FINITE DIMENSIONAL MODULES OF BORCHERDS-KAC-MOODY LIE SUPERALGEBRAS

URMIE RAY[1,2]

1. Introduction

Some finite dimensional simple Lie superalgebras have a Cartan decomposition with a Cartan matrix of Borcherds-Kac-Moody type. With respect to such a decomposition, the root system either has no roots of negative norm or has exactly one positive even root of negative norm. The first group contains the Lie superalgebras of type $A(m,0)$, $B(0,n)$, $B(m,1)$, $C(n)$ and $D(m,1)$. The second one has two subclasses, one with no odd roots of negative norm of type $A(m,1)$ and one with odd roots of negative norm, namely the exceptional classical Lie superalgebras, i.e. $G(3)$, $F(4)$ and $D(2,1;\alpha)$, $\alpha \neq 0,1$ [R2]. The character and super-character formulae are known for all the finite dimensional classical modules of finite dimensional classical Lie superalgebras which are not exceptional. In [K2], this was given by Kac for typical modules and his result in particular applies to finite dimensional typical modules of $G(3)$, $F(4)$ and $D(2,1;\alpha)$. For atypical modules of simple finite dimensional Lie superalgebras, they were given by several people [H,HM1, HM2, M, MSS, NRS, S1, S2, V, VHKT]. In the case of atypical modules of the exceptional Lie superalgebras, they are not yet known. The character and super-character formulae we gave in [R1,3] apply to integrable modules of infinite dimensional BKM superalgebras and to some finite dimensional modules – typical and atypical – of classical Lie superalgebras but in the case of exceptional Lie superalgebras aaply only to the trivial representation. In this paper we give a proof of the character and super-character formulae which applies to all finite dimensional modules of Borcherds-Kac-Moody Lie superalgebras except of type $A(m,1)$. In particular, we derive the character and super-character of all atypical finite dimensional representations of the exceptional classical Lie superalgebras.

2. The structure of Borcherds-Kac-Moody Lie superalgebras

In this section, we give the specific properties of the finite dimensional Lie superalgebras of Borcherds-Kac-Moody type which make the proof possible.

---

[1] on leave from Dept de Math., Univ. de Reims, France

[2] Supported by a grant from the Ministerio de Educación, Cultura y Deporte, España, Ref. n.: sab2003-0080

Most of the results can be found in [R2] and we only state them. Others are simple consequences.

Let I be a finite set indexing the simple roots and S a subset of I indexing the odd simple roots. The finite dimensional Borcherds-Kac-Moody Lie superalgebras have a Cartan decomposition of Borcherds-Kac-Moody type. In other words, they have an abelian even Lie algebra subalgebra H – called a Cartan subalgebra – with bilinear form $(.,.)$ containing elements h_i, $i \in I$ such that the Cartan matrix $A = (a_{ij})$, $a_{ij} = (h_i, h_j)$ satisfies

(i) $a_{ij} \leq 0$ if $i \neq j$;
(ii) $\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$;

and are generated by the Lie subalgebra H and elements $e_i, f_i, i \in I$ satisfying the following defining relations:

(1) $[e_i, f_j] = \delta_{ij} h_i$;
(2) $[h, e_i] = (h, h_i)e_i$, $[h, f_i] = -(h, h_i)f_i$;
(3) $\deg e_i = 0 = \deg f_i$ if $i \notin S$, $\deg e_i = 1 = \deg f_i$ if $i \in S$;
(4) $(\mathrm{ad}\,(e_i))^{1-\frac{2a_{ij}}{a_{ii}}} e_j = 0 = (\mathrm{ad}\,(f_i))^{1-\frac{2a_{ij}}{a_{ii}}} f_j$ if $a_{ii} > 0$ and $i \neq j$;
(5) $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

As we are concerned with finite dimensional simple Lie superalgebras, the conditions are stronger. The proof of the character and super-character formulae which we give is strongly based on the following properties, which are immediate from [R2, Lemma 2.4].

Lemma 2.1. *Let G be a finite dimensional simple BKM Lie superalgebra. Then for all $i \in I$, $a_{ii} \geq 0$. Furthermore, $a_{ii} = 0$ implies that $i \in S$.*

We next list the BKM finite dimensional Lie superalgebras and their BKM type Cartan matrices: The BKM Lie superalgebras of type $A(m,0) = sl(m+1,1)$, $A(m,1) = sl(m+1,2)$, $B(0,n) = osp(1,2n)$, $B(m,1) = osp(2m+1,2)$, $C(n) = osp(2,2n-2)$, $D(m,1) = osp(2m,2)$, $D(2,1;\alpha)$ for $\alpha \neq 0, -1$, $F(4)$, and $G(3)$ have a base with respect to which the Cartan matrix is respectively:

$$\begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -2 \\ 0 & 0 & 0 & \dots & -2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -1 & \dots & 0 & 0 \\ 0 & 2 & -1 & \dots & 0 & 0 \\ -1 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 & -\alpha \\ -1 & 2 & 0 \\ -\alpha & 0 & 2\alpha \end{pmatrix} \text{ for } \alpha > 1, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 6 \end{pmatrix},$$

where in all cases $i \in S$ if and only if $a_{ii} = 0$ or 1. Furthermore $D(2,1,\alpha)$ and $D(2,1;\beta)$ are isomorphic when α and β belong to the same orbit of the group of order 6 generated by $\alpha \mapsto -1-\alpha$, $\alpha \mapsto \frac{1}{\alpha}$.

Remarks 2.2. (i) The usual Cartan matrices (which we will write B) for example for $G(3)$, $F(4)$ and $D(2,1;\alpha)$ (given in [K1]) are respectively:

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

However in each case, there exists a diagonal matrix D with positive entries such that the matrix $A = DB$ is symmetric. With respect to these matrices, the Lie superalgebras G have similar types of generators e_i', f_i' as the ones given above, the only difference being that for $i,j \in I$, $(h_i', h_j') = d_i^{-1} d_j^{-1} a_{ij}$, where $h_i' = [e_i', f_i']$ and d_i is the (i,i)-th entry of the diagonal matrix D. However, we may as well take the symmetric version A of the Cartan matrix for reasons of simplicity since these generators are multiples of the above ones.

(ii) As the Lie superalgebras in question are simple, the elements h_i form a basis of the Cartan subalgebra H.

(iii) The reason we do not consider the case $A(m,1)$ is that it is the only finite dimensional BKM superalgebra with two odd simple roots. Part of proof applies to this case and it could be presumably completed to deal with it. However as the character and super-character formulae for it have already been computed in [VHKT], there is no need to do so.

Notation. Set G to be a BKM superalgebra and H a Cartan subalgebra, $G_{\overline{0}}$ (resp. $G_{\overline{1}}$) the even (resp. odd) part. Set G_E to be the Lie subalgebra generated by the root spaces $G_{\pm\alpha_i}$, $i \in I - S$.
The roots spaces will be written $G_\alpha = \{x \in G : [h,x] = \alpha(h)x\}$. The following will denote the usual nilpotent Lie sub-superalgebras: $N = \oplus_{\alpha\in\Delta^+} G_\alpha$, and $N_{\overline{0}} = N \cap G_{\overline{0}}$.
We will write Δ for the set of roots and respectively Δ^+, Δ_0, Δ_1, Δ_0^+, Δ_1^+ for the set of positive, even, odd, even positive, odd positive roots of G with respect to the above given Cartan decomposition. Furthermore, we will write Δ_E for the set of roots generated by the even simple roots, i.e. α_i with $i \in I - S$. In all cases, α_1 will denote the unique odd simple root. It has norm 0.
The Weyl group W is the group generated by the reflections induced by all the even roots. And W_E will denote the Weyl group induced by the even simple roots. For $i \in I - S$, r_i will denote the reflection corresponding to the simple root α_i.

We next give some further properties specific to finite dimensional BKM superalgebras, which will play an important role in the proof of the character and super-character formulae. They are easy consequences of the above Cartan matrices or else see [R2, Lemmas 2.2, 2.4, Corollary 2.5, Proposition 2.6].

Lemma 2.3.
(i) *All positive roots of norm 0 are conjugate to a simple root of norm 0 under the action of the group W_E.*
(ii) *If the roots $\gamma_1, \gamma_2 \in \Delta^+$ have non-positive norm, then $(\gamma_1, \gamma_2) \leq 0$. ;*
(iii) *There is at most one odd positive root of negative norm. It has norm -2. If such an odd root β exists, then the Lie superalgebra G is exceptional, 2β is the unique even positive root 2β of negative norm and there is a unique simple root of norm 0. Furthermore, 2β is the highest root of Δ.*
(iv) *If the root β given in (ii) exists, then $r_\beta w = w r_\beta$ for all $w \in W_E$, where r_β be the reflection induced by the even root 2β. Equivalently, $(\beta, \alpha) = 0$ for all $\alpha \in \Delta_E$.*
(v) *All positive odd roots of norm 0 are conjugate to an odd simple root under the action of the group W_E. Moreover, $W_E(\Delta_1^+) = \Delta_1^+$ and $r_\beta(\Delta_1^+) = -\Delta_1^+$ in the case when the root β exists.*
(vi) *All the root spaces have dimension 1.*

(vii) *For any $\alpha, \gamma \in \Delta_1^+$ of norm 0, either $[G_\alpha, G_\gamma] = 0$ or $[G_\alpha, G_\gamma] = G_{2\beta}$. Moreover if there are roots of negative norm, then for every $\alpha \in \Delta_1^+$, there $2\beta - \alpha \in \Delta_1^+$.*

Proof. We only prove (vii): Suppose that $\alpha, \gamma \in \Delta_1^+$ are of norm 0 and that $[G_\alpha, G_\gamma] \neq 0$. Then $\alpha + \beta$ is an even root containing 1 in its support. Therefore, it cannot have positive norm since it would then have to be be conjugate to a simple root of positive norm under the action of W_E [R2, Lemma 2.2]. As the Lie superalgebra G is finite dimensional, it does not have even roots of norm 0 [R2, Lemma 2.4] and hence $\alpha + \gamma$ has negative norm. The first part of (vii) then follows from (vi). Now, for any $\alpha \in \Delta_1^+$, $\alpha = \alpha_1 + \sum_{i>1} k_i \alpha_i$, and so by (ii) and (iv) $(\alpha, \beta) < 0$. Therefore $\alpha - 2\beta$ is a root, proving the second part of (vii). □

Notation. When there is an even root of negative norm, it will be called 2β and r_β will denote the corresponding reflection.

Remarks 2.4. **(i)** There is a a root of negative norm when the even part of the exceptional Lie superalgebras is not an irreducible Lie algebra and $S_{2\beta} = G_{-2\beta} \oplus [G_{-2\beta}, G_{2\beta}] \oplus G_{2\beta}$ is isomorphic to a A_1 summand of $G_{\bar{0}}$. Moreover, $G_{\bar{0}}$ has at most one simple summand corresponding to a root of negative norm. Indeed, $A(m,0)_{\bar{0}} = A_m \oplus \mathbf{C}$, $A(m,1)_{\bar{0}} = A_m \oplus A_1 \oplus \mathbf{C}$, $B(0,n)_{\bar{0}} = C_n$, $B(m,1) = B_m \oplus A_1$, $D(m,1)_{\bar{0}} = D_m \oplus A_1$, $D(2,1;\alpha)_{\bar{0}} = A_1 \oplus A_1 \oplus A_1$ for $\alpha \neq 0, -1$, $F(4) = B_3 \oplus A_1$, and $G(3) = G_2 \oplus A_1$ [K1].

(ii) Since an even root of negative norm is the highest root, it contains all the simple roots in its support.

(iii) The characteristic property of BKM superalgebras stated in Lemma 2.3 (ii) [R2, Proposition 2.6] is the reason why there is a proof valid for all simple finite dimensional Lie superalgebras with a BKM type Cartan decomposition. This condition does not hold for any other simple finite dimensional Lie superalgebra.

Example 2.5. The exceptional Lie superalgebra $G(3)$ has three simple roots $\alpha_1, \alpha_2, \alpha_3$ with $S = \{1\}$. The set of positive odd roots is

$$\Delta_1^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3,$$

$$\alpha_1 + 3\alpha_2 + \alpha_3, \alpha_1 + 3\alpha_2 + 2\alpha_3, \alpha_1 + 4\alpha_2 + 2\alpha_3\}$$

and the set of positive even roots generated by the simpler roots α_2, α_3 – in other words the set of positive roots of G_2 – is

$$\Delta_E^+ = \{\alpha_2, \alpha_3, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, 3\alpha_2 + \alpha_3, 3\alpha_2 + 2\alpha_3\}.$$

The unique positive odd root of negative norm is $\beta = \alpha_1 + 2\alpha_2 + \alpha_3$. All the other positive odd roots have norm 0.

3. The character and super-character formulae for finite dimensional representations

We first fix some more notation, which we keep standard.

Notation. For $w \in W$, set

$$\epsilon(w) = \begin{cases} 1 & \text{if } w \text{ is the product of an even number of reflections} \\ -1 & \text{otherwise} \end{cases}.$$

For any weight $\Lambda \in H^*$, $L(\Lambda)$ will denote the finite dimensional module of highest weight Λ and $P(\Lambda)$ the set of weights of $L(\Lambda)$.
Set ρ to be the Weyl vector, i.e. $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all $i \in I$. Set $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Delta_0^+} \alpha$ and $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Delta_1^+} \alpha$.

To start with, an elementary well known result about the Weyl vector of finite dimensional BKM superalgebras. It follows from the above given Cartan matrices.

Lemma 3.1. *Suppose that the BKM Lie superalgebra G is finite dimensional. Then, $\rho = \rho_0 - \rho_1$. Furthermore, $(\rho_0, \beta) = -2$.*

Proof. The equality follows from the fact that $(\rho_0, 2\beta) = \frac{1}{2}(2\beta, 2\beta)$ and that the root β has norm -2 (Lemma 2.3 (iii)). □

We are interested in integrable modules. This means that all finite type [R2, Definition 2.3] root vectors act in a locally nilpotent manner. As here our Lie superalgebra G has finite dimension, all its roots are of finite type. Clearly, all root vectors x corresponding to roots of norm 0 and finite type act in this manner on any G-module since $[x, x] = 0$. Furthermore as 2β is is the highest root (if roots of negative norm exist), its roots space commutes with every positive root space [R2, Lemma 2.4].

Lemma 3.2. *Let V be a G-module.*
(i) For any $v \in V$ and any $\alpha \in \Delta_1^+$ such that $(\alpha, \alpha) = 0$, $x^2v = 0$ for all $x \in G_\alpha$.
(ii) If there is an odd root of negative norm β, then $e_{2\beta}v = 2e_\beta v$ and $e_{2\beta}xv = xe_{2\beta}xv$ for all $e_\beta \in G_\beta$, $e_{2\beta} \in G_{2\alpha}$, $x \in N$ and $v \in V$.

Remarks 3.3. For Kac-Moody Lie superalgebras, integrable modules and irreducible ones are the same [K4, §10]. This is no longer necessarily true for BKM with simple roots of non-positive norm. Indeed there may be non-trivial vectors v which are not highest weight such that $e_i v = 0$ for all $i \in I$. The module $L(\Lambda)$ is integrable if and only if $(\Lambda, \alpha) \geq 0$ for all roots α of finite type. In [R1,3], we give the character and super-character formulae for irreducible integrable highest weight modules $L(\Lambda)$ of BKM superalgebras G satisfying $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$, so with the added technical condition $(\Lambda, \alpha_i) \geq 0$ when $a_{ii} \leq 0$. Hence, $(\Lambda, \alpha) \geq 0$ for all $\alpha \in \Delta_+$.

When G is finite dimensional, as we have seen in the previous section, it may have roots of negative norm and finite type. In this case, integrability forces $(\Lambda, \beta) \leq 0$. Therefore $(\Lambda, \beta) = 0$. As $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$ and all $i \in I$ appear in the support of β (see Remark 2.4 (iii)), this implies that $(\Lambda, \alpha_i) = 0$ for all $i \in I$ and so $\Lambda = 0$. So the formulae given in [R1,3] only apply to the trivial irreducible representation of finite dimensional BKM Lie superalgebras with roots of negative norm.

When G is finite dimensional with no roots of negative norm, G is either of type $A(m, 0)$ or $B(0, n)$. In the latter case, all roots have positive norm, and our formulae apply to all integrable modules, and we recover the results of [K3]. In the former case, there is a unique simple odd root α_1 of norm 0 and by definition of the Weyl vector $(\rho, \alpha_1) = 0$. And so our formulae apply to all typical integrable modules satisfying $(\Lambda, \alpha_1) > 0$ and to all atypical modules satisfying $(\Lambda, \alpha_1) = 0$.

When G is infinite dimensional, there are no roots of finite type and negative norm [R2, Corollary 2.5] and hence the formulae apply to the large class of irreducible integrable modules $L(\Lambda)$ for which $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$.

From now on G will be a finite dimensional BKM Lie superalgebra and the G-module $L(\Lambda)$ will be finite dimensional. We need to introduce some further notation.

Notation. Since $G_{\overline{0}}$ is a semisimple Lie algebra, as a $G_{\overline{0}}$-module $L(\Lambda)$ is a direct sum of irreducible $G_{\overline{0}}$-modules, which will be written V_i, $1 \leq i \leq m$. The highest weight of the module V_i will be denoted Λ_i.

Since ρ_0 is the Weyl vector of the Lie algebra $G_{\overline{0}}$, the next result gives the character formula of the $G_{\overline{0}}$-module $L(\Lambda)$.

Lemma 3.4.

$$e(\rho_0) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \mathrm{ch}\, L(\Lambda) = \sum_{w \in W} \sum_{i=1}^{m} \epsilon(w) e(\Lambda_i + \rho_0). \qquad (1)$$

We next find more information about the weights Λ_i.

Lemma 3.5. *For any $1 \leq i \leq m$, $\Lambda - \Lambda_i$ is a sum of distinct positive odd roots.*

Proof. Let $v \in L(\Lambda)$ be a highest weight vector of the $G_{\overline{0}}$-module V_i. Since

$$L(\Lambda) = U(N_0^-)U(N_1^-)v_\Lambda,$$

$$v = v_1 + v_2, \quad \text{where} \quad v_1 \in U(N_1^-)v_\Lambda \quad \text{and} \quad v_2 \in U(N_0^-)N_0^- U(N_1^-)v_\Lambda.$$

We first show that $v_1 \neq 0$. So, suppose that

$$v = v_2.$$

We may then write

$$v = \sum_{i>1} f_i u_i + f_{2\beta} u_\beta \quad \text{for some} \quad u_i, u \in L(\Lambda).$$

Since the weights of the vectors u_i and u_β are strictly greater than Λ_i, the vectors u_i, u_β belong to a direct sum of $\oplus V_k$ of $G_{\overline{0}}$-modules of highest weight strictly greater than Λ_i. It follows that $v \in \oplus V_k$. Hence, the weight of the vector v is strictly greater than Λ_i. This contradiction forces $v_1 \neq 0$.

We can therefore deduce that $\Lambda - \lambda$ is a sum of odd positive roots. Since $v_1 \in U(N_1^-)v_\Lambda$, by Lemmas 2.3 (vii) and 3.2, there are no repeats in this sum. □

Similarly as in [K3] or [R1], the character of the Verma module of highest weight Λ can be computed and the next equalities follow using standard arguments.

Proposition 3.6. *The character and super-character of the finite dimensional highest weight G-module $L(\Lambda)$ are given by:*

$$e(\rho_0) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \operatorname{ch} L(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c_\lambda e(\lambda + \rho) e(\rho_1) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha)) \quad (2)$$

and

$$e(\rho_0) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \operatorname{sch} L(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c'_\lambda e(\lambda + \rho) e(\rho_1) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha)) \quad (3)$$

where $c_\lambda, c'_\lambda \in \mathbf{Z}$ and $c_\Lambda = 1 = c'_\Lambda$.

The first step and crux of the proof is given in the following result.

Proposition 3.7. *We keep the notation of Proposition 3.6. Suppose that $\lambda \in H^*$ is a weight for which either $c_\lambda \neq 0$ or $c'_\lambda \neq 0$.*

(i) If there are no roots of negative norm, then there is an element $w \in W$ such that $w(\Lambda + \rho) - (\lambda + \rho)$ is a sum of positive odd roots.

(ii) If $(\Lambda + \rho, \beta) \leq 0$ (resp. $(\Lambda + \rho, \beta) \geq 0$), then there is an element $w \in W$ (resp. $w \in W_E$) such that $w(\Lambda + \rho) - (\lambda + \rho)$ is a sum of positive odd roots.

Proof. We prove case (ii). Suppose first that $(\Lambda + \rho, \beta) \leq 0$. Suppose that the result is false. Then, there is a weight $\lambda \in H^*$ such that $c_\lambda \neq 0$ and for which for any element $w \in W$, $w(\Lambda + \rho) - (\lambda + \rho)$ is never a sum of positive odd roots. We may choose a weight λ with this property and such that the height of $\Lambda - \lambda$ is minimal.

Suppose first that $c_\lambda + \sum_\mu c_{\lambda+\mu} \neq 0$, where μ runs over all possible sums of distinct positive odd roots. Equating the left hand sides of equations (1) and (2), we get

$$\lambda + \rho_0 = w(\Lambda_i + \rho_0)$$

for some $w \in W$ and $1 \leq i \leq m$. The definition of ρ_1 together with Lemma 2.3 (iv) imply that for any $w \in W_E$, $w(\rho_1) = \rho_1$ and $r_\beta(\rho_1) = -\rho_1$. Hence, since $2\rho_1 = \sum_{\alpha \in \Delta_1^+} \alpha$, there is an element $w \in W_E$ such that either

$$\lambda + \rho = w(\Lambda_i + \rho) \tag{i}$$

or

$$\lambda + \rho = r_\beta w(\Lambda_i + \rho) - \sum_{\alpha \in \Delta_1^+} \alpha. \tag{ii}$$

If equality (i) holds, then it follows immediately from Lemmas 3.5 and 2.3 (v) that $w(\Lambda + \rho) - (\lambda + \rho)$ is either 0 a sum of positive odd roots. Suppose that equality (ii) holds. Then, by Lemma 3.5, $\mu = \Lambda - \Lambda_i$ is a sum of distinct positive odd roots. Hence by Lemma 2.3 (v), so is $-wr_\beta(\mu)$ and thus either $wr_\beta(\mu) + \sum_{\alpha \in \Delta_1^+} \alpha$ is also a sum of distinct positive odd roots or 0. So again, $w(\Lambda + \rho) - (\lambda + \rho)$ is either 0 or a sum of positive odd roots. Therefore, in both cases, assumptions are contradicted.

Therefore, $c_\lambda + \sum_\mu c_{\lambda+\mu} = 0$, where μ runs through all sums of distinct positive odd roots. Hence there exists a sum of positive odd root μ such that $c_{\lambda+\mu} \neq 0$. By minimality of the weight λ, there is an element $w \in W$ for which $\lambda + \rho + \mu = w(\Lambda + \rho) - \gamma$ for some sum γ of positive odd roots. Hence $w(\Lambda + \rho) - (\lambda + \rho)$ is a sum of positive odd roots, again contradicting assumptions and proving the result when $(\Lambda + \rho, \beta) \leq 0$.

Suppose next that $(\Lambda+\rho,\beta)\geq 0$ and that the result is false. Then, there is a weight $\lambda\in H^*$ such that $c_\lambda\neq 0$ and for which for any element $w\in W_E$, $w(\Lambda+\rho)-(\lambda+\rho)$ is never a sum of positive odd roots. We may choose a weight λ with this property and such that the height of $\Lambda-\lambda$ is minimal. Again either equality (i) or (ii) holds. If (i) does, then we get a contradiction to the definition of λ as above. Suppose that (ii) holds. Then, by Lemma 2.3 (iv) and (v) together with Lemma 3.1,

$$\begin{aligned}\lambda+\rho &= wr_\beta(\Lambda_i+\rho_0)-\rho_1\\ &= w(\Lambda+\rho_0)+(\Lambda+\rho_0,\beta)\beta-wr_\beta(\mu)-\rho_1\\ &= w(\Lambda+\rho)+(\Lambda+\rho_0,\beta)\beta-wr_\beta(\mu).\end{aligned}$$

Since $\lambda+\rho\leq\Lambda+\rho$,

$$(\Lambda+\rho_0,\beta)\beta-wr_\beta(\mu)\leq\Lambda+\rho-w(\Lambda+\rho). \qquad (iii)$$

Since $w\in W_E$, by Lemma 3.8, $w(\Lambda+\rho)\leq\Lambda+\rho$, and so $\Lambda+\rho-w(\Lambda+\rho)$ is a positive sum of the simple roots α_i, $i\in I-S$. Since for any $\alpha\in\Delta_1^+$, $\alpha=\alpha_1+\sum_{i\notin S}k_i\alpha_i$, inequality (iii) implies that $-(\Lambda+\rho_0)$ is greater than the number of positive odd roots appearing in the sum $-wr_\beta(\mu)$. Moreover, as $L(\Lambda)$ is an integrable module, $\frac{2(\Lambda,2\beta)}{(2\beta,2\beta)}$ is an integer and so (Λ,β) is an even integer since by Lemma 2.3 (iii), the norm of β is -2. Hence, by Lemma 3.1, $(\Lambda+\rho_0,\beta)$ is also an even integer. Now, for any $\alpha\in\Delta_1^+$, $2\beta-\alpha\in\Delta_1^+$ by Lemma 2.3 (vii). We can therefore deduce that $-(\Lambda+\rho_0,\beta)\beta+wr_\beta(\mu)$ is a sum of positive odd roots. Therefore we again get a contradiction to the definition of λ. The result then follows in the same way as above.
The case when $c'_\lambda\neq 0$ is dealt with in the same manner so is Case 1. □

Before proving the second main step leading to a description of the weights λ with $c_\lambda\neq 0$ or $c'_\lambda\neq 0$, we need to see that the group W_E keeps the set of weights λ satisfying $c_\lambda\neq 0$ invariant and consider the effect of the reflection r_β on this set.

Lemma 3.8. *We keep the notation of Proposition 3.6. For all $\lambda\in H^*$ and all $w\in W_E$, $c_{w(\lambda+\rho)-\rho}=\epsilon(w)e_\lambda$ and $c'_{w(\lambda+\rho)-\rho}=\epsilon(w)c'_\lambda$.*

Proof. To prove the first equality, it suffices to show that for any weight $\lambda\in H^*$ satisfying $c_\lambda\neq 0$,

$$r_i(\lambda+\rho)\leq\Lambda+\rho\quad\forall i\neq 1. \qquad (i)$$

Indeed suppose this holds then $\lambda+\rho=r_i(\mu+\rho)$ for some weight $\mu\in H^*$ such that $\mu\leq\Lambda$ and $|\mu+\rho|=|\Lambda+\rho|$. Hence, we may write

$$\sum_{\substack{\lambda\leq\Lambda\\ |\lambda+\rho|^2=|\Lambda+\rho|^2}} d_\lambda e(\lambda+\rho)=\sum_{\substack{\lambda\leq\Lambda\\ |\lambda+\rho|^2=|\Lambda+\rho|^2}} c_\lambda e(r_i(\lambda+\rho)).$$

On the other hand, applying the reflection r_i to the left hand side of (2) multiplies it by -1. Now, by Lemma 2.3 (v) and by definition of ρ_1,

$$r_i(\rho_1) = \rho_1 \quad \text{and} \quad r_i(\prod_{\alpha\in\Delta_1^+}(1+e(-\alpha))) = \prod_{\alpha\in\Delta_1^+}(1+e(-\alpha)).$$

Hence applying r_i to the right hand side of equality (2), we can deduce that

$$\sum_{\substack{\lambda\leq\Lambda \\ \left|\lambda+\rho\right|^2=\left|\Lambda+\rho\right|^2}} (d_\lambda + c_\lambda)e(\lambda+\rho)e(\rho_1)\prod_{\alpha\in\Delta_1^+}(1+e(-\alpha)) = 0. \qquad (ii)$$

Suppose there exists a weight λ for which $d_\lambda \neq -c_\lambda$. We may take λ to be such that the height $\Lambda - \lambda$ is minimal. Then, equation (ii) gives

$$d_\lambda + c_\lambda + \sum_\mu d_{\lambda+\mu} + c_{\lambda+\mu} = 0,$$

the sum being taken over all distinct sums μ of positive odd roots. By minimality of the weight λ, it follows that

$$d_\lambda + c_\lambda = 0,$$

contradicting the definition of λ.

We now prove property (i). Let $\lambda \in H^*$ be a weight satisfying $c_\lambda \neq 0$ and $r_i(\lambda+\rho) \not\leq \Lambda+\rho$ for some $i \neq 1$. We may take a weight λ with this property such that the height of $\Lambda - \lambda$ is minimal. To complete the proof, we only need to show that this leads to a contradiction

Suppose that $c_\lambda + \sum_\mu c_{\lambda+\mu} \neq 0$, where μ runs over all sums of distinct positive odd roots. Equating the right hand sides of equations (1) and (2), we can deduce that

$$\lambda + \rho_0 = w(\Lambda_i + \rho_0)$$

for some index $1 \leq i \leq m$ and some element $w \in W$. It follows that

$$r_i(\lambda+\rho_0) \leq \Lambda + \rho_0.$$

Therefore, as $r_i(\rho_1) = \rho_1$,

$$r_i(\lambda+\rho) \leq \Lambda + \rho,$$

contradicting assumption. Hence,

$$c_\lambda + \sum_\mu c_{\lambda+\mu} = 0,$$

where μ runs over all sums of distinct positive odd roots. So, there is a sum μ of positive odd roots such that $c_{\lambda+\mu} \neq 0$. By minimality of the height of $\Lambda - \lambda$, $r_i(\lambda + \mu + \rho) \leq \Lambda + \rho$. By Lemma 2.3 (v), $r_i(\mu) > 0$. So,

$$r_i(\lambda + \rho) \leq \Lambda + \rho,$$

again contradicting assumption, and proving the first equality. The second equality is proved in the same way. □

Remark 3.9. Note that lemma 3.8 does not necessarily hold for all $w \in W$. Indeed if there is an even weight 2β of negative norm, then $(\rho, 2\beta) \geq 0$ since $(\rho, \alpha_i) \geq 0$ for all $i \in I$. The inequality is strict if there are simple roots of positive norm and so we may have $r_\beta(\Lambda+\rho) > \Lambda+\rho$ even though $r_\beta(\Lambda) \leq \Lambda$ since $L(\Lambda)$ being finite dimensional, $r_\beta(P(\Lambda)) = P(\Lambda)$. And as we will see later, even when $r_\beta(\Lambda+\rho) \leq \Lambda+\rho$, Lemma 3.8 may not hold for all weights $\lambda \in H^*$ such that $c_\lambda \neq 0$. However a restricted version of Lemma 3.8 holds for the reflection r_β.

Lemma 3.10. *We keep the notation of Proposition 3.6. Suppose that there is an even root 2β of negative norm. Then, for any weight $\lambda \in H^*$, $c_{r_\beta(\lambda+\rho)-\rho} = -c_\lambda \neq 0$ (resp $c'_{r_\beta(\lambda+\rho)-\rho} = -c'_\lambda$) if and only if $-\Lambda - \rho \leq \lambda + \rho \leq \Lambda + \rho$. Furthermore, such a weight λ exists if and only if $(\Lambda + \rho, \beta) \leq 0$.*

Proof. First notice that for any weight $\mu \in H^*$ such that $(\mu, \alpha_i) \geq 0$ for all $i \neq 1$ and $(\mu, \beta) \leq 0$,

$$-\mu \leq w(\mu) \leq \mu$$

for all $w \in W$ since there is an element $w_0 \in W$ such that $w_0(\mu) = -\mu$.

Suppose that for $\lambda \in H^*$, $c_{r_\beta(\lambda+\rho)-\rho} = -c_\lambda \neq 0$. Hence, Lemma 3.8 and the above imply that

$$-\Lambda - \rho \leq -\lambda - \rho \leq r_\beta(\lambda + \rho) \leq \lambda + \rho \leq \Lambda + \rho$$

or

$$-\Lambda - \rho \leq -r_\beta(\lambda + \rho) \leq \lambda + \rho \leq r_\beta(\lambda + \rho) \leq \Lambda + \rho$$

depending on the sign of $(\lambda + \rho, \beta)$. This proves the result. □

We next prove the second main step. We deal with two cases depending on the sign of $(\Lambda + \rho, \beta)$.

Proposition 3.11. *We keep the notation of Proposition 3.6. Suppose that either there is no roots of negative norm or* $(\Lambda+\rho,\beta)>0$*. If* $c_\lambda \neq 0$*, then there exits* $w \in W$ *such that either*
(i) $\lambda+\rho = w(\Lambda+\rho)$ *or*
(ii) there is an infinite chain of weights λ_i*:*

$$w(\Lambda+\rho) > \lambda+\rho > \lambda_1+\rho > ... > \lambda_n+\rho > ...$$

such that $c_{\lambda_i} \neq 0$*,* $\lambda_i - \lambda_{i+1}$ *is a sum of odd roots.*

Proof. Assume that there is a root β as the case when no such roots exist can be proved in a similar simplified manner. Suppose that there is a weight λ such that for all $w \in W$, $\lambda+\rho \neq w(\Lambda+\rho)$ and $c_\lambda \neq 0$. Set $\mu = \lambda - \sum_{\alpha\in\Delta_1^+}\alpha$. By Lemma 3.7, there is an element $w \in W_E$ and a sum γ of positive odd roots such that

$$\mu+\rho = w(\Lambda+\rho) - \gamma - \sum_{\alpha\in\Delta_1^+}\alpha. \qquad (i)$$

Claim: $c_\mu + \sum_\alpha c_{\mu+\alpha} = 0$, where the sum is over all sums α of distinct positive odd roots.

Suppose this is false. Then from equation (2), we get $\mu+\rho_0 = w_1(\Lambda_i+\rho_0)$ for some element $w_1 \in W$ and some index $1 \le i \le m$. So equation (i) becomes

$$w_1(\Lambda_i+\rho_0) - \rho_1 = w(\Lambda+\rho) - \gamma - \sum_{\alpha\in\Delta_1^+}\alpha.$$

By definition, $\sum_{\alpha\in\Delta_1^+}\alpha = 2\rho_1$. So,

$$\begin{aligned} w_1(\Lambda_i+\rho_0) &= w(\Lambda+\rho) - \gamma - \rho_1 \\ &= w(\Lambda+\rho_0) - \gamma - \sum_{\alpha\in\Delta_1^+}\alpha \end{aligned}$$

since $w \in W_E$. Setting $\delta = \Lambda - \Lambda_i$, the above can be re-written as

$$w_1(\Lambda+\rho_0) - w(\Lambda+\rho) = -\gamma - \sum_{\alpha\in\Delta_1^+}\alpha + w_1(\delta). \qquad (ii)$$

Now, by Lemma 3.5, δ is a sum of distinct positive odd roots. Hence if $w_1 \in W_E$, then by Lemma 2.3 (v), the right hand side of equation (ii) is a sum of positive odd roots, whereas this cannot be the case of the left hand side. So $w_1 \notin W_E$ and there is an element $w_2 \in W_E$ such that $w_1 = w_2 r_\beta$. Then,

$$\begin{aligned}\mu + \rho &= w_2 r_\beta(\Lambda_i + \rho_0) - \rho_1 \\ &= w_2 r_\beta(\Lambda_i + \rho) - 2\rho_1.\end{aligned}$$

Hence, the definition of μ gives

$$\lambda + \rho = w_2 r_\beta(\Lambda_i + \rho). \qquad (iii)$$

From Lemmas 2.3 (iv) and 3.5 and the assumption $(\Lambda + \rho, \beta) > 0$, $(w_2(\Lambda_i + \rho), \beta) > 0$. Hence, by equation (iii), $(\lambda + \rho, \beta) < 0$. On the other hand, Equation (i) together with Lemma 2.3 (ii) imply that $(\lambda + \rho, \beta) > 0$. This contradiction proves the Claim.

An immediate consequence of the above Claim is that there is some weight λ' such that $\lambda - \lambda'$ is a sum of distinct positive odd roots and $c_{\lambda'} \neq 0$. Hence, the result follows from Proposition 3.7 and repeating the argument for λ' and so on. □

The next result also applies to the case $(\Lambda + \rho, \beta) > 0$. Nevertheless, we give Proposition 3.11 to show that the case $(\Lambda + \rho, \beta) > 0$ can be dealt with in a straightforward manner.

Proposition 3.12. *Suppose that the G-module $L(\Lambda)$ is irreducible. We keep the notation of Proposition 3.6. Suppose that $c_\lambda \neq 0$ or $c'_\lambda \neq 0$. If for any $w \in W$, $w(\lambda + \rho) \neq \Lambda + \rho$, then there is some root $\alpha \in \Delta_1^+$ of norm 0 such that $(\lambda + \rho, \alpha) = 0$.*

Proof. We give the proof in the context of the character formula, the super-character formulae can be dealt with in the same manner.

From the denominator formula given in [R1], it follows that

$$\begin{aligned}& e(\rho_1)\Pi_{\alpha\in\Delta_1^+}(1 + e^{-\alpha}) \sum_{w\in W_E} \epsilon(w)(-1)^l e(w(\rho - l\alpha_1))chL(\Lambda) \\ &= e(\rho_0)\Pi_{\alpha\in\Delta_0^+}(1 - e^{-\alpha})chL(\Lambda) \\ &= e(\rho_1)\Pi_{\alpha\in\Delta_1^+}(1 + e^{-\alpha}) \sum_{\substack{\lambda\leq\Lambda \\ |\lambda+\rho|^2=|\Lambda+\rho|^2}} c_\lambda e(\lambda + \rho)\end{aligned} \quad .$$

Clearly,

$$\sum_{w\in W_E} \epsilon(w)(-1)^l e(w(\rho - l\alpha_1))chL(\Lambda) = \sum_{\lambda\leq\Lambda} d_\lambda e^{\lambda+\rho}$$

for some constants $d_\lambda \in \mathbf{Z}$. Suppose that $d_\lambda \neq c_\lambda$ for some weight λ. We may take λ such that the height $\Lambda - \lambda$ is minimal. The above equation then forces $d_\lambda - c_\lambda + \sum_\mu d_{\lambda+\mu} - c_{\lambda+\mu} = 0$, where the sum is taken over all distinct sums μ of positive odd roots. So, by minimality of λ, $d_\lambda = c_\lambda$, contradicting assumption. Hence,

$$\sum_{w\in W_E} \epsilon(w)(-1)^l e(w(\rho - l\alpha_1))chL(\Lambda) = \sum_{\substack{\lambda\leq\Lambda \\ \left|\lambda+\rho\right|^2=\left|\Lambda+\rho\right|^2}} c_\lambda e(\lambda+\rho). \qquad (i)$$

Let $\lambda \in H^*$ be a weight such that $c_\lambda \neq 0$ and such that for any $w \in W$, $w(\lambda+\rho) \neq \Lambda + \rho$. From equation (i), we can deduce that

$$\lambda + \rho = w(\mu + \rho - l\alpha_1)$$

for some $\mu \in P(\Lambda)$ and some integer $l \geq 0$. We show that when $w = 1$, $(\lambda, \alpha_1) = 0$, which by Lemma 2.3 (v) is sufficient to prove the Proposition. So suppose that $w = 1$ and $(\lambda, \alpha_1) \neq 0$.

Case 1: Assume that for all $v \in L(\Lambda)$ of weight μ, either

$$e_1(v) \neq 0 \quad \text{or} \quad f_1(v) \neq 0.$$

Claim: The μ-weight space of the module $L(\Lambda)$ has a basis consisting of vectors v for which either $f_1 v = 0$ or $e_1 v = 0$.

Take a basis v_i, $1 \leq i \leq s$ of the μ-weight space of the module $L(\Lambda)$. By assumption, $(\lambda, \alpha_1) \neq 0$ and so $(\mu, \alpha_1) \neq 0$. Hence, as

$$(\mu, \alpha_1)v_i = f_1 e_1 v_i + e_1 f_1 v_i,$$

the μ-weight space is generated by the vectors $f_1 e_1 v_i$, $e_1 f_1 v_i$, $1 \leq i \leq s$, which proves our Claim.

Set v_i, $1 \leq i \leq s$ to be a basis satisfying this condition. Consider the left hand side of (i). Since $\lambda - l\alpha_1 = \lambda - \alpha_1 - (l-1)\alpha_1$, it follows from the above Claim that $l = 0$ and so $\mu = \lambda$ and c_λ is at most the number of basis vectors v_k for which $e_1 v_k = 0$. So, as $c_\lambda \neq 0$, there exists basis vectors v_k for which $e_1(v_k) = 0$. Let such basis vectors be labelled by indices $1 \leq k \leq r$. Let $J \subset I - S$ be the indexing set for simple root α_j such that $(\alpha_j, \alpha_1) = 0$. For all $j \in J$, $r_j(\rho) = \rho - \alpha_j$, $r_j(\alpha_1) = \alpha_1$ and $e_1 e_j v_k = e_j e_1 v_k = 0$ for all $1 \leq k \leq r$. Hence, if there is some index $j \in J$ such that $e_j v_k \neq 0$ and $e_j(v_k - v_i) \neq 0$ for all $1 \leq i, k \leq r$, then the left hand side of equation (i) forces $c_\lambda = 0$, contradicting assumption. Now, except for the case when the Lie superalgebra G is of type $D(2,1;\alpha)$ when $J=\emptyset$, $J=\{\alpha_i : i\geq 3\}$. Suppose that $J\neq\emptyset$. Therefore replacing v_k by $v_k - v_i$ if necessary, we may assume that there are basis vectors v_k with $1\leq k\leq r$ for which $e_3v_k=0$. Let the indexes $1\leq k\leq r_3$ label all such basis vectors v_k. Then, repeating this argument as many times as necessary, we may assume that there are basis vectors v_k for which

$$e_1v_k=0 \quad \text{and} \quad e_jv_k=0 \quad \forall j\in J. \tag{ii}$$

Let B the set of all vectors satisfying conditions (ii). Since the G-module $L(\Lambda)$ is irreducible, for all

$$u,v\in B, \quad e_2(u)\neq 0 \quad \text{and} \quad e_2(u-v)\neq 0.$$

(Note that when G is of type $D(2,1;\alpha)$, this may also be assumed). Set $e_{\alpha_1+\alpha_2}=[e_1,e_2]$.
Claim. Either, $e_{\alpha_1+\alpha_2}e_2=0$ or $e_{\alpha_1+\alpha_2}f_1v\neq 0$.
Suppose that

$$e_{\alpha_1+\alpha_2}e_2v\neq 0 \tag{iii}$$

and that

$$u=e_{\alpha_1+\alpha_2}f_1v=0. \tag{iv}$$

Equation (iv) can be re-written as

$$e_1e_2f_1v=e_2e_1f_1v=(\lambda,\alpha_1)e_2v.$$

Hence, by Lemma 3.2 (i),

$$0=(\lambda,\alpha_1)e_1e_2v.$$

So, as $(\lambda,\alpha_1)\neq 0$,

$$e_{\alpha_1+\alpha_2}v=0$$

since $e_1v=0$. By Lemma 2.3 (vii), $e_2e_{\alpha_1+\alpha_2}v=e_{\alpha_1+\alpha_2}e_2v$. Hence we get a contradiction to assumption (iii) which proves the Claim.

Suppose that $u,v\in B$ such that $e_{\alpha_1+\alpha_2}e_2v=0$ and $e_{\alpha_1+\alpha_2}e_2u\neq 0$. We show that

$$e_{\alpha_1+\alpha_2}f_1u\neq e_2v. \tag{v}$$

So suppose that inequality (v) does not hold. Then,

$$\begin{aligned} e_{\alpha_1+\alpha_2}v &= e_1e_2v \\ &= e_{\alpha_1+\alpha_2}e_1f_1u \\ &= e_{\alpha_1+\alpha_2}(\lambda,\alpha_1)u. \end{aligned}$$

Hence, as $(\lambda, \alpha_1) \neq 0$,

$$e_2 e_{\alpha_1+\alpha_2} u = 0,$$

contradicting assumption.

If $u, v \in B$ such that $e_{\alpha_1+\alpha_2} e_2 v \neq 0$ and $e_{\alpha_1+\alpha_2} e_2 u \neq 0$, then replace u by $u - v$. Hence, considering the left hand side of equality (i), we can conclude that $c_\lambda = 0$, which contradicts assumption. Hence, $(\lambda, \alpha_1) = 0$ holds, proving the result in Case 1.

Case 2. Assume next that

$$e_1 v_\mu = 0 = f_1 v_\mu.$$

for some non-trivial vector v_μ of weight μ.

Then,

$$0 = f_1 e_1 v_\mu = (\lambda, \alpha_1) v_\mu$$

and so $(\mu, \alpha_1) = 0$. Hence, from the definition of the weight λ, it follows that $(\lambda, \alpha_1) = 0$, which proves the result in Case 2. □

As a consequence of the last two results, we can now find more precise crucial information on the sum $\Lambda - \lambda$ for all weights λ for which $c_\lambda \neq 0$ or $c'_\lambda \neq 0$. We do this in couple of stages as this will be useful later for technical reasons.

Lemma 3.13. *Suppose that $\lambda \in H^*$ is a weight satisfying:*
(i) $\Lambda - \lambda$ *is a sum of positive odd roots,*
(ii) $(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho)$
(iii) $(\lambda + \rho, \mu) = 0$ *for some root* $\mu \in \Delta_1^+$ *of norm 0.*

Then, there is an element $w \in W_E$ and a root $\alpha \in \Delta_1^+$ orthogonal to $\Lambda + \rho$ of norm 0 such that $\lambda + \rho = w(\Lambda + \rho - l\alpha)$ for some non-negative integer l.

Proof. Let λ be a weight satisfying the conditions stated in the Lemma. For any non-negative integer s, set $\lambda_s = \lambda - s\mu$. Due to conditions (ii) and (iii),

$$(\lambda_s + \rho, \lambda_s + \rho) = (\Lambda + \rho, \Lambda + \rho). \qquad (a)$$

For any $w \in W_E$, $w(\lambda_s + \rho) \leq \Lambda + \rho$ follows from condition (i) and Lemma 2.3 (v). Hence, without loss of generality, we may assume that $(\lambda + \rho, \alpha_i) \geq 0$ for all $i > 1$. Furthermore, for each non-negative integer s, there is an element $w_s \in W_E$ such that the height of $\Lambda + \rho - w_s(\lambda_s + \rho)$ is minimal. Then, $(w_s(\lambda_s + \rho), \alpha_i) \geq 0$ for all $i > 1$.

Claim: For $s > 0$, $\lambda + \rho - w_s(\lambda_s + \rho)$ is a sum of positive odd roots.

Suppose that there is some $i \neq 1$ such that $(\lambda_s+\rho, \alpha_i) < 0$. Equivalently,

$$(\lambda + \rho, \alpha_i) < s(\mu, \alpha_i).$$

Hence our assumption implies that $(\mu, \alpha_i) > 0$ and $s(\mu-(\mu, \alpha_i)\alpha_i)+(\lambda+\rho)\alpha_i$ is therefore a sum of positive odd roots since for any integer $0 \leq t \leq (\mu, \alpha_i)$, $\mu - t\alpha_i$ is a root which is necessarily odd since so is μ whereas the simple root α_i is even. This proves our Claim.

Hence,

$$\lambda + \rho - w_s(\lambda_s + \rho) = s\alpha_1 + \gamma \qquad (b)$$

where $\gamma \in \mathbf{Z}_+\Delta_E$ and s can be taken as large as wanted. It follows that for s large enough,

$$(w_s(\lambda_s + \rho), \alpha_i) \geq 0 \quad \forall i \in I.$$

In particular, $(w_s(\lambda_s+\rho), w_s(\mu)) = 0$ implies that $(w_s(\lambda_s+\rho), \alpha_1) = 0$. So, equations (a) and (b) imply that

$$(\lambda + \rho, \lambda + \rho) = (\lambda + \rho - \gamma, \lambda + \rho - \gamma).$$

Since $\gamma \in \mathbf{Z}_+\Delta_E$, considering $w(\lambda + \rho)$ for $w \in W_E$ such that the height of $\lambda+\rho-w(\gamma+\rho)$ is as small as possible, we can conclude that $w(\Lambda+\rho-\gamma) = \Lambda + \rho$. As a result, equality (b) becomes

$$w_s(\lambda_s + \rho) = w(\lambda + \rho) - s\alpha_1. \qquad (c)$$

A similar equation holds with respect to Λ. Namely that for some non-negative integer $t \geq s$ and some element $w_1 \in W_E$,

$$w_s(\lambda_s + \rho) = w_1(\Lambda + \rho) - t\alpha_1. \qquad (d)$$

From (c) and (d), we immediately get

$$(\lambda+\rho) = w^{-1}w_1(\Lambda+\rho) - (t-s)w^{-1}(\alpha_1) = w^{-1}w_1(\Lambda+\rho-(t-s)w_1^{-1}(\alpha_1))$$

and as $w_1 \in W_E$, $w_1^{-1}(\alpha_1)$ is an odd positive root of norm 0. Moreover, from (a) it follows that the root $w_1^{-1}(\alpha_1)$ is orthogonal to $\Lambda+\rho$. This proves the result. $\square$

Further information on weights λ with $c_\lambda \neq 0$ or $c'_\lambda \neq 0$ now follows immediately from Proposition 3.12 and Lemma 3.13.

Corollary 3.14. *We keep the notation of Proposition 3.6. Suppose that λ is a weight such that $c_\lambda \neq 0$ or $c'_\lambda \neq 0$.*

(i) If there are no roots of negative norm or if $(\Lambda+\rho,\beta) < 0$, then there exists $w \in W$ such that $w(\Lambda+\rho) - (\lambda+\rho) = s\alpha$ for some positive odd root $\alpha \in \Delta_1^+$ of norm 0 orthogonal to $w(\Lambda+\rho)$ and some non-negative integer s.

(ii) If $(\Lambda+\rho,\beta) > 0$, then there exists $w \in W_E$ such that $w(\Lambda+\rho)-(\lambda+\rho) = s\alpha$ for some positive odd root $\alpha \in \Delta_1^+$ of norm 0 orthogonal to $w(\Lambda+\rho)$ and some non-negative integer s.

Proof. We only need to notice two facts: first that in the case when there are roots of negative norm and $(\Lambda+\rho,\beta) < 0$, then Lemma 3.13 can be applied to a weight $\lambda \in H^*$ such that $r_\beta(\Lambda+\rho) - (\lambda+\rho)$ is a sum of positive odd roots; second that when the finite dimensional module $L(\Lambda)$ is not irreducible, Lemma 3.13 still implies this result. □

Remark 3.15. Notice that the proof of Lemma 3.13 can be modified in a straightforward manner in order to derive Corollary 3.14 for the case when either there are no roots of negative norm or $(\Lambda+\rho,\beta) > 0$ from Proposition 3.11. This shows that when either of these conditions are fulfilled, the situation is much simpler and Proposition 3.12 is not needed.

It now remains to calculate the exact values of the coefficients c_λ. First, we need to notice that the degree of atypicality of the Lie superalgebra G is at most 1.

Lemma 3.16. *There is at most one positive odd root α of norm 0 orthogonal to $\Lambda+\rho$.*

Proof. Suppose that $(\Lambda+\rho,\alpha) = 0 = (\Lambda+\rho,\gamma)$ for $\alpha,\gamma \in \Delta_1^+$. Hence, $(\Lambda+\rho,\gamma-\alpha) = 0$ Without loss of generality, we may assume that $\gamma \geq \alpha$. As both α and γ are odd roots, the simple root α_1 appears exactly once in the expression of α or γ as a sum of simple roots. So, $\gamma-\alpha$ is then a positive sum of the simple roots α_i, $i \notin S$. Hence, as $(\Lambda+\rho,\alpha_i) \geq 0$ for all $i \notin S$, if $\gamma \neq \alpha$, then there is some $i \notin S$ for which $(\Lambda+\rho,\alpha_i) = 0$. However $(\rho,\alpha_i) > 0$ and as $L(\Lambda)$ is an integrable module, $(\Lambda,\alpha_i) \geq 0$ and thus, we get a contradiction. It follows that $\gamma = \alpha$. □

We are almost ready to give the character and super-character formulae for the G-module $L(\Lambda)$. We first remind the reader about the Casimir operator [K3,R1]. We need to fix some further notation. The Cartan matrix of G being symmetric and non-degenerate, it follows that there is a non-degenerate super-symmetric bilinear form – which we will write $(.,.)$ – on G (namely, the super-Killing form [K1]). So for each positive root $\alpha \in D^+$, we can set $e_\alpha \in G_\alpha$ and $f_\alpha \in G_{-\alpha}$ be such that

$$(e_\alpha, f_\alpha) = 1.$$

In particular, for $\alpha \in \Delta_1^+$ such that $\alpha = \alpha_1 + \gamma$ for some $\gamma \in \Delta_0^+$, we take

$$e_\alpha = [e_1, e_\gamma].$$

Lemma 3.17. *For all $\gamma \in \Delta_0^+$ such that $\gamma + \alpha_1 \in \Delta$, $[e_1, f_{\alpha_1+\gamma}] = -f_\gamma$*

Proof. Let $c \in \mathbf{C}$ such that $[e_1, f_{\alpha_1+\gamma}] = cf_\gamma$. By definition of the vector f_γ, $c = (e_\gamma,$

$$[e_1, f_{\alpha_1+\gamma}]) = -(e_{\alpha_1+\gamma}, f_{\alpha_1+\gamma}) = -1,$$

proving the result. □

The Casimir operator Ω is a linear operator acting on all H^*-graded G-modules in the following manner: If v is vector of weight λ, then by definition

$$\Omega(v) = ((\lambda + \rho, \lambda + \rho) - (\rho, \rho))v + \sum_{\alpha \in \Delta^+} f_\alpha e_\alpha v.$$

Furthermore, as $L(\Lambda)$ is a highest weight module, as shown in [R1]

$$\Omega(v) = ((\Lambda + \rho, \Lambda + \rho) - (\rho, \rho))v.$$

For reasons of clearness of presentation, we next split the proof of the main Theorem by first giving a result using properties of the Casimir operator.

Lemma 3.18. *We keep the notation of Proposition 3.6. Suppose that the G-module $L(\Lambda)$ is irreducible. Let $\lambda \in H^*$ be a weight such that $c_\lambda \neq 0$ or $c'_\lambda \neq 0$. Then, if there exits $w \in W$, $1 \leq i \leq m$ such that*

$$\lambda + \rho_0 \neq w(\Lambda_i + \rho_0),$$

then there is an odd root of negative norm, $\Lambda_i = 0$, and $\Lambda = (\rho, \beta)\beta$. In particular, $(\Lambda, \alpha_j) = 0$ for all $j \neq 1$.

Proof. Suppose the Lemma is false. So there is an index $1 \leq i \leq m$ such that

$$(\Lambda_i + \rho, \Lambda_i + \rho) = (\Lambda + \rho, \Lambda + \rho) \quad \text{and} \qquad (i)$$

and

$$\lambda + \rho_0 = w(\Lambda_i + \rho_0) \quad \text{for some} \quad w \in W. \qquad (i')$$

Furthermore, there is a vector $v \in L(\Lambda)$ of weight Λ_i satisfying

$$xv = 0 \quad \forall \quad x \in N_0^+. \tag{ii}$$

Hence, applying the Casimir operator Ω,

$$\Omega(v) = \sum_{\mu \in \Delta_1^+} f_\mu e_\mu v + ((\Lambda_i + \rho, \Lambda_i + \rho) - (\rho, \rho))v = ((\Lambda + \rho, \Lambda + \rho) - (\rho, \rho))v. \tag{iii}$$

Equations (i) and (iii) imply that

$$\sum_{\mu \in \Delta_1^+} f_\mu e_\mu v = 0. \tag{iv}$$

Hence, using the above definition of the vectors e_μ together with Lemma 3.1 and equality (ii), (iv) can be re-written as

$$f_1 e_1 v - \sum_{\substack{\gamma \in \Delta_0^+ \\ \alpha_1 + \gamma \in \Delta}} f_{\alpha_1 + \gamma} e_\gamma e_1 v + f_{2\beta - \alpha_1} e_2^2 e_1 v = 0. \tag{v}$$

Next, apply the operator e_1 to both sides of (v). By Lemma 3.17, this leads to:

$$(\Lambda_i, \alpha_1) e_1 v + \sum_{\gamma \in \Delta_+} f_\gamma e_\gamma e_1 v = 0$$

since when $\alpha_1 + \gamma \notin \Delta$, $f_\gamma e_\gamma e_1 v = f_\gamma e_1 e_\gamma v = 0$. In other words,

$$\Omega(e_1 v) = ((\Lambda_i + \alpha_1 + \rho, \Lambda_i + \alpha_1 + \rho) - (\rho, \rho) - (\Lambda_i, \alpha_1)) e_1 v.$$

Now, $e_1 v \neq 0$ for otherwise $xv = 0$ for all $x \in N_+$ and the G-module $L(\Lambda)$ would have a non-trivial submodule, contradicting the irreducibility of $L(\Lambda)$. Hence, the above mentioned property of the Casimir operator together with equality (i) force

$$(\Lambda_i + \alpha_1 + \rho, \Lambda_i + \alpha_1 + \rho) - (\Lambda_i, \alpha_1) = (\Lambda_i + \rho, \Lambda_i + \rho).$$

Thus,

$$2(\Lambda_i + \rho, \alpha_1) = (\Lambda_i, \alpha_1),$$

which in turn gives

$$(\Lambda_i, \alpha_1) = 0 \tag{vi}$$

since $(\rho, \alpha_1) = 0$. As Λ_i is a highest weight of the irreducible finite dimensional $G_{\bar{0}}$-module $V(\Lambda_i)$, $(\Lambda_i, \alpha_i) \geq 0$ for all $i > 1$ and $(\Lambda_i, \beta) \leq 0$. So from (vi), we can deduce that $(\Lambda_i, \beta) = 0$ and so $(\Lambda_i, \alpha_i) = 0$ for all $i \in I$. Therefore

$$\Lambda_i = 0.$$

Hence, Lemma 3.5 tells us that Λ is a sum of distinct positive odd roots. We next need to deal with various cases.

Case 1: There is an odd root β of negative norm.

Then, we may write $\Lambda = s\beta \pm \gamma$, where s is a positive integer, γ is a non-negative sum of the simple roots α_i, $i \neq 1$ and $0 \leq \gamma \leq 2(\beta - \alpha_1)$. Hence, as $(\Lambda, \alpha_i) \geq 0$ and $(\beta, \alpha_i) = 0$ for all $i \neq 1$ and there are no positive roots in Δ_E^+ such that $(\gamma, \alpha_i) \leq 0$ for all $i \neq 1$, $g \geq 0$ and either $\gamma = 0$, $\beta - \alpha_1$ or $2(\beta - \alpha_1)$. Furthermore because of (i), we must have

$$(\Lambda + 2\rho, \Lambda) = 0.$$

Equivalently,

$$-2s^2 + (\gamma, \gamma) + 2s(\rho, \beta) + (\rho, \gamma) = 0.$$

Suppose that $\gamma = \beta - \alpha_1$. Then,

$$-2s^2 + 2 + (2s + 1)(\rho, \beta) = 0.$$

However (ρ, β) being an odd integer, we get a contradiction. Next suppose that $\gamma = 2(\beta - \alpha_1)$. Then,

$$-s^2 + 4 + (s + 1)(\rho, \beta) = 0.$$

Again, s cannot be even since then $(s+1)(\rho, \beta)$ is odd and s cannot be odd, since then $4 + (s + 1)(\rho, \beta)$ is even. Hence,

$$\gamma = 0 \quad \text{and} \quad s = (\rho, \beta).$$

Since $(\beta, \alpha_j) = 0$ for all $j \neq 1$, the result follows is this case.

Case 2: There are no odd roots of negative norm, but there is an even root 2β of negative norm, i.e. G is of type $D(2, 1; \alpha)$.

In this case, write $\Lambda = s\alpha_1 + \gamma$, where s is a positive integer, γ is a non-negative sum of the simple roots α_i, $i \neq 1$. Since $(\alpha_1, \alpha_i) = -(\rho, \alpha_i)$ for all $i \in I$, we get

$$(1 - 2s)(\rho, \gamma) = 0.$$

Since s is an integer, it follows that $(\rho, \gamma) = 0$. In other words, $\gamma = 0$ and so $\Lambda = \alpha_1$. However this implies that $(\Lambda, \alpha_i) < 0$ for all $i \neq 1$, contradicting the integrability of the module $L(\Lambda)$.

Case 3: There are no roots of negative norm, i.e. G is of type $A(m, 0)$.

In this case, write $\Lambda = \alpha_1$ or $\Lambda = s\alpha_1 + \sum_{j=1}^{s} \gamma_j$, where s is a positive integer and each γ_j is a positive even root and furthermore $(\gamma_j, \gamma_k) = 1$ for all $j \neq k$ and $(\alpha_1, \gamma_j) = -1$. However

$$s(s-1) + (\rho, \sum_{j=1}^{s} \gamma_j) = 0$$

is not possible as both summands are non-negative and the second one is positive. Hence, $\Lambda = \alpha_1$, in which case $(\Lambda, \alpha_2) < 0$, again giving a contradiction and proving the Lemma. □

We are now ready to prove the character and super-character formulae.

Theorem 3.19. *Let $L(\Lambda)$ be a finite dimensional highest weight G-module of highest weight Λ. Set*

$$R = e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha))^{-1}$$

and

$$R' = e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))^{-1}$$

(i) Suppose that for all positive odd roots α of norm 0, $(\Lambda + \rho, \alpha) \neq 0$. Then,

$$R \operatorname{ch} L(\Lambda) = \sum_{w \in W} \epsilon(w) e(w(\Lambda + \rho))$$

and

$$R' \operatorname{sch} L(\Lambda) = \sum_{w \in W} \epsilon(w) e(w(\Lambda + \rho))$$

(ii) Suppose that α is a positive odd root such that $(\Lambda + \rho, \alpha) = 0$. If $(\Lambda + \rho, \beta) > 0$, then

$$R \operatorname{ch} L(\Lambda) = \sum_{w \in W_E} \epsilon(w)(-1)^l e(w(\Lambda + \rho - l\alpha))$$

and

$$R' \operatorname{sch} L(\Lambda) = \sum_{w \in W_E} \epsilon(w)(-1)^l e(w(\Lambda + \rho - l\alpha)).$$

If $(\Lambda+\rho,\beta)<0$, *then*

$$R\,\mathrm{ch}\,L(\Lambda)=\sum_{w\in W_E}\epsilon(w)(-1)^l(e(w(\Lambda+\rho-l\alpha))-e(w(r_\beta(\Lambda+\rho+l\alpha)))$$

and

$$R'\,\mathrm{sch}\,L(\Lambda)=\sum_{w\in W_E}\epsilon(w)(-1)^l(e(w(\Lambda+\rho-l\alpha))-e(w(r_\beta(\Lambda+\rho+l\alpha))).$$

Proof. Part (i) is a direct consequence of Proposition 3.7 and Corollary 3.14. So we only need to prove Part (ii). The proof for the character and the super-character formulas being the same, we only prove the former. Suppose first the module $L(\Lambda)$ is irreducible. Let the root $\alpha\in\Delta_1^+$ of norm 0 satisfy $(\Lambda+\rho,\alpha)=0$. By Lemma 3.8 and Corollary 3.14, we only need to show that

$$c_{\Lambda-l\alpha}=(-1)^l \qquad (i)$$

and that

$$c_{r_\beta(\Lambda+\rho)-\rho-l\alpha}=-(-1)^l \qquad (ii)$$

when there are roots of negative norm and $(\Lambda+\rho,\beta)<0$. We prove equality (i) as (ii) will follow from similar arguments. We do this by induction of l. This is clearly true for $l=0$. So suppose this holds for l. Let us consider $\lambda=\Lambda-(l+1)\alpha$.

Suppose first that the weight λ can also be written as follows:

$$\lambda+\rho=w(\Lambda+\rho-s\alpha)-\delta, \qquad (iii)$$

where $w\in W$ and δ is a sum of distinct positive odd roots or $\delta=0$. Furthermore, we may assume that $w\in W_E$ for otherwise c_λ will follow from equality (ii). From Lemmas 3.13 and 3.16 we get that $\Lambda+\rho-w^{-1}(\delta)=w_1(\Lambda+\rho-t\alpha)$ for some non-negative integer t. Hence, as δ is a sum of distinct positive odd roots, $t=1$. Hence equation (iii) becomes

$$\Lambda+\rho-(l+1)\alpha=w(w_1(\Lambda+\rho-\alpha)-s\alpha).$$

Since the left hand side has the same norm as $\Lambda+\rho$, by Lemma 3.16, we can deduce that $w_1=1$. This in turn forces $s=l+1$ and so

$$w(\lambda+\rho)=\lambda+\rho.$$

Let $r\in W_E$ be such that $(r(\lambda+\rho),\alpha_i)\geq 0$ for all $i>1$. Therefore, rw is the product of reflections r_i leaving $r(\lambda+\rho)$ invariant [K4, proposition

3.12.a]. The subgroup W_λ of W_E generated by these reflections is either trivial or has even order. Suppose it is not trivial. Then by Lemma 3.8, $\epsilon(r_i)c_{r(\lambda+\rho)-\rho} = c_{r(\lambda+\rho)-\rho}$ and so as $\epsilon(r_i) = -1$, $c_\lambda = 0$. However, the subgroup W_λ having even order,

$$\sum_{w\in W_\lambda} \epsilon(w)\epsilon(r)e^{rw(\lambda+\rho)} = 0.$$

As a result we may take any values for c_λ, in particular, we may set $c_{\Lambda-(l+1)\alpha} = (-1)^{l+1}$ as wanted.

Suppose next that if the weight λ satisfies an equation of type (iii), then $w = 1$ and $\delta = 0$. Hence there is some $i \neq 1$ such that $(\Lambda, \alpha_i) > 0$ since there is some $i \neq 1$ for which $(\alpha, \alpha_i) \geq 0$ and $\alpha - \alpha_i \in \Delta_1^+$. Hence, as a result of Lemma 3.17, equality (2) implies that

$$c_\lambda + \sum_\mu c_{\lambda+\mu} = 0, \qquad (iv)$$

where the sum is taken over all distinct sums μ of positive odd roots. Equality (iv) implies that $c_{\Lambda-(l+1)\alpha} = (-1)^{l+1}$. This proves the Theorem when the module $L(\Lambda)$ is irreducible.

Suppose next that the module $L(\Lambda)$ is finite dimensional but not necessarily irreducible. Then there may be non-trivial vectors in $L(\Lambda)$ which are not highest weight vectors such that $e_i v = 0$ for all $i \in I$. In this case the Theorem in the irreducible case together with Corollary 3.14 give the character formula. □

In particular this Theorem answers a question asked in [K2] about atypical finite dimensional representations of exceptional Lie superalgebras and thus improves [K2, Theorem 2]

Corollary 3.20. *The condition $(\Lambda + \rho, \alpha) \neq 0$ for $\alpha \in \Delta_1^+$ is sufficient for the finite dimensional highest weight module $L(\Lambda)$ to be typical in the case of the exceptional Lie superalgebras $D(2, 1; \alpha)$, $F(4)$ and $G(3)$*

Acknowledgements The author is most grateful to the staff of the CRM and in particular to its director, Manuel Castellet, for their extremely kind hospitality. She would also like to thank Vera Serganova for useful discussions.

REFERENCES

[HM1]. J.P. Hurni, B. Morel, Irreducible representations of superalgebras of type II, *J. Math. Phys.* **23** (1982), 2296

[HM2]. J.P. Hurni, B. Morel, Irreducible representations of $su(m|n)$, *J. Math. Phys.* **24** (1983), 157

[K1]. V.G. Kac, Lie Superalgebras, *Advances in Mathematics* **26**, (1977) 8–96

[K2]. V. G. Kac, Characters of typical representations of classical Lie superalgebras, *Communications in Algebra* **5** (8), (1977) 889–897

[K3]. V.G. Kac, Infinite-Dimensional Algebras, Dedekind's ν-Function, Classical Möbius Function and the Very Strange Formula, *Advances in Mathematics* **30**, (1978) 85–136

[K4]. V.G. Kac, "Infinite dimensional Lie algebras", third ed., Cambridge University Press 1990

[M]. M. Marcu The representations of $spl(2|1)$: an example of representations of basic superalgebras, *J. Math. Phys.* **21** (1980), 1277

[MSS]. B. Morel, A.Sciarrino, P. Sorba, Representations of $osp(M|2n)$ and Young supertableaux, *J. Phys. A Math. Gen.* **18** (1985), 1597

[NRS]. W. Nahm, V. Rittenberg, M. Scheunert, Irreducible representations of the $osp(1,2)$ and $spl(1,2)$ graded Lie algebras, *J. Math. Phys.* **18** (1977), 155

[R1]. U. Ray, A character formula for generalized Kac-Moody superalgebras, *Journal of Algebra* **177** (1995), 154–163

[R2]. U. Ray, A characterization theorem for a certain class of graded Lie superalgebras, *Journal of Algebra* **229** (2000), 405–434

[R3]. U. Ray, The super-character formula for integrable irreducible infinite dimensional modules of generalized Kac-Moody superalgebras, preprint

[S1]. V. Serganova, A reduction method for atypical representations of classical Lie superalgebras, *adv. Math.* **180** (2003) (no1), 248–274

[S2]. V. Serganova, Characters of irreducible representations of simple Lie superalgebras, *Proceedings of the International Congress of Mathematicians* (Vol II), Berlin 1998,

[V]. J. Van der Jeugt, Irreducible representations of the exceptional Lie superalgebras $D(2,1;\alpha)$ *J. Math. Phys.* **26** (1985), 913

[VHKT]. J. Van der Jeugt, J.W.B. Hughes, R.C. King, J. Thierry-Mieg, A character formula for singly atypical modules of the Lie superalgebras $sl(m,n)$. *J. Math. Phys.* **31** (1990), 2278

CRM, Apartat 50, 08193 Bellaterra, Spain