\rho}$ factors through Γ. Hence $\bar{\rho}$ coincides with the composition of a representation

$$\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{Z})$$

with the surjection

$$\mathrm{Gal}(\bar{k} \mid k) \to \Gamma.$$

The representation ρ defines an action of the group Γ on the $\mathbb{Z}$-module

$$M := \mathrm{Hom}(T_L, G_{m,L})\,;$$

let $\check{M}$ be the $\mathbb{Z}[\Gamma]$-module dual to M and let $\check{\rho}$ be the integral representation contragredient to ρ. We choose a $\mathbb{Z}$ - basis $\{e_1, \ldots, e_d\}$ of $\check{M}$ and write

$$\check{\rho}(g)\, e_i = \sum_{j=1}^{d} e_j\; r(g)_{ji},\; 1 \le i \le d, \tag{2}$$

with $r(g) \in \mathfrak{M}_{dd}$. Further, let us choose a basis $\{\omega_1, \ldots, \omega_n\}$ of the extension $L \mid k$ and a system of independent variables

$$x := (\ldots, x_j^{(i)}, \ldots),\; 0 \le i \le d,\; 1 \le j \le n$$

and consider the following $d+1$ linear forms:

$$t_i := x_1^{(i)}\omega_1 + \cdots + x_n^{(i)}\omega_n\;,\; 0 \le i \le d,$$

in $L[x]$. Let

$$g t_i := \sum_{j=1}^{n} x_j^{(i)} g\omega_j \tag{3}$$

for $1 \le i \le d,\; g \in \Gamma$. The equations

$$\prod_{i=0}^{d} t_i - 1 = \sum_{j=1}^{n} P_j^{(0)}(x)\,\omega_j$$

and

$$g t_i \prod_{j=1}^{d} t_j^{r(g)_{ji}^-} - \prod_{j=1}^{d} t_j^{r(g)_{ji}^+} = \sum_{j=1}^{n} P_j^{(i,g)}(x)\,\omega_j,\; 1 \le i \le d,\; g \in \Gamma,$$

uniquely define the following system of polynomials in $k[x]$:

$$\mathcal{P} := \{P_j^{(0)}(x),\; P_j^{(i,g)}(x) \mid 1 \le j \le n,\; 1 \le i \le d,\; g \in \Gamma\}.$$

Let I be the ideal, generated in $k[x]$ by the set of polynomials $\mathcal{P}$, and let

$$B_0 := k[x]/I.$$

It follows from the basic definitions that

$$T = \mathrm{Spec}\; B_0, \tag{4}$$

see, for instance, [18] or [14, Section 2, Corollary 1]. It is clear that

$$B_0 \otimes_k L = L[t, t^{-1}],$$

where $t^{-1} := (t_1^{-1}, \ \ldots, \ t_d^{-1})$, and

$$T_L = \text{Spec } L[t, t^{-1}]. \tag{5}$$

For $g \in \Gamma$, one can extend the Galois action

$$g \colon L \to L$$

to an automorphism

$$g \colon L[t, t^{-1}] \to L[t, t^{-1}]$$

by letting

$$gt_i = \prod_{j=1}^{d} t_j^{r(g)_{ji}}. \tag{6}$$

It follows then that

$$B_0 = (L[t, t^{-1}])^{\Gamma}. \tag{7}$$

2. A normal variety X defined over a field K of characteristic zero is called a $G_{m,K}^d$ - *toric variety* if X contains a dense open subset isomorphic to the torus $G_{m,K}^d$ and is equipped with an action

$$G_{m,K}^d \times X \to X$$

that extends the natural action of the torus $G_{m,K}^d$ on itself; cf. [9, p. 3].

Let $\sigma_{\mathbb{Q}}$ be a strongly convex rational polyhedral cone (=: scrp-cone) in the $\mathbb{Q}$-vector space $V := M \otimes_{\mathbb{Z}} \mathbb{Q}$ and let

$$\check{\sigma}_{\mathbb{Q}} = \{ v \ \mid \ v \in \check{V}, \ (v \mid \ u) \geq 0 \text{ for } u \in \sigma_{\mathbb{Q}} \}$$

be the cone in the vector space $\check{V} := \check{M} \otimes_{\mathbb{Z}} \mathbb{Q}$, dual to $\sigma_{\mathbb{Q}}$. One defines two semigroups

$$\sigma := \sigma_{\mathbb{Q}} \cap M$$

and

$$\check{\sigma} := \check{\sigma}_{\mathbb{Q}} \cap \check{M}.$$

Let $\{u_1, \ \ldots, \ u_l\}$ be the minimal set of generators of $\check{\sigma}$ (cf. [9, p. 14]), and let

$$\mathfrak{R}(\sigma) := \{ \mathbf{a} \mid \ \mathbf{a} \in \mathbb{Z}^l, \ \sum_{j=1}^{l} a_j u_j = 0 \}$$

stand for the group of the relations between those generators. There is a matrix c in $\mathfrak{M}_{dl}$ such that

$$u_j = \sum_{i=1}^{d} e_i\, c_{ij},\ 1 \le j \le l;$$

clearly,

$$\mathfrak{R}(\sigma) := \{\mathbf{a} \mid\ \mathbf{a} \in \mathbb{Z}^l,\ c \cdot \mathbf{a} = 0\}. \tag{8}$$

We introduce the independent variables $s := (s_1,\ \ldots,\ s_l)$ and write, for brevity, $s^{-1} := (s_1^{-1},\ \ldots,\ s_l^{-1})$. Let $\bar{J}_0(\sigma)$ stand for the ideal in $L[s, s^{-1}]$, generated by the following set of rational functions:

$$\{h_{\mathbf{a}}(s) \mid\ h_{\mathbf{a}}(s) := \prod_{j=1}^{l} s_j^{a_j} - 1,\ \mathbf{a} \in \mathfrak{R}(\sigma)\},$$

and consider the L-algebra

$$B_L := L[s, s^{-1}]/\bar{J}_0(\sigma).$$

Since the set $\{u_j \mid\ 1 \le j \le l\}$ generates the $\mathbb{Z}$-module $\check{M}$, the set

$$\mathfrak{C} := \{b \mid\ b \in \mathfrak{M}_{ld},\ c \cdot b = I_d\}$$

is not empty. One defines the L - algebra homomorphisms

$$\varphi_0\colon L[s, s^{-1}] \to L[t, t^{-1}],\ s_j \mapsto \prod_{i=1}^{d} t_i^{c_{ij}},\ 1 \le j \le l,$$

and

$$\psi_0\colon L[t, t^{-1}] \to L[s, s^{-1}],\ t_i \mapsto \prod_{j=1}^{l} s_j^{b_{ji}},\ 1 \le i \le d,$$

for *some* b in $\mathfrak{C}$.

Lemma 1. *The homomorphism ψ_0 gives rise to an L - algebra isomorphism*

$$\psi\colon L[t, t^{-1}] \to B_L. \tag{9}$$

The definition of the map ψ does not depend on the choice of b in $\mathfrak{C}$.

Proof. Since $\varphi_0 \circ \psi_0 = 1$, the map φ_0 is surjective. It follows from (8) and the definition of φ_0 that

$$\bar{J}_0(\sigma) \subseteq \operatorname{Ker} \varphi_0.$$

Suppose that

$$q(s) \in \operatorname{Ker} \varphi_0.$$

Write
$$q(s) = \sum_{\substack{1\le i\le R\\ 1\le j\le S}} \alpha_{ij} m_{ij}(s),$$
where $\alpha_{ij} \in L$ and $\{m_{ij}(s) \mid \ 1 \le j \le S\}$ is a set of Laurent monomials in $L[s, s^{-1}]$ such that
$$\varphi_0(m_{ij}(s)) = n_i(t),\ 1 \le i \le R,\ 1 \le j \le S, \tag{10}$$
for some Laurent monomials $n_i(t)$ in $L[t, t^{-1}]$, satisfying the following condition:
$$n_\nu(t) \neq n_\mu(t) \text{ for } \nu \neq \mu.$$
Since $\varphi_0(q(s)) = 0$, it follows that
$$\sum_{1\le j\le S} \alpha_{ij} = 0,\ 1 \le i \le R. \tag{11}$$
On the other hand, it follows from (8), (10), and the definition of φ_0 that
$$m_{ij}(s) = m_{i1}(s) \prod_{h=1}^{l} s_h^{a_h^{(ij)}} \tag{12}$$
with
$$\mathbf{a}^{(ij)} \in \mathfrak{R}(\sigma). \tag{13}$$
In view of (11) and (12), one concludes that
$$q(s) = \sum_{\substack{1\le i\le R\\ 1\le j\le S}} \alpha_{ij} m_{i1}(s)(\prod_{h=1}^{l} s_h^{a_h^{(ij)}} - 1). \tag{14}$$
Relations (13) and (14) show that $q(s) \in \bar{J}_0(\sigma)$. Thus
$$\bar{J}_0(\sigma) = \text{Ker } \varphi_0. \tag{15}$$
Therefore the map ψ_0 gives rise to an L - algebra isomorphism (9), as asserted. It can be easily checked that the definition of that isomorphism does not depend on the choice of b in $\mathfrak{C}$. This concludes the proof of the lemma.

Corollary 1. *We have*
$$T_L \cong \text{Spec } B_L. \tag{16}$$

Proof. It follows from Lemma 1 and relation (5).

Let $J_0(\sigma)$ denote the ideal of the polynomial ring $L[s]$, generated by the following set of binomials:

$$\{f_{\mathbf{a}}(s) \mid \ f_{\mathbf{a}}(s) := \prod_{j=1}^{l} s_j^{a_j^+} - \prod_{j=1}^{l} s_j^{a_j^-}\ ,\ \mathbf{a} \in \mathfrak{R}(\sigma)\}.$$

Corollary 2. *Let $p(s) \in L[s]$, let $m_1(s)$ and $m_2(s)$ be two monomials in $L[s]$, and let $q(s) := p(s)m_2(s) + m_1(s)$. If $q(s) \in J_0(\sigma)$, then*

$$\frac{m_1(s)}{m_2(s)} = m_3(s) \prod_{j=1}^{l} s_j^{a_j} \tag{17}$$

for some monomial $m_3(s)$ in $L[s]$ and some $\mathbf{a}$ in $\mathfrak{R}(\sigma)$.

Proof. Since $q(s) \in J_0(\sigma)$, it follows that

$$\varphi_0(\frac{m_1(s)}{m_2(s)} + p(s)) = 0 \tag{18}$$

in $L[t, t^{-1}]$. Write

$$p(s) = \sum_{i=1}^{N} a_i n_i(s) \tag{19}$$

for some monomials $n_i(s)$ in $L[s]$ and some a_i in L. It follows from (18) and (19) that

$$\varphi_0(\frac{m_1(s)}{m_2(s)}) = \varphi_0(m_3(s)) \tag{20}$$

for some $m_3(s)$ in $\{n_i(s) \mid 1 \leq i \leq N\}$. As in the proof of Lemma 1, the assertion of Corollary 2 follows from (8), (20), and the definition of φ_0.

Let

$$A_L := L[s]/J_0(\sigma),$$

and consider the affine L - scheme

$$X_L(\sigma) := \text{Spec } A_L. \tag{21}$$

The scheme Spec B_L is clearly isomorphic to a dense open subset of the scheme $X_L(\sigma)$, and therefore it follows from Corollary 1 that the torus T_L can be imbedded into the variety $X_L(\sigma)$ as a dense open orbit. Let

$$\lambda_1 \colon T_L \to X_L(\sigma)$$

be the dominant open immersion, describing that imbedding. We shall say that the T_L - toric variety $X_L(\sigma)$ corresponds to the scrp-cone $\sigma_{\mathbb{Q}}$ (cf. [9, p. 19]).

Example 1. It can be easily seen that $X_L(\{0\}) \cong T_L$.

One can show that any *affine* T_L - toric variety corresponds to a scrp - cone (see, for instance, [19]).

3. For $b \in \mathfrak{M}_{ld}$, let

$$\tilde{r}(g,b) := b \cdot r(g) \cdot c, \; g \in \Gamma, \tag{22}$$

the matrix $r(g)$ being defined by (2). Clearly,

$$\check{\rho}(g) \; u_i = \sum_{j=1}^{l} u_j \; \tilde{r}(g,b)_{ji}, \; 1 \leq i \leq l, \tag{23}$$

for $b \in \mathfrak{C}$.

Lemma 2. *Let $g \in \Gamma$ and $m \in \mathbb{Z}$, $1 \leq m \leq l$. For every $\mathbf{a}$ in $\mathfrak{R}(\sigma)$, there is a matrix b in $\mathfrak{M}_{ld}$ such that*

$$c \cdot b = 0 \;\; \textit{and} \;\; \tilde{r}(g,b)_{jm} = a_j \; \textit{for} \; 1 \leq j \leq l.$$

Proof. Let $\{v_1, \ldots, v_t\}$ be a $\mathbb{Z}$ - basis of $\mathfrak{R}(\sigma)$ and let

$$\mathbf{a} = \sum_{j=1}^{t} \alpha_j v_j. \tag{24}$$

Since the map $b \mapsto b \cdot r(g)$ is an automorphism of the $\mathbb{Z}$ - module

$$N := \{b \mid \; b \in \mathfrak{M}_{ld}, \; c \cdot b = 0\},$$

it suffices to find a matrix b in N satisfying the following condition:

$$(b \cdot c)_{jm} = a_j, \; 1 \leq j \leq l.$$

Since $\{u_1, \; \ldots, \; u_l\}$ is the minimal basis of the saturated semi-group $\check{\sigma}$, it follows that h.c.f.$(c_{1m}, \ldots, c_{dm}) = 1$. Therefore there is a matrix w in $\mathfrak{M}_{ld}$ such that

$$\alpha_i = \sum_{j=1}^{d} c_{jm} w_{ji}, \; 1 \leq i \leq t. \tag{25}$$

Let

$$b^{(i)} = \sum_{j=1}^{d} w_{ij} v_j, \; 1 \leq i \leq d, \tag{26}$$

and let $b = (b^{(1)}, \; \ldots, \; b^{(d)})$ be the matrix with columns $b^{(i)}$. By construction, $b \in N$. On the other hand, it follows from equations (24) - (26) that

$$\mathbf{a} = \sum_{i=1}^{t} b^{(i)} c_{im};$$

consequently,

$$a_j = \sum_{i=1}^{t} b_{ji} c_{im} = (b \cdot c)_{jm},\ 1 \le j \le d,$$

as required.

Corollary 3. *The set of rational functions*

$$\{s_j - \prod_{j=1}^{l} s_i^{(b\cdot c)_{ij}} \mid b \in \mathfrak{C},\ 1 \le j \le l\}$$

generates the ideal $\bar{J}_0(\sigma)$ *in* $L[s, s^{-1}]$.

Proof. Let $b_0 \in \mathfrak{C}$. Since

$$u_j = \sum_{i=1}^{t} u_i (b_0 \cdot c)_{ij},\ 1 \le j \le l,$$

it follows from (8) that

$$(b_0 \cdot c)_{ij} = \delta_{ij} + a_i^{(j)},\ 1 \le i,\ j \le l,$$

for some $\mathbf{a}^{(j)}$ in $\mathfrak{R}(\sigma)$. By Lemma 2, for every j there is a matrix β in $\mathfrak{M}_{ld}$ such that $c \cdot \beta = 0$ and

$$(\beta \cdot c)_{ij} = a_i - a_i^{(j)},\ 1 \le i \le l.$$

Let $b = b_0 + \beta$, then $b \in \mathfrak{C}$ and

$$h_{\mathbf{a}}(s) = s_j^{-1}(s_j - \prod_{j=1}^{l} s_i^{(b\cdot c)_{ij}}).$$

This proves the corollary, since the polynomials $h_{\mathbf{a}}(s)$ generate the ideal $\bar{J}_0(\sigma)$, as $\mathbf{a}$ runs through $\mathfrak{R}(\sigma)$.

In view of (8), one can extend the Galois action

$$g \colon L \to L$$

to an automorphism

$$g \colon B_L \to B_L$$

by letting, for some b in $\mathfrak{C}$,

$$g s_j = \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)_{ij}},\ 1 \le j \le l,\ g \in \Gamma. \tag{27}$$

Lemma 3. *The map (9) is a* Γ *- isomorphism and, moreover,*

$$B_0 \cong (B_L)^{\Gamma}. \tag{28}$$

Proof. The first assertion is an immediate consequence of the relations (6), (22), (27), and the definition of the map ψ_0. Since ψ is a Γ - isomorphism, relation (28) follows from (7).

4. The following definition describes one of the main objects of this work.

Definition 1. Let T be an algebraic k-torus and let $\sigma_{\mathbb{Q}}$ be an scrp - cone in the $\mathbb{Q}$ - vector space V, as above. An *affine T - toric variety*, corresponding to the scrp - cone $\sigma_{\mathbb{Q}}$, is a separated k - scheme Y satisfying the following conditions:

(i) There is a k-immersion

$$\lambda_2 \colon T \to Y;$$

(ii) There are a splitting field L of the torus T and an L-isomorphism

$$\varphi_1 \colon Y \times_k L \to X_L(\sigma);$$

(iii) The diagram

$$\begin{array}{ccc} T & \xrightarrow{\lambda_2} & Y \\ {\scriptstyle p_1}\uparrow & & {\scriptstyle q}\uparrow \\ T_L & \xrightarrow{\lambda_1} & X_L(\sigma) \end{array} \tag{29}$$

commutes (here

$$p_1 \colon T_L \to T \text{ and } p_2 \colon Y \times_k L \to Y$$

are the natural projections, and $q := p_2 \circ \varphi_1^{-1}$).

Lemma 4. *An affine T - toric variety is an affine k - scheme of finite type.*

Proof. Let Y be an affine T - toric variety. Then there is a finite extension of fields $L \mid k$ such that the scheme $Y \times_k L$ is an affine L - scheme of finite type. Therefore Y is an affine k - scheme of finite type [11, p. 20] (cf. also [20, Example 2 on p. 23]).

Theorem 1. *The following assertions hold true:*

(i) If there exists an affine T - toric variety, corresponding to an scrp - cone $\sigma_{\mathbb{Q}}$, then the cone $\sigma_{\mathbb{Q}}$ is Γ - invariant.

(ii) There exists one and, up to an isomorphism, only one affine T - toric variety corresponding to a Γ - invariant scrp - cone.

(iii) An affine T - toric variety is an affine k-scheme of finite type, containing (a subscheme isomorphic to) the torus T as a dense open subset; such a variety satisfies conditions (ii) and (iii) of Definition 1 for any splitting field L.

Proof. 1) Let Y be an affine T - toric variety, corresponding to an scrp - cone $\sigma_{\mathbb{Q}}$. It follows from Lemma 4 that

$$Y = \text{Spec } B \tag{30}$$

for a suitable commutative finitely generated k - algebra B. In view of Definition 1(i), relation (4), and relation (28), there is an injective homomorphism

$$B \to (B_L)^{\Gamma}.$$

Therefore commutative diagram (29) gives rise to the following commutative diagram:

$$\begin{array}{ccc} A_L & \longrightarrow & B_L \\ \uparrow & & \uparrow \\ B & \longrightarrow & (B_L)^{\Gamma} \end{array} \tag{31}$$

Since all the maps in (31) are injective, it may be assumed, without loss of generality, that

$$B \subseteq A_L \subseteq B_L, \;\; B \subseteq (B_L)^{\Gamma} \subseteq B_L.$$

It follows then that $B \subseteq A_L \cap (B_L)^{\Gamma}$, and therefore

$$B \subseteq (A_L)^{\Gamma} \tag{32}$$

since $A_L \cap (B_L)^{\Gamma} = (A_L)^{\Gamma}$. Moreover, in accordance with Definition 1(ii) and relation (21), we shall assume that

$$B \otimes_k L = A_L. \tag{33}$$

Since A_L is an integral domain, one can choose a system of independent variables

$$y := (\ldots, y_j^{(i)}, \ldots), \; 1 \leq i \leq l, \; 1 \leq j \leq n,$$

in such a way that

$$s_i = y_1^{(i)} \omega_1 + \cdots + y_n^{(i)} \omega_n \; , \; 1 \leq i \leq l,$$

and

$$B = k[y]/\mathfrak{A} \tag{34}$$

for some ideal $\mathfrak{A}$ of $k[y]$. The choice of variables y determines the action of the group Galois group Γ on B_L:

$$gs_i = y_1^{(i)} g\omega_1 + \cdots + y_n^{(i)} g\omega_n \; , \; 1 \leq i \leq l, \; g \in \Gamma. \tag{35}$$

In view of (32), (34), and (35), we conclude that

$$B = (A_L)^{\Gamma}. \tag{36}$$

Let us enlarge our set of independent variables y to the set

$$\bar{y} := (\ldots, y_j^{(i)}, \ldots), \; 0 \leq i \leq l, \; 1 \leq j \leq n,$$

and let
$$s_0 = y_1^{(0)}\omega_1 + \cdots + y_n^{(0)}\omega_n.$$
The relations
$$\prod_{i=0}^{l} s_i - 1 = \sum_{j=1}^{n} Q_j^{(0)}(\bar{y})\,\omega_j,\ Q_h^{(0)}(\bar{y}) \in k[\bar{y}],\ 1 \le h \le l,$$
and
$$g s_j \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^-_{ij}} - \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^+_{ij}} = \sum_{h=1}^{n} Q_h^{(j,b,g)}(y)\,\omega_h,$$
$$Q_h^{(j,b,g)}(y) \in k[y],\ 1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma, \tag{37}$$
uniquely define the following system of polynomials in $k[\bar{y}]$:
$$\mathcal{P}_1 := \{Q_h^{(j,b,g)}(y),\ Q_h^{(0)}(\bar{y}) \mid 1 \le h \le n,\ 1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma\}.$$
Let $\mathfrak{A}_1$ be the ideal of the ring $k[\bar{y}]$, generated by the set $\mathcal{P}_1$, and let
$$B_1 := k[\bar{y}]/\mathfrak{A}_1. \tag{38}$$
It is clear that
$$B_1 \otimes_k L \cong L[s, s^{-1}]/\mathfrak{A}_0, \tag{39}$$
where the ideal $\mathfrak{A}_0$ is generated in the ring $L[s, s^{-1}]$ by the following set of rational functions:
$$\{s_j - \prod_{j=1}^{l} s_i^{(b\cdot c)_{ij}} \mid\ b \in \mathfrak{C},\ 1 \le j \le l\}.$$
By Corollary 3,
$$\mathfrak{A}_0 = \bar{J}_0(\sigma);$$
therefore it follows from (39) that
$$B_1 \otimes_k L \cong B_L. \tag{40}$$
On identifying the k-algebra B_1 with a subalgebra of B_L, one infers from (40) that
$$B_1 = (B_L)^{\Gamma}. \tag{41}$$
It follows from relations (34), (36), (38), and (41) that
$$\mathfrak{A} = A_L \cap \mathfrak{A}_1.$$
Therefore the ideal $\mathfrak{A}$ is generated in $k[y]$ by the set of polynomials
$$\mathcal{P}_2 := \{Q_h^{(j,b,g)}(y) \mid 1 \le h \le n,\ 1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma\} \tag{42}$$
defined by relations (37). Thus equations (30) and (34) define the scheme Y uniquely up to an isomorphism, in terms of the torus T and the scrp -

cone $\sigma_{\mathbb{Q}}$. Moreover, it follows from relations (4), (28), (30), (38), and (41) that the torus T is (isomorphic to) a dense open subset of the scheme Y.

2) By definition, $gs_j \in A_L$; therefore there is a polynomial $h_{g,j}(s)$ in $L[s]$ such that

$$gs_j = h_{g,j}(s) \pmod{J_0(\sigma)}.$$

Let

$$f_{g,j,b}(s) := h_{g,j}(s)\prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^-_{ij}} - \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^+_{ij}}.$$

Since $B \subseteq A_L$, we may conclude that

$$f_{g,j,b}(s) \in J_0(\sigma),\ 1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma. \tag{43}$$

In view of Corollary 2, it follows from (43) that

$$\tilde{r}(g,b)_{ij} = n(g,b,j)_i + a(g,b,j)_i,\ n(g,b,j)_i \in \mathbb{N},\ 1 \le i \le l, \tag{44}$$

with $\mathbf{a}(g,b,j) \in \mathfrak{R}(\sigma)$ for every j in the interval $1 \le j \le l$, every b in $\mathfrak{C}$, and each g in Γ. Since

$$\sum_{i=1}^{l} a(g,b,j)_i u_i = 0,$$

relations (23) and (44) show that

$$g \cdot u_j \in \check{\sigma},\ 1 \le j \le l,\ g \in \Gamma.$$

Thus the cone $\check{\sigma}_{\mathbb{Q}}$ and, therefore, the cone $\sigma_{\mathbb{Q}}$ are Γ - invariant.

3) Conversely, suppose the scrp-cone $\sigma_{\mathbb{Q}}$ and, therefore, the cone $\check{\sigma}_{\mathbb{Q}}$ be Γ - invariant. We shall construct an affine T - toric variety, corresponding to the cone $\sigma_{\mathbb{Q}}$. Let us choose a system of independent variables

$$y := (\ldots, y_j^{(i)}, \ldots),\ 1 \le i \le l,\ 1 \le j \le n,$$

and write, for brevity,

$$gs_i := y_1^{(i)} g\omega_1 + \cdots + y_n^{(i)} g\omega_n\ ,\ 1 \le i \le l,\ g \in \Gamma. \tag{45}$$

We define the set $\mathcal{P}_2$ by (37) and (42) as above, and *denote* by $\mathfrak{A}$ the ideal in $k[y]$ generated by the set $\mathcal{P}_2$. Let the k-algebra B be *defined* by (34), then

$$B \otimes_k L = L[y]/\bar{\mathfrak{A}},$$

where $\bar{\mathfrak{A}}$ is the ideal generated by the set $\mathcal{P}_2$ in $L[y]$. Since the cone $\check{\sigma}_{\mathbb{Q}}$ and, therefore, the semigroup $\check{\sigma}$ are Γ - invariant, one may write

$$gu_i = \sum_{j=1}^{l} u_j \beta(g)_{ji},\ \beta(g)_{ij} \in \mathbb{N},\ 1 \le i,j \le l,\ g \in \Gamma. \tag{46}$$

It follows from (23) and (46) that

$$\tilde{r}(g,b)_{ji} = \beta(g)_{ji} + a(g,b,i)_j,\ 1 \le j \le l,\ \mathbf{a}(g,b,i) \in \mathfrak{R}(\sigma) \qquad (47)$$

for every i in the interval $1 \le i \le l$, every b in $\mathfrak{C}$, and each g in Γ . By Lemma 2, one can find a matrix b_1 in $\mathfrak{M}_{ld}$ such that $c \cdot b_1 = 0$ and

$$\tilde{r}(g,b_1)_{ji} = -a(g,b,i)_j,\ 1 \le j \le l.$$

Let $b_2 = b + b_1$, then $b_2 \in \mathfrak{C}$ and

$$\tilde{r}(g,b_2)_{ji} = \tilde{r}(g,b_2)^+_{ji} = \beta(g)_{ji}.$$

Therefore the ideal $\bar{\mathfrak{A}}$ contains the set of polynomials

$$\mathcal{P}_3 := \{gs_i - \prod_{j=1}^{l} s_j^{\beta(g)_{ji}} \mid 1 \le i \le l,\ g \in \Gamma\};$$

moreover, it follows from Corollary 3 that $J_0(\sigma) \subseteq \bar{\mathfrak{A}}$. Let $\mathfrak{A}_2$ be the ideal generated in $L[y]$ by the set $\mathcal{P}_3 \cup J_0(\sigma)$, then $\mathfrak{A}_2 \subseteq \bar{\mathfrak{A}}$. On the other hand, it follows from (47) and the definition of the ideal $\mathfrak{A}_2$ that

$$gs_j \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^-_{ij}} - \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^+_{ij}} = \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^-_{ij} + \beta(g)_{ij}} - \prod_{i=1}^{l} s_i^{\tilde{r}(g,b)^+_{ij}} = 0 \pmod{\mathfrak{A}_2},$$

therefore $\mathcal{P}_2 \subseteq \mathfrak{A}_2$ and consequently $\bar{\mathfrak{A}} \subseteq \mathfrak{A}_2$. Thus $\mathfrak{A}_2 = \bar{\mathfrak{A}}$, and we conclude that

$$B \otimes_k L = L[y]/\mathfrak{A}_2. \qquad (48)$$

Since

$$det(g\omega_i)_{1 \le i \le l,\ g \in \Gamma} \ne 0,$$

it follows from (45) and the definition of the set $\mathcal{P}_3$ that

$$y_j^{(i)} = p_j^{(i)}(s) \pmod{\mathfrak{A}_2},\ p_j^{(i)}(s) \in L[s],\ 1 \le i \le l,\ 1 \le j \le n. \qquad (49)$$

Combining relations (48) and (49), one concludes that

$$B \otimes_k L = L[s]/J_0(\sigma) = A_L. \qquad (50)$$

Commutative diagram (31) is a straightforward consequence of equations (50) and the definitions. Since commutative diagram (29), with $Y =$ Spec B, is equivalent to (31), it follows that the k - scheme Spec B is an affine T - toric variety corresponding to the scrp-cone $\sigma_{\mathbb{Q}}$. This completes the proof of Theorem 1.

Example 2. It is clear that the torus T itself is an affine T-toric variety corresponding to the *trivial* scrp-cone $\{0\}$ since

$$T \times_k L \cong X_L(\{0\}),$$

cf. Example 1.

Definition 2. The torus T is said to be *isotropic* if the vector space V (or equivalently, the vector space $\check{V}$) contains a non-zero Γ-invariant vector.

Lemma 5.*(cf.* [18]*). The vector space V contains a non-trivial Γ-invariant scrp-cone if and only if the torus T is isotropic.*

Proof. Suppose that the torus T is isotropic. Then there is an element z in V such that $z \neq 0$ and $g \cdot z = z$. The non-trivial scrp-cone

$$\{a \cdot z \mid a \in \mathbb{Q},\ a \geq 0\}$$

is clearly Γ-invariant. Conversely, let $\sigma_{\mathbb{Q}}$ be a non-trivial Γ-invariant scrp-cone, let $u \in \sigma_{\mathbb{Q}} \setminus \{0\}$, and let

$$z = \sum_{g \in \Gamma} g \cdot u.$$

It is clear that $z \in \sigma_{\mathbb{Q}}$ and that z is a Γ-invariant vector. Suppose that $z = 0$, then

$$u = -\sum_{g \in \Gamma \setminus \{1\}} g \cdot u.$$

Therefore $u \in \sigma_{\mathbb{Q}} \cap (-\sigma_{\mathbb{Q}})$ and, consequently, $\sigma_{\mathbb{Q}} \cap (-\sigma_{\mathbb{Q}}) \neq 0$, in contradiction with the strict convexity of the cone $\sigma_{\mathbb{Q}}$. Thus $z \neq 0$, and the torus T is isotropic.

§3. Integral models.

1. Let now k be a number field, let $\mathfrak{o}$ be the ring of integers of k, and let K be the *minimal* splitting field of our k-torus T. Thus $K \mid k$ is a finite normal extension, and the torus T splits over any finite extension of K. We start with the following corollary of a recent theorem of M. V. Bondarko [2], alluded to in §1.

Proposition 1. *There is a finite normal extension $L \mid k$ satisfying the following two conditions:*
(i) $K \subseteq L$
and
(ii) every fractional ideal of L is a free $\mathfrak{o}$-module.

Proof. Let H be the Hilbert class field of k and suppose that $F \mid H$ is a finite extension of even degree. Then any fractional ideal of F is an $\mathfrak{o}$-free module [2, Theorem 1]. Therefore it suffices to let L be a finite normal extension of k such that $H \cdot K \subseteq L$ and the degree $[L : H]$ is even.

Let the number field L be chosen to satisfy conditions (i) and (ii) of Proposition 1. Then L is a splitting field of the torus T. Let $\mathfrak{O}$ be the ring of integers of L, and suppose $\{\omega_1, \ldots, \omega_n\}$ be an integral basis of the extension $L \mid k$, so that

$$\mathfrak{O} = \mathfrak{o}\omega_1 \oplus \ldots \oplus \mathfrak{o}\omega_n.$$

Then the set of polynomials $\mathcal{P}$, defined in §2, is contained in the ring $\mathfrak{o}[x]$. Let J be the ideal generated in $\mathfrak{o}[x]$ by the set $\mathcal{P}$, let $A := \mathfrak{o}[x]/J$, and let

$$\mathcal{T} = \operatorname{Spec} A. \tag{51}$$

It is clear that the scheme $\mathcal{T}$ is of finite type over $\mathfrak{o}$ and that

$$\mathcal{T} \times_{\mathfrak{o}} k \cong T.$$

It follows from (51) and the definition of the ideal J that

$$\mathcal{T}(\mathcal{A}) = \{\vec{\alpha} \mid \vec{\alpha} \in [(\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O})^*]^d,\ g\alpha_j = \prod_{i=1}^{d} \alpha_i^{r(g)_{ij}},\ 1 \leq j \leq n,\ g \in \Gamma\} \tag{52}$$

for any commutative $\mathfrak{o}$-algebra $\mathcal{A}$. In view of (52), one concludes that $\mathcal{T}$ is an $\mathfrak{o}$-group scheme.

Proposition 2. *(cf.* [14, Proposition 5]*). The scheme $\mathcal{T}$ is a reduced faithfully flat $\mathfrak{o}$-scheme.*

Proof. 1) Since $det(g\omega_j)_{1\leq j\leq n,\, g\in\Gamma} \neq 0$, there are a system of d independent variables $t := (t_1,\ \ldots,\ t_d)$ and a system of $d \cdot n$ polynomials

$$p(t) := (\ldots,\ p(t)_{ij},\ \ldots),\ p(t)_{ij} \in L[t],\ 1 \leq i \leq d,\ 1 \leq j \leq n,$$

satisfying the relations

$$gt_i := \sum_{j=1}^{d} p(t)_{ij} \cdot \omega_j\ ,\ 1 \leq j \leq n,\ g \in \Gamma.$$

Therefore it follows from the definition of the ideal J that

$$A \otimes_{\mathfrak{o}} \mathfrak{O} = \mathfrak{O}[t,\ t^{-1},\ p(t)],$$

where $t^{-1} := (t_1^{-1},\ \ldots,\ t_d^{-1})$. Hence $\mathcal{T} \times_{\mathfrak{o}} \mathfrak{O}$ is a reduced scheme and, since $\mathfrak{O}$ is a free $\mathfrak{o}$-module, it follows that the scheme $\mathcal{T}$ is a reduced.

2) Let $\mathfrak{p}$ be a prime ideal in $\mathfrak{o}$ and let $\mathfrak{P}$ be a fixed prime ideal in $\mathfrak{O}$ lying above $\mathfrak{p}$ (so that $\mathfrak{P}|\mathfrak{p}$). Let $k_{\mathfrak{p}}$, $\mathfrak{o}_{\mathfrak{p}}$, $L_{\mathfrak{P}}$, and $\mathfrak{O}_{\mathfrak{P}}$ denote the $\mathfrak{p}$-completion of the field k, the ring of integers of $k_{\mathfrak{p}}$, the $\mathfrak{P}$-completion of the field L, and the ring of integers of $L_{\mathfrak{P}}$ respectively. Let $\Gamma_{\mathfrak{p}} = \operatorname{Gal}(L_{\mathfrak{P}} \mid k_{\mathfrak{p}})$ and $n_{\mathfrak{p}} := [L_{\mathfrak{P}} : k_{\mathfrak{p}}]$ be the Galois group and the degree of the extension $L_{\mathfrak{P}} \mid k_{\mathfrak{p}}$. Since "being a reduced faithfully flat scheme" is a local property (cf. [4, Chap. II, § 3, Corollary to Proposition 15]), it suffices to prove that the scheme

$$\mathcal{T}^{(\mathfrak{p})} := \mathcal{T} \times_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}$$

is a faithfully flat $\mathfrak{o}_{\mathfrak{p}}$-scheme. Let $\mathcal{A}$ be a commutative $\mathfrak{o}_{\mathfrak{p}}$-algebra. The following well-known identity

$$\mathfrak{O}_{\mathfrak{o}_{\mathfrak{p}}} \otimes_{\mathfrak{o}} \mathfrak{O} = \sum_{g \in \Gamma/\Gamma_{\mathfrak{p}}} \oplus\, \mathfrak{O}_{g\mathfrak{P}},$$

[8, Chap. III, eq.(1.8)], gives:

$$\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O} = \mathcal{A} \otimes_{\mathfrak{o}_{\mathfrak{p}}} \mathfrak{o}_{\mathfrak{p}} \otimes_{\mathfrak{o}} \mathfrak{O} = \sum_{g \in \Gamma/\Gamma_{\mathfrak{p}}} \oplus (\mathcal{A} \otimes_{\mathfrak{o}_{\mathfrak{p}}} \mathfrak{O}_{g\mathfrak{P}}),$$

so that

$$[(\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O})^*]^d = [\sum_{g \in \Gamma/\Gamma_{\mathfrak{p}}} \oplus (\mathcal{A} \otimes_{\mathfrak{o}_{\mathfrak{p}}} \mathfrak{O}_{g\mathfrak{P}})^*]^d \tag{53}$$

Since

$$\mathcal{T}^{(\mathfrak{p})} = \mathrm{Spec}\ (A \otimes_{\mathfrak{o}} \mathfrak{o}_{\mathfrak{p}}),$$

the sets of $\mathcal{A}$-points $\mathcal{T}(\mathcal{A})$ and $\mathcal{T}^{(\mathfrak{p})}(\mathcal{A})$ may be identified. Therefore it follows from (52) and (53) that

$$\mathcal{T}^{(\mathfrak{p})}(\mathcal{A}) = \{\vec{\alpha} \mid \vec{\alpha} \in [(\mathcal{A} \otimes_{\mathfrak{o}_{\mathfrak{p}}} \mathfrak{O}_{\mathfrak{P}})^*]^d,\ g\alpha_j = \prod_{i=1}^{d} \alpha_i^{r(g)_{ij}},\ 1 \le j \le n_{\mathfrak{p}},\ g \in \Gamma_{\mathfrak{p}}\}.$$

Consequently, as in 1), we can find an $\mathfrak{o}_{\mathfrak{p}}$-algebra $A_{\mathfrak{p}}$ such that

$$\mathcal{T}^{(\mathfrak{p})} = \mathrm{Spec}\ A_{\mathfrak{p}}$$

and

$$A_{\mathfrak{p}} \otimes_{\mathfrak{o}_{\mathfrak{p}}} \mathfrak{O}_{\mathfrak{P}} = \mathfrak{O}_{\mathfrak{P}}[t,\ t^{-1},\ q(t)], \tag{54}$$

where $t := (t_1,\ \ldots,\ t_d)$ is a system of d independent variables, $t^{-1} := (t_1^{-1},\ \ldots,\ t_d^{-1})$, and $q(t)$ is a system of $d \cdot n$ polynomials with coefficients in $L_{\mathfrak{P}}$. Since $\mathfrak{O}_{\mathfrak{P}}$ is a free $\mathfrak{o}_{\mathfrak{p}}$-module, it follows from (54) that the $\mathfrak{o}_{\mathfrak{p}}$-module $A_{\mathfrak{p}}$ is torsion-free. The ring $\mathfrak{o}_{\mathfrak{p}}$ is a principal ideal domain, therefore $A_{\mathfrak{p}}$ is a flat module, [12, Example 9.1.3 on p.254]. Moreover, since $\mathfrak{p}A_{\mathfrak{p}} \neq A_{\mathfrak{p}}$, the module $A_{\mathfrak{p}}$ is faithfully flat [4, Chap. I, § 3, Proposition 1]. This completes the proof of Proposition 2.

Remark. In [14, Propositions 2 and 4], we have proved that the $\mathfrak{o}$-group scheme $\mathcal{T}$ is isomorphic to the scheme-theoretic closure of the k-torus T with respect to the natural imbedding of that torus into the Néron–Raynaud model of the quasi-split k-torus $R_{L|k}G^d_{m,L}$. This assertion can be used to give a shorter proof of the flatness of the scheme $\mathcal{T}$, cf. [3, p.291].

The $\mathfrak{o}$-group scheme $\mathcal{T}$ may be regarded as a natural $\mathfrak{o}$-integral model of the k-torus T; in [13], [14], we have called $\mathcal{T}$ the *standard model* of T. If the extension $K \mid k$ is at most *tamely ramified*, then the identity component of the scheme $\mathcal{T}$ is isomorphic to the identity component of the Néron–Raynaud model of the torus T [14, Theorem 3]. In general, one can obtain a smooth integral model of finite type of the torus T from the standard model by a suitable process of resolution of singularities. In the following example, such a model is described for the norm-torus, defined over $\mathbb{Q}$ by the equation $x^2 - 2y^2 = 1$.

Example 3. Let $k = \mathbb{Q}$, let $K = \mathbb{Q}(\sqrt{2})$, and consider the norm-torus

$$T = R^1_{K|k} G_{m,k} := \text{Ker } (N_{K|k} : R_{K|k} G_{m,K} \to G_{m,k}).$$

The extension $K \mid k$ is wildly ramified at the prime 2. We have

$$\mathcal{T} = \text{Spec } \mathbb{Z}[x, y]/(x^2 - 2y^2 - 1).$$

The reduction $\mathcal{T}_2 = \mathcal{T} \times_{\mathbb{Z}} \mathbf{F}_2$ of $\mathcal{T}$ modulo 2 is not a reduced scheme; indeed,

$$\mathcal{T}_2 = \text{Spec } \mathbf{F}_2[x, y]/((x - 1)^2).$$

Thus the standard model $\mathcal{T}$ is not a smooth scheme in this case. It follows that the identity component $\mathcal{N}^{(0)}$ of the Néron–Raynaud model $\mathcal{N}$ of the torus T is isomorphic to the identity component $\mathcal{N}_1^{(0)}$ of the following "smoothening"

$$\mathcal{N}_1 = \text{Spec } \mathbb{Z}[x, y]/(x^2 + x - 2y^2)$$

of the standard model $\mathcal{T}$; cf. [6, Example 4.3], [14, Example 4], [15, Proposition 5.6]. One observes that

$$\mathcal{N}_1^{(0)} = \mathcal{N}_1 \setminus \mathcal{S},$$

where

$$\mathcal{S} = \text{Spec } \mathbb{Z}[x, y]/(x + 1, 2);$$

clearly, $\mathcal{S} \cong G_{a,\mathbf{F}_2}$. A simple calculation shows that

$$\mathcal{N}^{(1)} \setminus \mathcal{S} \cong \text{Spec } \mathbb{Z}[x, y]/(2x^2 + x - 4y^2);$$

thus

$$\mathcal{N}^{(0)} \cong \text{Spec } \mathbb{Z}[x, y]/(2x^2 + x - 4y^2).$$

At the end of our joint work with V.E. Voskresenskiĭ [14], we ask whether the identity component of the Néron–Raynaud model of the torus T is an affine scheme. Professor Q. Liu [16] and Professor D. Lorenzini [17] tell us that this is indeed the case.

Proposition 3. ([16], [17]). *The identity component $\mathcal{N}^{(0)}$ of the Néron–Raynaud model $\mathcal{N}$ of an algebraic torus defined over a number field k is an affine scheme.*

Proof. Since $\mathcal{N}$ is a smooth separated scheme locally of finite type over $\mathfrak{o}$ [3, Definition 1 on p. 289], it follows from the general theory [10, Proposition 5.5.1 on p.136], [3, Proposition 2.4.8 on p.53] that $\mathcal{N}^{(0)}$ is a flat separated group scheme of finite type over $\mathfrak{o}$. But $\mathfrak{o}$ is a Dedekind domain and the generic fibre T of $\mathcal{N}^{(0)}$ is affine, therefore $\mathcal{N}^{(0)}$ is an affine scheme [1, Proposition 2.3.1 on p.30].

The reader may consult [14] and the works cited there for a further discussion of the arithmetic properties and applications of integral models of algebraic tori.

2. Let X be an affine T - toric variety corresponding to an scrp - cone $\sigma_{\mathbb{Q}}$, and suppose that $X = \text{Spec } k[y]/\mathfrak{A}$, where the ideal $\mathfrak{A}$ is generated in $k[y]$ by the set of polynomials $\mathcal{P}_2$. As in [13], let $\vec{\mathfrak{B}} := (\mathfrak{B}_1, \ldots, \mathfrak{B}_l)$ be a sequence of fractional ideals of L such that

$$g \cdot \mathfrak{B}_j = \prod_{i=1}^{l} \mathfrak{B}_i^{\tilde{r}(g,b)_{ij}},\ 1 \le j \le l,\ g \in \Gamma, \tag{55}$$

for some (and therefore for every) b in $\mathfrak{C}$.

In view of Proposition 1, every fractional ideal in L is a free $\mathfrak{o}$-module. Let us fix an $\mathfrak{o}$-basis $\{\omega_{1j}, \ldots, \omega_{nj}\}$ of the ideal $\mathfrak{B}_j$, write, for brevity,

$$gz_j := y_1^{(j)} g\omega_{1j} + \cdots + y_n^{(j)} g\omega_{nj},\ 1 \le j \le l,\ g \in \Gamma,$$

and consider the set of polynomials

$$\mathcal{P}_4(\vec{\mathfrak{B}}) := \{R_h^{(j,b,g)}(y) \mid 1 \le h \le n,\ 1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma\}$$

defined by the following conditions:

$$gz_j \prod_{i=1}^{l} z_i^{\tilde{r}(g,b)_{ij}^-} - \prod_{i=1}^{l} z_i^{\tilde{r}(g,b)_{ij}^+} = \sum_{h=1}^{n} R_h^{(j,b,g)}(y)\, \omega_h^{(j,b,g)},$$

$$R_h^{(j,b,g)}(y) \in \mathfrak{o}[y],\ 1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma,$$

where $\{\omega_1^{(j,b,g)}, \ldots, \omega_n^{(j,b,g)}\}$ is an $\mathfrak{o}$-basis of the ideal

$$\prod_{i=1}^{l} \mathfrak{B}_i^{\tilde{r}(g,b)_{ij}^+} (= g\mathfrak{B}_j \prod_{i=1}^{l} \mathfrak{B}_i^{\tilde{r}(g,b)_{ij}^-}).$$

In view of (55), the polynomials $R_h^{(j,b,g)}(y)$ are well-defined. By construction, $\mathcal{P}_4(\vec{\mathfrak{B}}) \subseteq \mathfrak{o}[y]$.

Let $J_4(\vec{\mathfrak{B}})$ be the ideal generated in $\mathfrak{o}[x]$ by the set $\mathcal{P}_4(\vec{\mathfrak{B}})$ and let

$$\mathcal{X}_{\vec{\mathfrak{B}}} = \text{Spec } \mathfrak{o}[x]/J_4(\vec{\mathfrak{B}}). \tag{56}$$

It follows from (56) and the definition of X that

$$\mathcal{X}_{\vec{\mathfrak{B}}} \times_{\mathfrak{o}} k \cong X;$$

therefore the scheme $\mathcal{X}_{\vec{\mathfrak{B}}}$ is an $\mathfrak{o}$-integral model of X. Moreover, there are a natural imbedding

$$F:\ \mathcal{T} \hookrightarrow \mathcal{X}_{\vec{\mathfrak{B}}} \tag{57}$$

and a natural action

$$G:\ \mathcal{T} \times_{\mathfrak{o}} \mathcal{X}_{\vec{\mathfrak{B}}} \to \mathcal{X}_{\vec{\mathfrak{B}}}\ , \tag{58}$$

although, in general, $\mathcal{T}$ is not isomorphic to an open subscheme of the scheme $\mathcal{X}_{\vec{\mathfrak{B}}}$.

Let $\mathcal{A}$ be a commutative $\mathfrak{o}$-algebra. It follows from the definition of the $\mathfrak{o}$-scheme $\mathcal{X}_{\vec{\mathfrak{B}}}$ that

$$\mathcal{X}_{\vec{\mathfrak{B}}}(\mathcal{A}) = \{\vec{\beta} \mid \beta_j \in \mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{B}_j,\ g\beta_j \prod_{i=1}^{l} \beta_i^{\tilde{r}(g,b)^-_{ij}} = \prod_{i=1}^{l} \beta_i^{\tilde{r}(g,b)^+_{ij}},$$
$$1 \le j \le l,\ b \in \mathfrak{C},\ g \in \Gamma\}.$$

Morphisms (57) and (58) can be pointwise given by the following maps:

$$F(\mathcal{A}) : \mathcal{T}(\mathcal{A}) \hookrightarrow \mathcal{X}_{\vec{\mathfrak{B}}}(\mathcal{A}),\ F(\mathcal{A}) : \vec{\alpha} \mapsto \vec{\beta},$$

where

$$\beta_j = \prod_{i=1}^{d} \alpha_i^{c_{ij}},\ 1 \le j \le l,$$

and

$$G(\mathcal{A}) : \mathcal{T}(\mathcal{A}) \times \mathcal{X}_{\vec{\mathfrak{B}}}(\mathcal{A}) \to \mathcal{X}_{\vec{\mathfrak{B}}}(\mathcal{A}),\ G(\mathcal{A}) : (\vec{\alpha}, \vec{\beta}) \mapsto F(\mathcal{A})(\vec{\alpha}) \cdot \vec{\beta};$$

(here we let $\vec{\beta} \cdot \vec{\gamma} = \vec{\delta}$ with $\delta_j := \beta_j \gamma_j,\ 1 \le j \le l$, for $\vec{\beta},\ \vec{\gamma}$ in $\mathcal{X}_{\vec{\mathfrak{B}}}(\mathcal{A})$).

As in [13], we introduce the group

$$\mathfrak{I}(\sigma) := \{\vec{\mathfrak{B}} \mid g \cdot \mathfrak{B}_j = \prod_{i=1}^{l} \mathfrak{B}_i^{\tilde{r}(g,b)_{ij}},\ 1 \le j \le l,\ g \in \Gamma\}$$

of sequences of fractional ideals, with componentwise multiplication, and its subgroup

$$\mathfrak{I}(\sigma)_{pr} := \{(\vec{\alpha}) \mid \vec{\alpha} \in (L^*)^l,\ g \cdot \alpha_j = \prod_{i=1}^{l} \alpha_i^{\tilde{r}(g,b)_{ij}},\ 1 \le j \le l,\ g \in \Gamma,\ b \in \mathfrak{C}\},$$

where $(\vec{\alpha}) := ((\alpha_1), \ldots, (\alpha_l))$ stands for a sequence of principal ideals. Let

$$H(\sigma) = \mathfrak{I}(\sigma)/\mathfrak{I}(\sigma)_{pr}.$$

One can prove [13, Proposition 1] that the group $H(\sigma)$ is finite and, moreover, that if $\vec{\mathfrak{B}}^{-1}\vec{\mathfrak{B}}' \in \mathfrak{I}(\sigma)_{pr}$, then $\mathcal{X}_{\vec{\mathfrak{B}}} \cong \mathcal{X}_{\vec{\mathfrak{B}}'}$. In particular, it follows that there are only finitely many pairwise non-isomorphic $\mathfrak{o}$-integral schemes $\mathcal{X}_{\vec{\mathfrak{B}}}$ for a given T-toric variety X.

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