

**THE SUPER-CHARACTER FORMULA FOR A CLASS
OF INTEGRABLE INFINITE DIMENSIONAL MODULES
OF BORCHERDS-KAC-MOODY LIE SUPERALGEBRAS**

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In [R1] we gave the character formula for integrable infinite dimensional highest weight modules of Borchersds-Kac-Moody (BKM) Lie superalgebras. In this short note, we give a formula for their super-character. The proof is similar to the one in [R1]. However in [R1] we omitted certain details which ought to be emphasized for otherwise one of the main arguments would not hold.

We first remind the reader of the definition of a BKM Lie superalgebra.

Let I be a finite set indexing the simple roots and S a subset of I indexing the odd simple roots. The finite dimensional Borchersds-Kac-Moody Lie superalgebras have a Cartan decomposition of Borchersds-Kac-Moody type. In other words, they have an abelian even Lie algebra subalgebra H – called a Cartan subalgebra – with bilinear form (\cdot, \cdot) containing elements $h_i, i \in I$ such that the Cartan matrix $A = (a_{ij}), a_{ij} = (h_i, h_j)$ satisfies

- (i) $a_{ij} \leq 0$ if $i \neq j$;
- (ii) $\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$;

and are generated by the Lie subalgebra H and elements $e_i, f_i, i \in I$ satisfying the following defining relations:

- (1) $[e_i, f_j] = \delta_{ij} h_i$;
- (2) $[h, e_i] = (h, h_i) e_i, [h, f_i] = -(h, h_i) f_i$;
- (3) $\deg e_i = 0 = \deg f_i$ if $i \notin S, \deg e_i = 1 = \deg f_i$ if $i \in S$;
- (4) $(\text{ad}(e_i))^{1-\frac{2a_{ij}}{a_{ii}}} e_j = 0 = (\text{ad}(f_i))^{1-\frac{2a_{ij}}{a_{ii}}} f_j$ if $a_{ii} > 0$ and $i \neq j$;
- (5) $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

We next fix some more notation, which we keep standard.

Notation. Set G to be a BKM superalgebra and H a Cartan subalgebra. The roots spaces will be written $G_\alpha = \{x \in G : [h, x] = \alpha(h)x\}$. We will write Δ for the set of roots and respectively $\Delta^+, \Delta_0, \Delta_1, \Delta_0^+, \Delta_1^+$ for the set of positive, even, odd, even positive, odd positive roots of

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G . We also set $\Delta_E^+ = \{\alpha \in \Delta_0^+ : \frac{1}{2}\alpha \notin \Delta\}$, $I_< = \{i \in I : a_{ii} \leq 0\}$ and $S_0 = \{i \in S : a_{ii} = 0\}$.

The Weyl group W is the group generated by the reflections induced by all the simple roots of positive norm.

For $w \in W$, set

$$\epsilon(w) = \begin{cases} 1 & \text{if } w \text{ is the product of an even number of reflections} \\ -1 & \text{otherwise} \end{cases}.$$

For any weight $\Lambda \in H^*$, $L(\Lambda)$ will denote the G -module of highest weight Λ .

Set ρ to be the Weyl vector, i.e. $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all $i \in I$.

We now state a result proved in [R2, Lemma 2.2] which we will be frequently using:

Lemma 1. *For any root $\alpha \in \Delta^+$ of norm 0, there exists $w \in W$ such that $w(\alpha)$ is a simple odd root. Furthermore for all $w \in W$, $w(\alpha) \in \Delta^+$.*

We are interested in integrable modules. This means that all finite type [R2, Definition 2.3] root vectors act in a locally nilpotent manner. Hence, $(\Lambda, \alpha_i) \geq 0$ whenever $a_{ii} > 0$. In this paper, we add a technical condition and so consider irreducible highest weight modules $L(\Lambda)$ for which that

$$(\Lambda, \alpha_i) \geq 0 \quad \forall i \in I. \tag{1}$$

We are mostly interested in infinite dimensional BKM superalgebras. Hence all roots of negative norm are of infinite type [R2, Corollary 2.5]. Therefore, the super-character formula we give in this paper applies to a large class of modules.

Remark 2. When the BKM Lie superalgebra G is finite dimensional, because there may be roots of negative type, which would then necessarily be of finite type, the formula we give applies to a class of highest weight modules which are in general of infinite dimension: Indeed, the module $L(\Lambda)$ is integrable if and only if $\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \in \mathbf{Z}_+$ for all roots α of finite type with non-trivial norm. Hence, if there is a positive root α of negative norm, then the above condition (1) implies that $(\Lambda, \alpha) \geq 0$, whereas integrability forces $(\Lambda, \alpha) \leq 0$. Therefore $(\Lambda, \alpha) = 0$. As $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$ and all $i \in I$ appear in the support of β [R2, Corollary 2.5], this implies that $(\Lambda, \alpha_i) = 0$ for all $i \in I$ and so $\Lambda = 0$. So the only integrable module the formula we give here applies to is the trivial irreducible representation of finite dimensional BKM Lie superalgebras with roots of negative norm. See [R3] for arbitrary integrable modules.

When G is finite dimensional with no roots of negative norm, G is either of type $A(m, 0)$ or $B(0, n)$. In the latter case, all roots have positive norm, and our formula applies to all integrable modules, and we recover the results of [K1]. In the former case, there is a unique simple odd root α_1 of norm 0 and by definition of the Weyl vector $(\rho, \alpha_1) = 0$. And so our formula applies in this case, to all typical integrable modules satisfying $(\Lambda, \alpha_1) > 0$ and to all atypical modules satisfying $(\Lambda, \alpha_1) = 0$.

So, when G is finite dimensional, our formula applies to highest weight modules which are not in general integrable, and in particular are infinite dimensional.

Similarly as in [K3] or [R1], the super-character of the Verma module of highest weight Λ can be computed and the next equalities follow using standard arguments.

Proposition 3. *The super-character of the highest weight G -module $L(\Lambda)$ is given by:*

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \text{sch } L(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} c_\lambda e(\lambda + \rho) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha)) \quad (2)$$

where $c_\lambda \in \mathbf{Z}$ and $c_\Lambda = 1$.

Before proving the super-character formula, we need to see that the Weyl group W keeps the set of weights λ satisfying $c_\lambda \neq 0$ invariant as it is not completely obvious in the case when there are odd roots of norm 0.

Lemma 4. *We keep the notation of Proposition 3. For all $\lambda \in H^*$ and all $w \in W_E$, $c_{w(\lambda + \rho) - \rho} = \epsilon(w)e_\lambda$.*

Proof. To prove the Lemma, it suffices to show that for any weight $\lambda \in H^*$ satisfying $c_\lambda \neq 0$,

$$r_i(\lambda + \rho) \leq \Lambda + \rho \quad \forall a_{ii} > 0. \quad (i)$$

Indeed suppose this holds. Then $\lambda + \rho = r_i(\mu + \rho)$ for some weight $\mu \in H^*$ such that $\mu \leq \Lambda$ and $|\mu + \rho| = |\Lambda + \rho|$. Hence, we may write

$$\sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} d_\lambda e(\lambda + \rho) = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} c_\lambda e(r_i(\lambda + \rho)).$$

If $\alpha \in D_1^+$ has non-zero norm, then $2\alpha \in D^+$. Hence, equation (2) may be re-written as:

$$\begin{aligned} e(\rho) \prod_{\alpha \in \Delta_E^+} (1 - e(-\alpha)) & \prod_{\alpha \in \Delta_0^+ - \Delta_E^+} (1 + e(-\alpha)) \text{sch } L(\Lambda) \\ & = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} c_\lambda e(\lambda + \rho) \prod_{\substack{\alpha \in \Delta_1^+ \\ (\alpha, \alpha) = 0}} (1 - e(-\alpha)) \end{aligned} \quad (2')$$

Applying the reflection r_i to the left hand side of (2') multiplies it by -1 . Furthermore by Lemma 1,

$$r_i \left(\prod_{\substack{\alpha \in \Delta_1^+ \\ (\alpha, \alpha) = 0}} (1 - e(-\alpha)) \right) = \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha)).$$

Hence applying r_i to the right hand side of equality (2'), we can deduce that

$$\sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} (d_\lambda + c_\lambda) e(\lambda + \rho) \prod_{\substack{\alpha \in \Delta_1^+ \\ (\alpha, \alpha) = 0}} (1 + e(-\alpha)) = 0. \quad (ii)$$

Suppose there exists a weight λ for which $d_\lambda \neq -c_\lambda$. We may take λ to be such that the height $\Lambda - \lambda$ is minimal. Then, equation (ii) gives

$$d_\lambda + c_\lambda + \sum_{\mu} d_{\lambda + \mu} + c_{\lambda + \mu} = 0,$$

the sum being taken over all distinct sums μ of positive odd roots of norm 0. By minimality of the weight λ , it follows that

$$d_\lambda + c_\lambda = 0,$$

contradicting the definition of λ .

We now prove property (i). Let $\lambda \in H^*$ be a weight satisfying $c_\lambda \neq 0$ and $r_i(\lambda + \rho) \not\leq \Lambda + \rho$ for some $i \in I$ such that $a_{ii} > 0$. We may take a weight λ with this property such that the height of $\Lambda - \lambda$ is minimal. To complete the proof, we only need to show that this leads to a contradiction

Suppose that $c_\lambda + \sum_{\mu} c_{\lambda + \mu} \neq 0$, where μ runs over all sums of distinct positive odd roots of norm 0. Considering the left hand side of (2'), we then get a contradiction. Hence,

$$c_\lambda + \sum_{\mu} c_{\lambda + \mu} = 0,$$

where μ runs over all sums of distinct positive odd roots of norm 0. So, there is a sum μ of positive odd roots of norm 0 such that $c_{\lambda+\mu} \neq 0$. By minimality of the height of $\Lambda - \lambda$, $r_i(\lambda + \mu + \rho) \leq \Lambda + \rho$. By Lemma 1, $r_i(\mu) > 0$. So,

$$r_i(\lambda + \rho) \leq \Lambda + \rho,$$

again contradicting assumption, and proving the result. \square

Let $\mu = \sum_{i \in I} k_i \alpha_i$, where $k_i \geq 0$ for all $i \in I$ and write

$$\text{ht}_0(\mu) = \sum_{i \in I-S} k_i.$$

Set

$$T_\Lambda = e(\Lambda + \rho) \sum \epsilon(\mu) e(\mu)$$

with

$$\epsilon(\mu) = \begin{cases} (-1)^{\text{ht}_0(\mu)}, & \text{if } \mu = \sum_{i \in I} k_i \alpha_i \in \mathbf{Z}_+ \Delta^+, \quad k_i \leq 1 \quad \text{unless } i \in S_0 \\ & \alpha_{ij} = 0, (\Lambda, \alpha_i) = 0 \\ 0 & \text{otherwise} \end{cases}$$

We are now ready to prove the super-character formula.

Theorem 5. *Let $L(\Lambda)$ be an integrable G -module satisfying $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$. Set*

$$R = e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))^{-1}$$

The super-character formula for the module $L(\Lambda)$ is:

$$\text{sch } L(\Lambda) = e(\rho) \sum_{w \in W} \epsilon(w) w(T'_\Lambda) L'^{-1}.$$

Proof. Let Λ be a weight satisfying inequalities (1). From Proposition 3, we get

$$e(\rho) \text{ch } L(\Lambda) L = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda - \rho|^2 = |\Lambda - \rho|^2}} c_\lambda e(\lambda + \rho). \quad (i)$$

Lemma 4 gives $c_\lambda = \epsilon(w) c_\mu$ if $w(\lambda + \rho) = \mu + \rho$ for some $w \in W$. Let λ be such that $c_\lambda \neq 0$. Hence, for all $w \in W$, $c_{w(\lambda+\rho)-\rho} \neq 0$ and so $w(\lambda + \rho) \leq \Lambda + \rho$. Let $\mu \in \{w(\lambda + \rho) - \rho | w \in W\}$ be such that $\text{ht}(\Lambda - \mu)$

is minimal. Then $(\mu + \rho, \alpha_i) \geq 0$ for all $i \in I$ such that $a_{ii} > 0$, and $|\mu + \rho|^2 = |\Lambda + \rho|^2$. Setting $\mu = \Lambda - \sum_{i \in I} x_i \alpha_i$, $x_i \in \mathbf{Z}_+$, the latter equation gives

$$\sum_{i \in I} x_i (\Lambda + \mu + 2\rho, \alpha_i) = 0. \quad (ii)$$

From the above and the definition of ρ and Λ , $(\Lambda + \mu + 2\rho, \alpha_i) > 0$ for all $i \in I$ such that $a_{ii} > 0$. Next, consider $i \in I$ such that $a_{ii} \leq 0$. Now, $(\Lambda + \mu + 2\rho, \alpha_i) = (\Lambda + \mu + \alpha_i, \alpha_i)$. Suppose that $x_i \neq 0$. Then $\mu + \alpha_i = \Lambda - \sum_{i \in I} y_i \alpha_i$ and $y_i \geq 0$ for all $i \in I$, and so $(\Lambda + \mu + 2\rho, \alpha_i) \geq 0$. Hence, equation (ii) forces $x_i \neq 0$ only if $a_{ii} \leq 0$ and $(\Lambda + \mu + 2\rho, \alpha_i) = 0$. Therefore, for all i such that $x_i \neq 0$,

$$(\Lambda, \alpha_i) = 0, \quad i \neq j \Rightarrow (\alpha_i, \alpha_j) = 0 \quad x_i \geq 2 \Rightarrow (\alpha_i, \alpha_i) = 0. \quad (iii)$$

We need to compute the sum T_1 of all terms in the right hand side of (i) for which $(\lambda + \rho, \alpha_i) \geq 0$ for all $i \in I$ with $a_{ii} > 0$. Equation (2) can be re-written as

$$e(\rho) \text{ch } L(\Lambda) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha))^{\text{mult}\alpha} = \sum_{w \in W} w(T) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))^{\text{mult}\alpha}. \quad (iv)$$

Let $c_\lambda e(\lambda + \rho)$ be a term in T_1 . Set

$$\lambda = \Lambda - \alpha - \beta,$$

where $\alpha = \sum_{i \in I-S} x_i \alpha_i$ and $\beta = \sum_{i \in S} z_i \alpha_i$, $x_i, z_i \in \mathbf{Z}_+$. For $x_i \neq 0$ and $z_i \neq 0$, α_i satisfies conditions (iii).

Claim: If $e(w(\lambda + \rho) + \gamma)$ is a term in T_1 for $0 \neq \gamma = \sum_{i \in S} z_i \alpha_i$, $z_i \in \mathbf{Z}_+$ and $w \in W$, then $w(\lambda + \rho) = \lambda + \rho$.

From the above, $(\lambda + \rho, \alpha_i) \geq 0$ for all $i \in I$. Therefore

$$w(\lambda + \rho) = \lambda + \rho - \sum_{i, a_{ii} > 0} c_i h_i \quad (v),$$

where $c_i \in \mathbf{Z}_+$. Since $e(w(\lambda + \rho) + \beta)$ is a term in T_1 , $\sum_{i \in I-S} (x_i + c_i) \alpha_i + \sum_{i \in S} (y_i + z_i) \alpha_i$ must satisfy conditions (iii). Hence $c_i \neq 0 \Rightarrow a_{ii} \leq 0$, and so from (v) we can deduce that $w(\lambda + \rho) = \lambda + \rho$.

We now show by induction on $\text{ht}(\beta)$ that $c_\lambda = \epsilon(\alpha)$, where $\epsilon(\alpha) = (-1)^n$ if α is the sum of n distinct pairwise perpendicular simple even roots α_i perpendicular to Λ , and $\epsilon(\alpha) = 0$ otherwise.

Suppose that $\text{ht}(\beta) = 0$. Then, there must be a term on the left hand side of (iv) equal to $c_\lambda e(\lambda + \rho)$. However, $\text{ch } L(\Lambda)$ does not have λ as a term since $(\Lambda, \alpha_i) = 0$ for all $i \in I$ such that $x_i \neq 0$ [K2, Proposition 11.2]. Hence, $\lambda = \Lambda - \sum_{\mu_i \in \Delta^+} \mu_i$ and the roots μ_i in the sum are all distinct. The support of a root being connected [K2, §1], without loss of generality $\mu_i = \alpha_i$ and it follows that $x_i \leq 1$ for all $i \in I$ and so we get the desired answer for c_λ .

Next assume that $\text{ht}(\beta) > 0$ and that the result holds for all weights λ with β of smaller height. Then, no term on the left hand side of (iv) equals $c_\lambda e(\lambda + \rho)$. Since β is the sum of mutually orthogonal simple roots and if any appear more than once, it has norm 0, no sub-sum of β is a root. Hence the above Claim gives

$$c_\lambda e(\lambda) + \sum_{s=1}^r (-1)^s c_{\lambda + \alpha_{i_1} + \dots + \alpha_{i_s}} e(\lambda + \alpha_{i_1} + \dots + \alpha_{i_s}) e(-\alpha_{i_1}) \dots e(-\alpha_{i_s}) = 0,$$

where r is the number of distinct simple roots α_i with $z_i \neq 0$ since in each term, the α_i 's must be distinct. It follows by induction that

$$c_\lambda + \epsilon(\alpha) \sum_{s=1}^r \binom{t}{s} (-1)^s = 0,$$

giving the desired answer for c_λ . □

The super-character formula for the trivial module leads to one of the most important formulae associated to a BKM superalgebra:

Theorem 3.4. *For any BKM superalgebra G , the super-denominator formula is*

$$\frac{\prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha))^{\text{mult}\alpha - o(\alpha)}}{\prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))^{o(\alpha)}} = e(\rho) \sum_{w \in W} \epsilon(w) w(T').$$

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