MATHEMATICS VERSUS METAMATHEMATICS IN RAMSEY THEORY OF THE REAL NUMBERS

CARLOS AUGUSTO DI PRISCO

Abstract. The study of partition properties of the set of real numbers in several of its different presentations has been a very active field of research with interesting and sometimes surprising results. These partition properties are of the following form: if the set of real numbers (or a related space) is partitioned into a finite collection of pieces, there is a "large" collection of reals contained in one of the pieces. Different notions of largeness have been considered, and they give rise to properties of varied combinatorial character. The interplay between metamathematical questions and combinatorial problems has been present throughout the development of the theory. Most of these properties are, in their full generality, inconsistent with the axiom of choice, but versions of them where only partitions into simple pieces are considered, for example, Borel pieces, can be proved to be true. Nevertheless, the unrestricted versions are consistent with weak forms of the axiom of choice. We present here an overview of results about some of these partition properties, some old, some recent, and we mention several open problems.

1. Introduction

The theory of partitions is an area of combinatorial set theory originated in the 1930's with F. P Ramsey's famous theorem [34]. Some earlier results of Schur and Van der Waerden have some of the same combinatorial flavor, but it was Ramsey's theorem which attracted wide interest in partitions (see [15]). In the 1950's Erdös and Rado gave shape to the theory and extended Ramsey's result in several different directions with their partition calculus [12].

Ramsey's Theorem asserts that given positive integers n and k, for every partition of $\mathbb{N}^{[n]} = \{a \subseteq \mathbb{N} : |a| = n\}$ into k many pieces, there is an infinite $H \subseteq \mathbb{N}$ such that $H^{[n]}$, the collection of all of its n-element subsets, is contained in one piece. Such a set H is said to be homogeneous for the

This article was written while the author was a Visiting Researcher of Institució Catalana de Recerca i Estudis Avançats (ICREA) at the Centre de Recerca Matemàtica (CRM), Barcelona.

partition. A finitary version of Ramsey's theorem is stated as follows, given positive integers n, k, m, there is a positive integer N such that for every set A of N elements and every partition of $A^{[n]} = \{a \subseteq A : |a| = n\}$ into k many pieces, there is a subset $H \subseteq A$ with $|H| = m$, for which $H^{[n]}$ is contained in one of the pieces (i.e. H is homogeneous for the partition).

Ramsey proved his theorem as a technical tool to answer a question about decidability of first order logic [34]. This technical result turned out to be extremely interesting in itself, many different applications have been found, and the multiple ways it has been generalized or adapted to other contexts constitute a rich theory with applications in other areas of mathematics. The books [15, 31] provide a good sample of the degree of development of Ramsey Theory. Several other developments in this area have also started from metamathematical considerations. In these other cases as well, the solution of a metamathematical problem led to the development of concepts and theories of a purely combinatorial character. On the other hand, some natural questions about partitions have required a metamathematical analysis, for example to establish consistency results.

The obvious infinite dimensional generalization of Ramsey's theorem to partitions of the set $\mathbb{N}^{[\infty]}$ of infinite subsets of $\mathbb{N}$, instead of the collection of subsets of a specified finite size, is false. Nevertheless, partitions of $\mathbb{N}^{[\infty]}$ into a finite number of "topologically simple" pieces always admit an infinite homogeneous set, i.e. an infinite set $H \subseteq \mathbb{N}$ such that $H^{[\infty]} = \{A \subseteq H : A \text{ infinite}\}$ is contained in one of the pieces.

Since we can identify subsets of $\mathbb{N}$ with their characteristic functions, the set $\mathbb{N}^{[\infty]}$ corresponds to a subset of the Cantor space $2^{\mathbb{N}}$, which, with the inherited topology, is homeomorphic to the set of irrational numbers, and also to the Baire space, the set $\mathbb{N}^{\infty}$ of all infinite sequences of natural numbers endowed with the product topology. In this sense, we are dealing with partitions of the set of real numbers, considering these numbers represented by infinite subsets of $\mathbb{N}$, or infinite sequences of natural numbers.

Partition properties are frequently stated in terms of colorings. A k-coloring of a set S is simply a function $c : S \to K$ where K is a set of size k. Clearly, every k-coloring of S determines a partition of S into k pieces. For example, Ramsey's Theorem states that for every $n, k \in \mathbb{N}$ and every k-coloring of $\mathbb{N}^{[n]}$, there is $H \in \mathbb{N}^{[\infty]}$ such that $H^{[n]}$ is monochromatic.

We will consider partitions (or colorings) of $\mathbb{N}^{[\infty]}$ the set of infinite subsets of $\mathbb{N}$, and partitions of infinite products of various structures, finite or infinite, and the existence of different types of monochromatic sets for them. Different types of monochromatic sets usually give rise to corresponding partition properties of different strengths. We will be interested in the interrelationship between them, looking both into metamathematical and purely combinatorial aspects.

We mention some open questions, some quite old, like the questions about the necessity of inaccessible cardinals for the consistency of the Ramsey property $\omega \to (\omega)^\omega$, or if this partition property follows from the axiom of determinacy AD; and some more recent questions, perhaps easier to answer.

The notation used is standard. $\mathbb{N}$ is the set of natural numbers, which is identified with ω, the first infinite ordinal. $\mathbb{N}^\infty$ is the set of infinite sequences of natural numbers; the topological space obtained giving it the product topology (obtained from $\mathbb{N}$ with the discrete topology) is called the Baire space. $\mathbb{N}^{<\infty} = \bigcup_{n=0}^\infty \mathbb{N}^n$ is the set of finite sequences of natural numbers. Given a set A and $n \in \mathbb{N}$, $A^{[n]}$ is the collection of those subsets of A which have exactly n elements. The collection of finite subsets of $\mathbb{N}$ is denoted by $\mathbb{N}^{[<\infty]}$. For every infinite $A \subseteq \mathbb{N}$, we use $A^{[\infty]}$ to denote the collection of infinite subsets of A, accordingly, $\mathbb{N}^{[\infty]}$ is the set of infinite subsets of $\mathbb{N}$. If $A \in \mathbb{N}^{[\infty]}$ and $a \in \mathbb{N}^{[<\infty]}$, then $A/a = \{n \in A : \max a < n\}$.

We will use the letters $n, m, k, l, \ldots$ to denote natural numbers, and $A, B, C, H, X, Y, \ldots$ to denote infinite sets of natural numbers. The letters $a, b, c, \ldots$ will be used to denote finite sets of natural numbers, and $s, t, r, \ldots$ to denote finite sequences.

The author would like to thank Stevo Todorcevic for suggestions and comments which helped to improve this article.

2. The Ramsey property

The partition symbol
$$\omega \to (\omega)^\omega,$$
stands for the statement

"For every coloring $c : \mathbb{N}^{[\infty]} \to \{0, 1\}$ there is an infinite $H \subseteq \mathbb{N}$ such that $H^{[\infty]}$ is monochromatic".

If we restrict ourselves to partitions
$$c : \mathbb{N}^{[\infty]} \to \{0, 1\}$$
measurable with respect to a certain σ-field $\mathcal{C}$ of subsets of $\mathbb{N}^{[\infty]}$, we use the notation
$$\omega \to_{\mathcal{C}} (\omega)^\omega,$$
for the corresponding property.

As mentioned in the introduction, using the axiom of choice a partition of $\mathbb{N}^{[\infty]}$ can be given for which there is no infinite homogeneous set. Consider, for example, the equivalence relation defined on $\mathbb{N}^{[\infty]}$ by $A \sim B$ if and only if $A \Delta B$, the symmetric difference of A and B, is finite. Pick one element from each equivalence class, and define a partition of $\mathbb{N}^{[\infty]} = \mathcal{A} \cup \mathcal{B}$ as follows. Put A in $\mathcal{A}$ if and only if it differs from the chosen representative of its class in a (finite) set of even cardinality. Clearly, no set of the form $H^{[\infty]} = \{Y \subseteq H : Y \text{ infinite}\}$ is included in one of the pieces, as for every $Y \in \mathbb{N}^{[\infty]}$, Y and $Y \setminus \{\min Y\}$ lie in different pieces [1].

The non constructive character of counterexamples like this one, or the one given by Erdös and Rado in [11], suggests asking if such a counterexample can be given without using the axiom of choice. This question was in fact posed in a seminar on Ramsey theory conducted by D. Scott at Stanford University in 1967. Mathias recalls[2] that shortly after, several people, including P. Cohen, A. Ehrenfeucht and F. Galvin, had shown that for every open subset $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ there is an infinite set $A \in \mathbb{N}^{[\infty]}$, such that $A^{[\infty]} \subseteq \mathcal{O}$ or $A^{[\infty]} \cap \mathcal{O} = \emptyset$, and thus a partition into an open set and its complement cannot be a counterexample.

Nash-Williams [30] proved a result that implies that if $\mathbb{N}^{[\infty]}$ is partitioned into a finite number of sets which are simultaneously open and closed, there is $H \in \mathbb{N}^{[\infty]}$ such that $H^{[\infty]}$ is contained in one of the pieces. In [13], Galvin extends Nash-Williams' result to partitions into an open set and its complement, a step towards the proof of the following theorem of [14] from which follows that infinite homogeneous sets exist for partitions of $\mathbb{N}^{[\infty]}$ into a finite number of Borel subsets.

Theorem 1. *(Galvin and Prikry* [14]*) For every Borel-measurable 2-coloring of $\mathbb{N}^{[\infty]}$ there is a set $H \in \mathbb{N}^{[\infty]}$ such that $H^{[\infty]}$ is monochromatic. In symbols,*

$$\omega \rightarrow_{\mathcal{B}orel} (\omega)^{\omega}.$$

Note that this theorem can be stated as a property of Borel subsets of $\mathbb{N}^{[\infty]}$, namely, for every Borel set $\mathcal{A} \subseteq \mathbb{N}^{[\infty]}$ there is an infinite set $H \subseteq \mathbb{N}$ such that $H^{[\infty]} \subseteq \mathcal{A}$ or $H^{[\infty]} \cap \mathcal{A} = \emptyset$.

Consider the sets of the form

$$[a, B] = \{X : a \subseteq X \subseteq a \cup B\},$$

with $a \in \mathbb{N}^{[<\infty]}, B \in (\mathbb{N}/a)^{[\infty]}$.

Definition 1. *A set $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is Ramsey if for every $[a, A]$ there is $B \in A^{[\infty]}$ such that*

$$[a, B] \subseteq \mathcal{X} \text{ or } [a, B] \cap \mathcal{X} = \emptyset.$$

Galvin and Prikry actually showed in [14] that every Borel subset of $\mathbb{N}^{[\infty]}$ is Ramsey (they used "completely Ramsey" to name this property), from where Theorem 1 follows applying the definition of Ramsey to $[\emptyset, \mathbb{N}]$.

---

[1]This example is due to A. R. D. Mathias. In his thesis it appears in a slightly different presentation which requires only the use of the axiom of choice for pairs

[2]In a communication presented at the CRM, Barcelona, January of 2004.

Silver ([36]) extended this result proving that analytic sets are Ramsey. Recall that a subset of $\mathbb{N}^{[\infty]}$ is analytic if it is the image of $\mathbb{N}^{[\infty]}$ by a continuous function from $\mathbb{N}^{[\infty]}$ to itself. Silver's proof uses metamathematical methods; Ellentuck ([10]) gave a topological proof of Silver's result using the topology on $\mathbb{N}^{[\infty]}$ generated by the basic sets $[a, A]$. This topology, frequently called Ellentuck's Topology, refines the product topology, and it characterizes the Ramsey sets as follows.

Theorem 2. *(Ellentuck)*[10] *A subset of* $\mathbb{N}^{[\infty]}$ *is Ramsey if and only if it has the Baire property (with respect to Ellentuck's topology).*

The σ-field of Ramsey sets is closed under Souslin's operations, and therefore contains the analytic sets.

The Ramsey property can be viewed as another regularity property of subsets of $\mathbb{N}^{[\infty]}$, as Lebesgue measurability, the Baire property, or the perfect subset property. The existence of non-measurable sets of reals, of sets without the property of Baire, and of uncountable sets which do not contain perfect subsets are consequences of the axiom of choice, just as the existence of non-Ramsey sets. Solovay [37] constructed a model of set theory where every set of real numbers is Lebesgue measurable, has the property of Baire and, if uncountable, contains a perfect subset. Obviously, the axiom of choice does not hold in this model, but only a weak version of this axiom called "Axiom of Dependent Choices" (DC). The construction of the model relies on the assumption of the existence of an inaccessible cardinal (see [20, 21]).

Mathias [28] showed that in Solovay's model all subsets of $\mathbb{N}^{[\infty]}$ are Ramsey. Therefore, the partition relation $\omega \to (\omega)^\omega$ is consistent in the following precise sense.

Theorem 3. *(Mathias)*[28]

$$ZF + DC + \omega \to (\omega)^\omega$$

is consistent provided

$$ZFC + \text{ “There exists an inaccessible cardinal”}$$

is consistent.

Question 1. *One of the problems of the theory, which has remained open for several decades, is whether the hypothesis about inaccessible cardinals is necessary in Theorem 3.*

Shelah has shown that this hypothesis is necessary for the consistency of "all sets of reals are Lebesgue measurable", but not for the consistency of "all sets of real numbers have the property of Baire". It was known earlier that the hypothesis is necessary for the consistency of "every uncountable set of real numbers contains a perfect subset".

An argument of [27] shows that the existence of a non-principal ultrafilter on $\mathbb{N}$, which is a consequence of the axiom of choice, also provides a counterexample for the partition relation $\omega \to (\omega)^\omega$:

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Every set $X \in \mathbb{N}^{[\infty]}$ determines an infinite collection of consecutive intervals of $\mathbb{N}$ as follows: if $X = \{x_0, x_1, \dots\}$ is the increasing enumeration of X, let for every $n \in \mathbb{N}$, $I_n = [x_n, x_{n+1}) = \{k : x_n \leq k < x_{n+1}\}$. Define a partition $\mathbb{N}^{[\infty]} = \mathcal{A} \cup \mathcal{B}$ putting $X \in \mathcal{A}$ if and only if $\bigcup_{n\in\mathbb{N}}[x_{2n}, x_{2n+1}) \in \mathcal{U}$. For no $H \in \mathbb{N}^{[\infty]}$ the set $H^{[\infty]}$ is homogeneous, because $X \in \mathcal{A}$ if and only if $X \setminus \{\min X\} \in \mathcal{B}$. This is so because if Y is obtained removing from X its first element, then $y_n = x_{n+1}$, so $\bigcup_i [y_{2i}, y_{2i+1})$ differs from the complement of $\bigcup_i [x_{2i}, x_{2i+1})$ only on a finite set, so exactly one of those two unions belong to the ultrafilter $\mathcal{U}$. The fact that the existence of non-principal ultrafilters on $\mathbb{N}$ implies the negation of $\omega \to (\omega)^\omega$ will be used below.

3. Perfect set properties

As we have seen, the Ramsey property concerns the existence of monochromatic sets of the form $H^{[\infty]}$, with $H \in \mathbb{N}^{[\infty]}$, for finite partitions of $\mathbb{N}^{[\infty]}$. We will now consider a weaker property, stated in terms of the existence of monochromatic perfect sets.

The symbol

$$\omega \to (\text{perfect})^\omega$$

stands for the statement

"for every coloring $c : \mathbb{N}^{[\infty]} \to \{0, 1\}$ there is a perfect monochromatic subset of $\mathbb{N}^{[\infty]}$".

It is well known that $\mathbb{N}^{[\infty]}$ can be partitioned in two pieces neither of which contain a perfect set [1]. Nevertheless, every Borel-measurable coloring of $c : \mathbb{N}^{[\infty]} \to \{0, 1\}$ there is a monochromatic perfect set. This is expressed by the partition symbol

$$\omega \to_{\mathcal{B}orel} (\text{perfect})^\omega.$$

Notice that $\omega \to (\omega)^\omega$ implies $\omega \to (\text{perfect})^\omega$, since sets of the form $H^{[\infty]}$ are perfect. We saw in the previous section that the existence of a non principal ultrafilter on $\mathbb{N}$ gives a counterexample to $\omega \to (\omega)^\omega$. On the other hand,

$\omega \to (\text{perfect})^\omega$ is consistent with $ZF + DC +$ "there is an ultrafilter on $\mathbb{N}$ (see [3]), and therefore it is strictly weaker than $\omega \to (\omega)^\omega$ even in terms of consistency.

Question 2. *Is there a choice principle equivalent to the negation of $\omega \to$ (perfect)$^\omega$?*

Halpern and Läuchli proved in [17] a deep and powerful partition property about perfect trees. We need some definitions before stating the theorem.

A tree is a partially ordered set $(T, \prec)$ such that for every $u \in T$, the set $\{v : v \prec u\}$ is well ordered by $\prec$. The order type of $\{v : v \prec u\}$ is called the height of u in the tree. We consider trees in which every element has finite height. The height of the tree is the supremum of the heights of its elements. For every $n \in \mathbb{N}$, $T(n)$ is the collection of elements of T of height n, or the n-th level of the tree. A tree is perfect if for every $u \in T$, there are $v, w \in T$ such that $u \prec v$, $u \prec w$, with v and w incomparable with respect to the order $\prec$. If $A \subseteq \mathbb{N}$, $T \restriction A$ is the collection $\{u \in T : \text{height}(u) \in A\}$.

If $d \in \mathbb{N}$ and T_i is a tree for each $i < d$, then $\otimes_{i<d} T_i$ is the tree

$$\{(t_0, \dots, t_{d-1}) \in \prod_{i<d} T_i : \text{height}(t_0) = \cdots = \text{height}(t_{d-1})\}$$

with the ordering $(t_0, \dots, t_{d-1}) \prec (t'_0, \dots, t'_{d-1})$ if for every $i < d$ $t_i \prec_i t'_i$ (where $\prec_i$ is the ordering of the tree T_i).

Theorem 4. *([17]) Let $d \in \mathbb{N}$, and for every $i < d$ let T_i be a perfect tree of height ω. For every*

$$c : \otimes_{i<d} T_i \to \{0, 1\},$$

there is $A \in \mathbb{N}^{[\infty]}$, and a perfect subtree $R_i \subseteq T_i$ for every $i < d$ such that c is constant on $\otimes_{i<d}(R_i \restriction A)$.

This result was proved to solve a question regarding the Axiom of Choice and one of its consequences, the Boolean prime ideal theorem, which says that there is a prime ideal in every Boolean algebra. The Halpern-Laüchli theorem was obtained in order to construct a model of set theory where the Boolean prime ideal theorem holds but not the axiom of choice (see [18]). This is, then, another example of a deep combinatorial principle obtained to answer a metamathematical question.

Laver in [22] extended the Halpern-Laüchli theorem to infinite products of perfect trees.

4. Monochromatic sublattices of $\mathcal{P}(\mathbb{N})$

We consider now a partition property defined in terms of a different type of homogeneity for partitions of $\mathbb{N}^{[\infty]}$. Instead of requiring for every coloring

$c : \mathbb{N}^{[\infty]} \to \{0,1\}$ the existence of an infinite set B with all of its infinite subsets of the same color, we only require that all subsets of B containing a fixed subset A have the same color.

The partition symbol

$$\omega \to ((\omega))^\omega$$

means that for every coloring $c : \mathbb{N}^{[\infty]} \to \{0,1\}$ there is a pair (A, B) with $A \subseteq B \in \mathbb{N}^{[\infty]}$ such that

$$[A, B] = \{X : A \subseteq X \subseteq B\}$$

is monochromatic (we put no requirements on the size of A). This type of homogeneous sets were studied in [6]. Partition properties of certain classes of lattices has been considered previously from various points of view, see, for example, [33].

Since every such sublattice $[A, B]$ is a perfect subset of $\mathbb{N}^{[\infty]}$, it follows immediately from the definitions that

$$\omega \to (\omega)^\omega \text{ implies } \omega \to ((\omega))^\omega \text{ implies } \omega \to (\text{perfect})^\omega.$$

Using the same argument presented at the end of Section 2 for the property $\omega \to (\omega)^\omega$, it can be shown that a non-principal ultrafilter on $\mathbb{N}$ provides a counterexample for the property $\omega \to ((\omega))^\omega$, and so, the second implication is strict. Although the exact relation between $\omega \to (\omega)^\omega$ and $\omega \to ((\omega))^\omega$ is still unknown, most likely the first implication is also strict, since the consistency of $(ZF + DC + \omega \to ((\omega))^\omega)$ follows just from the consistency of ZFC, with no hypothesis involving inaccessible cardinals. This is so since $\omega \to_{Baire} ((\omega))^\omega$ holds (see [6]), and the consistency of "every subset of $\mathbb{N}^\infty$ has the Baire property" has been established by Shelah assuming just the consistency of ZFC [35].

J. Brendle, L. Halbeisen and B. Lowe have studied the property $\omega \to ((\omega))^\omega$ restricted to $\mathbf{\Sigma}^1_2$ sets, $\mathbf{\Delta}^1_2$ sets, and projective sets in general (see, for example, [2], and [16], where it is shown that $\omega \to_{\mathbf{\Sigma}^1_2} ((\omega))^\omega$ does not imply $\omega \to_{\mathbf{\Sigma}^1_2} (\omega)^\omega$).

Question 3. *What is the exact relationship between the properties $\omega \to (\omega)^\omega$ and $\omega \to ((\omega))^\omega$?*

5. Polarized partitions

Dealing with partitions of the space $\mathbb{N}^{[\infty]}$, we have considered several types of homogeneous sets. A different form is obtained as follows. Given a finite coloring of $\mathbb{N}^{[\infty]}$, we require the existence of a sequence of finite sets $\{H_i\}_{i=0}^\infty$ of certain required cardinalities such that for every i, $\max(H_i) < \min(H_{i+1})$, and such that the collection $\{X \in \mathbb{N}^{[\infty]} : \forall i\, |X \cap H_i| = 1\}$ is monochromatic. A convenient way to treat this is to consider partitions of the Baire space $\mathbb{N}^\infty = \mathbb{N} \times \mathbb{N} \times \dots$, the set of infinite sequences of natural numbers with the product topology obtained when $\mathbb{N}$ is considered as a discrete space.

Given

$$c : \mathbb{N}^\infty \to \{1, 2\},$$

we want a monochromatic product $\prod_{i=0}^{\infty} H_i$, with $H_i \subseteq \mathbb{N}$ of some specified size for every i. ([4]).

We use the partition symbol

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

to express that for every coloring $c : \mathbb{N}^\infty \to 2$, there is a monochromatic product $\prod_{i=0}^{\infty} H_i$ with $H_i \subseteq \mathbb{N}$ and $|H_i| = m_i$ for every i.

Notice that here we do not require the sequence $\{H_i\}_{i=0}^{\infty}$ to be increasing, but as shown in [4] this is not essential.

It should be clear that we cannot require H_0 and H_1 to be both infinite: let $c : \mathbb{N}^\infty \to \{0, 1\}$ be defined by $c(x) = 0$ if and only if $x(0) < x(1)$. Clearly, a product $\prod_{i=0}^{\infty} H_i$ with $H_i \subseteq \mathbb{N}$ and with H_0 and H_1 both infinite cannot be monochromatic for c. In the same fashion a counterexample can be given if we require infinite sets in any two specified (fixed) coordinates. If no coordinates are specified, it is possible to get for every $c : \mathbb{N}^\infty \to \{0, 1\}$ a monochromatic product $\prod_{i=0}^{\infty} H_i$ with at least two infinite factors. The position of the infinite factors depending on the partition. This follows from results of G. Moran and D. Strauss [29]. They prove that for every $k \in \mathbb{N}$ and every partition of $\mathbb{N}^\infty$ into two pieces, there is a monochromatic product $\prod_{i=0}^{\infty} H_i$ with k factors that are infinite (in fact, each of them is the whole set $\mathbb{N}$) and the rest of the factors are singletons. It is not known if we can also require the finite factors to be non-trivial, having at least two elements each.

Question 4. *Is the following statement consistent? (We mean consistent with ZF, provided that ZF is consistent). For every partition $c : \mathbb{N}^\infty \to \{1, 2\}$, there is a homogeneous product $\prod_{i=0}^{\infty} H_i$ such that for every i, $|H_i| > 1$ and there are $m, n \in \mathbb{N}$, $m \neq n$ with $|H_m| = |H_n| = \aleph_0$. (see* [6]*).*

Henle proved in [19] that $\omega \to (\omega)^\omega$ implies that every such partition admits a homogeneous product $\prod_{i=0}^{\infty} H_i$ such that for every i, $|H_i| > 1$ and H_0 is infinite.

Identifying infinite subsets of $\mathbb{N}$ with their increasing enumeration, it is easy to verify that $\omega \to (\omega)^\omega$ implies $\begin{pmatrix}\omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}m_0\\ m_1\\ \vdots\end{pmatrix}$ for every sequence $\{m_i\}_{i=0}^\infty$ of positive integers.

The exact relationship between the partition properties $\omega \to (\omega)^\omega$ and $\begin{pmatrix}\omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}m_0\\ m_1\\ \vdots\end{pmatrix}$ (for a given sequence $\{m_i\}_{i=0}^\infty$ of positive integers) turned out to be an interesting problem. The two partition properties are not equivalent since polarized partition property is consistent with the existence of non principal ultrafilters on $\mathbb{N}$ (see section 9). To prove this, it was necessary to develop a theory of partitions of products of finite sets and their parametrized versions. This theory, of intrinsic combinatorial interest, is another example of a combinatorial development motivated by a metamathematical question. The next three sections are devoted to present some aspects of this theory (see [7, 8, 38]).

6. Products of finite sets

In the previous section we considered partitions of $\mathbb{N}^\infty$. Here we will deal with partitions of subspaces of $\mathbb{N}^\infty$ of the form $\prod_{i=0}^\infty n_i$, where $\{n_i\}_{i=0}^\infty$ is a sequence of positive integers and, as usual, we identify each n with the set $\{0, 1, \ldots, n-1\}$. In other words, we will consider partitions of the set of infinite sequences of positive integers bounded by some sequence $\{n_i\}_{i=0}^\infty$.

Given a sequence $\{n_i\}_{i=0}^\infty$, invoking the axiom of choice we can define a coloring $c : \prod_{i=0}^\infty n_i \to \{0,1\}$ such that for no sequence $\{H_i\}_{i=0}^\infty$ with $H_i \subseteq n_i$ and $|H_i| = 2$ for every i, the product $\prod_{i=0}^\infty H_i$ is monochromatic. Restricting the class of colorings to be considered, some positive results are obtained. For example, there is a sequence $\{n_i\}_{i=0}^\infty$ such that every Borel coloring $c : \prod_{i=0}^\infty n_i \to \{0,1\}$ admits a monochromatic product of pairs ([24, 7]). An interesting connection appeared here between these partition relations restricted to definable classes of colorings and the Grzegorczyk hierarchy of primitive recursive functions (see Section 8 below).

In more general terms, for every sequence $\{m_i\}_{i=0}^\infty$ of positive integers, there is a sequence $\{n_i\}_{i=0}^\infty$ of positive integers such that for every Souslin-measurable coloring $c : \prod_{i=0}^\infty n_i \to \{0,1\}$ there is a monochromatic product $\prod_{i=0}^\infty H_i$ where for every i, $H_i \subseteq n_i$ has size m_i (Corollary 1).

Partitions of infinite products of finite sets were considered in [4, 23], with particular interest in the question of the consistency of the existence some sequence $\{n_i\}_{i=0}^\infty$ such that for every partition of $\prod_{i=0}^\infty n_i$ into two pieces there is a monochromatic product of pairs. In Section 9 we come back to this question, and see how Solovay's model provides a positive answer.

In this section we describe, for each sequence $\{m_i\}_{i=0}^{\infty}$, a class of products of finite sets and a class of colorings which admit homogeneous products whose factors have sizes determined by $\{m_i\}_{i=0}^{\infty}$. First we need some definitions. Given a sequence $\vec{n} = \{n_i\}_{i=0}^{\infty}$ of natural numbers, a sequence $\vec{H} = \{H_i\}_{i=0}^{\infty}$ of finite sets of natural numbers is said to be of type $\vec{n}$, or a $\vec{n}$-sequence, if for every i, $|H_i| = n_i$. We also say in this case that the product $\prod_{i=0}^{\infty} H_i$ is an $\vec{n}$-product.

Whenever $\{H_i\}_{i=0}^{\infty}$ and $\{J_i\}_{i=0}^{\infty}$ are sequences of finite sets of natural numbers we say that $\prod_{i=0}^{\infty} J_i$ is a sub-product of $\prod_{i=0}^{\infty} H_i$ if $J_i \subseteq H_i$ for every i. We write $\vec{J} \leq_k \vec{H}$ if $J_i \subseteq H_i$ for every $i \in \mathbb{N}$ and $J_i = H_i$ for $i < k$.

Given sequences of positive integers $\vec{m} = \{m_i\}_{i=0}^{\infty}$ and $\vec{n} = \{n_i\}_{i=0}^{\infty}$,

$$(\vec{n}) \to (\vec{m})$$

expresses that for every partition $c : \prod_{i=0}^{\infty} H_i \to \{0,1\}$ such that $|H_i| = n_i$ for all i, there is a sequence $\{J_i\}_{i=0}^{\infty}$ with $J_i \subseteq H_i$, and $|J_i| = m_i$ for all i, such that c is constant on $\prod_{i=0}^{\infty} J_i$. More concisely, for every 2-coloring of a product of type $\vec{n}$, there is a monochromatic sub-product of type $\vec{m}$. As before, if we restrict ourselves to colorings measurable with respect to a certain σ-field $\mathcal{C}$ of subsets of $\mathbb{N}^{[\infty]}$, we use the notation

$$(\vec{n}) \to_{\mathcal{C}} (\vec{m})$$

for the corresponding property.

Notice that $(\vec{n}) \to (\vec{m})$ does not follow from $\omega \to (\omega)^{\omega}$, as

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

does (for sequences $\{m_i\}_{i=0}^{\infty}$ and $\{n_i\}_{i=0}^{\infty}$ of positive integers).

For partitions of finite products we use the notation

$$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_k \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_k \end{pmatrix}$$

which means that for every partition $c : \prod_{i=0}^{k} H_i \to \{0,1\}$ such that $|H_i| = n_i$ for all $i \leq k$, there is a sequence $\{J_i\}_{i=0}^{k}$ with $J_i \subseteq H_i$, and $|J_i| = m_i$ for all $i \leq k$, such that c is constant on $\prod_{i=0}^{k} J_i$.

Definition 2. *Let* $S : \mathbb{N}^{<\infty} \to \mathbb{N}$ *be defined as follows*

$$S(m_0) = 2m_0 - 1$$

$$S(m_0, ..., m_{i+1}) = 2(m_{i+1} - 1) \left[\prod_{k=0}^{i} \binom{m_k}{S(m_0, ..., m_k)} \right] + 1.$$

The function S has the following property which can be verified by induction.

Lemma 1. *Let* $(m_i) \in \mathbb{N}^\infty$ *and let* $n_i = S(m_0, ..., m_i)$ *for all* i. *Then for every* k

$$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_k \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_k \end{pmatrix}.$$

It follows that we can get monochromatic sub-products of type $\vec{m}$ for every continuous 2-coloring of $\prod_{i=0}^{\infty} S(m_o, \ldots, m_i)$ (note that a continuous coloring gives a partition into clopen pieces).

The existence of monochromatic $\vec{m}$-sub-products for semicontinuous 2-colorings, i.e. partitions into a closed set and its complement, also follows from the Lemma (see [7]). To extend this result to a wider class of colorings, for example to Borel-measurable colorings, iterates of the function S are defined in order to carry out certain diagonalization arguments.

The iterates of the function S are defined recursively as follows.

$$S^{(0)}(m_0, ..., m_i) = S(m_0, ..., m_i),$$
$$S^{(p+1)}(m_0, ..., m_i) =$$
$$S(S^{(p)}(m_0), S^{(p)}(m_0, m_1), ..., S^{(p)}(m_0, ..., m_i)).$$

Using the function S and its iterates, for every sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers a family $\mathcal{H}(\vec{m})$ of $\vec{m}$-sequences of finite sets with the following properties can be defined :

(1) For every closed $\mathcal{X} \subseteq \mathbb{N}^\infty$, every $\vec{H} \in \mathcal{H}(\vec{m})$ and every $k \in \mathbb{N}$, there is $\vec{J} \in \mathcal{H}(\vec{m})$ such that $\vec{J} \leq_k \vec{H}$ and $\mathcal{X} \cap \prod_{i=0}^{\infty} J_i$ is clopen (in $\prod_{i=0}^{\infty} J_i$).

(2) Given a sequence

$$\vec{H}^0 \leq_{l_0} \vec{H}^1 \leq_{l_1} \cdots \leq_{l_{j-1}} \vec{H}^j \leq_{l_j} \ldots$$

of elements of $\mathcal{H}(\vec{m})$, there is $\vec{H} \in \mathcal{H}(\vec{m})$ such that for every j, $\vec{H} \leq_{l_j} \vec{H}^j$.

Now, for each sequence $\vec{m}$, we can define a corresponding σ-field of subsets of $\mathbb{N}^\infty$: $\mathcal{C}(\vec{m})$ is the collection of all $\mathcal{X} \subseteq \mathbb{N}^\infty$ such that for every $\vec{H} \in \mathcal{H}(\vec{m})$ and for every $n \in \mathbb{N}$ there is $\vec{J} \in \mathcal{H}(\vec{m})$ such that $\vec{J} \leq_n \vec{H}$ and $\mathcal{X} \cap \prod_i J_i$ is clopen in $\prod_i J_i$.

Theorem 5. *([9]) For every sequence $\vec{m}$, $\mathcal{C}(\vec{m})$ is a σ-field which contains the closed sets and is closed under Souslin's operation.*

The argument given in [9] to prove that $\mathcal{C}(\vec{m})$ is closed under Souslin's operation is metamathematical. It uses the decomposition of analytic sets into Borel sets and a forcing notion preserving $\aleph_1$. Once analytic sets are shown to be in $\mathcal{C}(\vec{m})$, it follows that $\mathcal{C}(\vec{m})$ is closed under Souslin's operation. S. Todorcevic has recently found a combinatorial proof of this fact ([39]).

By the comments in the paragraphs following Lemma 1, we have the following.

Corollary 1. *For every sequence $\vec{m} = \{m_i\}_{i=0}^\infty$, there is a sequence $\vec{n} = \{n_i\}_{i=0}^\infty$ such that*

$$(\vec{n}) \rightarrow_{\mathcal{C}(\vec{m})} (\vec{m}).$$

In particular, for any Souslin-measurable

$$c : \prod_{i=0}^{\infty} n_i \rightarrow \{0,1\}$$

there exist $H_i \subseteq N_i$ with $|H_i| = m_i$ for all i, such that c is constant on $\prod_{i=0}^\infty H_i$.

7. Parametrized partitions of products of finite sets

The following lemma from [8] is a more uniform version of Lemma 1. Besides being interesting in its own right, it is useful to parametrize the partition property $\omega \rightarrow (\omega)^\omega$ with partition properties of the form $\vec{n} \rightarrow (\vec{m})$. The proof of the lemma given in [8] uses the hypothesis "all $\mathbf{\Sigma}_2^1$-sets are Ramsey", which for some applications, like the one given in Section 9 can then be eliminated.

Lemma 2. *There is $R : \mathbb{N}^{<\infty} \rightarrow \mathbb{N}$ such that for every infinite sequence $\vec{m} = \{m_i\}_{i=0}^\infty$ of positive integers and every coloring*

$$c : \bigcup_k \prod_{i<k} R(m_0, \ldots, m_i) \rightarrow \{0,1\},$$

there exist $H_i \subseteq R(m_0, \ldots, m_i)$, $|H_i| = m_i$ for all i, and an infinite $A \subseteq \mathbb{N}$, such that c is constant on $\bigcup_{k \in A} \prod_{i<k} H_i$.

We cannot hope to get even more uniformity, since it is easy to define a partition for which there is no $\vec{m}$-sequence $\vec{H}$ such that for every k, the sub-product $\prod_{i=0}^{k} H_i$ monochromatic.

The iterates of the function R given by Lemma 2 are defined as those for S in the previous section, and for every sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers, a family $\mathcal{H}_R(\vec{m})$ of sequences of finite sets can be defined diagonalizing through the function R and its iterates applied to $\vec{m}$. Using this collection $\mathcal{H}_R(\vec{m})$, a field $\mathcal{PC}(\vec{m})$ of subsets of $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ is defined as follows: a subset $\mathcal{X} \subseteq \mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ is in $\mathcal{PC}(\vec{m})$ if for every sequence $\vec{H} \in \mathcal{H}_R(\vec{m})$, every n, every $A \in \mathbb{N}^{[\infty]}$, every $a \in \mathbb{N}^{[<\infty]}$, there exist a sequence $\vec{J} \in \mathcal{H}_R(\vec{m})$ such that $\vec{J} \leq_n \vec{H}$, $k \geq n$, and $B \subseteq A$ such that $\forall s \in \prod_{i<k} J_i$

$$[s, \vec{J}] \times [a, B] \subseteq \mathcal{X} \text{ or } [s, \vec{J}] \times [a, B] \cap \mathcal{X} = \emptyset.$$

Here, $[s, \vec{J}]$ is the collection of elements of $\prod_{i=0}^{\infty} J_i$ which extend s.

For partitions of the product space $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$, $\mathcal{PC}(\vec{m})$ plays a rôle analogous to that of the field $\mathcal{C}(\vec{m})$ of the previous section. The properties satisfied by the families $H_R(\vec{m})$ are then used to prove the following.

Theorem 6. ([9]) *For every sequence $\vec{m}$, $\mathcal{PC}(\vec{m})$ is a σ-field of subsets of $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ containing the open sets and closed under Souslin's operation.*

The proof of this theorem is much harder than the proof of Theorem 5. To show that open subsets of $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ are in $\mathcal{PC}(\vec{m})$, the notion of "barrier" of Nash-Williams is used, and a combinatorial forcing the developed along the lines of [14].

As for the case of the field $\mathcal{C}(\vec{m})$, the argument of [9] to show that every $\mathcal{PC}(\vec{m})$ is closed under Souslin's operation uses metamathematical tools. There is a recent combinatorial proof due to S. Todorcevic.

Corollary 2. *Every analytic subset of $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ is in $\mathcal{PC}(\vec{m})$.*

It can be verified ([8] 5.2) that for every sequence $\{m_i\}_{i=0}^{\infty}$ there is a sequence $\{n_i\}_{i=0}^{\infty}$ such that given a set $\mathcal{X} \subseteq \prod_{i=0}^{\infty} n_i \times \mathbb{N}^{[\infty]}$ in $\mathcal{PC}(\vec{m})$, there is an (m_i)-sequence $\{H_i\}_{i=0}^{\infty}$ and an infinite set H such that

$$\prod_{i=0}^{\infty} H_i \times H^{[\infty]} \subseteq \mathcal{X} \text{ or } \prod_{i=0}^{\infty} H_i \times H^{[\infty]} \cap \mathcal{X} = \emptyset.$$

Thus, we obtain the following.

Corollary 3. *For every sequence $\{m_i\}_{i=0}^{\infty}$, there is a sequence $\{n_i\}_{i=0}^{\infty}$ such that for every analytic $\mathcal{X} \subseteq \prod_{i=0}^{\infty} n_i \times \mathbb{N}^{[\infty]}$, there is an (m_i)-sequence*

$\{H_i\}_{i=0}^{\infty}$ *and an infinite set* H *such that*

$$\prod_{i=0}^{\infty} H_i \times H^{[\infty]} \subseteq \mathcal{X} \text{ or } \prod_{i=0}^{\infty} H_i \times H^{[\infty]} \cap \mathcal{X} = \emptyset.$$

Question 5. *Is the hypothesis "all* $\mathbf{\Sigma}_2^1$*-sets are Ramsey" necessary for Lemma 2?*

8. Rates of growth

It is shown in [7] that for the constant sequence $m_i = 2$ for all i, the sequence $n_i = 2^{2^{2i+1}}$ satisfies

$$(\vec{n}) \rightarrow_{Clopen} (\vec{m}).$$

More generally,

Theorem 7. *([38]) For every primitive recursive sequence* $\{m_i\}_{i=0}^{\infty}$*, there is a primitive recursive sequence* $\{n_i\}_{i=0}^{\infty}$ *such that*

$$(\vec{n}) \rightarrow_{\mathcal{B}orel} (\vec{m}).$$

Question 6. *Is there a primitive recursive sequence* $\{n_i\}_{i=0}^{\infty}$*such that for every Souslin measurable coloring*

$$c : (\prod_{i=0}^{\infty} n_i) \times \mathbb{N}^{[\infty]} \rightarrow \{0, 1\}$$

there exist $H_i \subseteq n_i$*,* $|H_i| = 2$ *for all* i*, and an infinite* $H \subseteq \mathbb{N}$*, such that the product* $(\prod_{i=0}^{\infty} H_i) \times H^{[\infty]}$ *is monochromatic?*

9. Some consistency results

In this section we indicate how to obtain a model where the partition property

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

holds for every sequence $\{m_i\}_{i=0}^{\infty}$, together with the existence of a non-principal ultrafilter on $\mathbb{N}$. We assume familiarity with Solovay's model where all sets of real numbers are Lebesgue measurable [37]. The reader can consult [20] for a presentation of this model and its main properties.

We say that a model M is a Solovay model over a ground model V if $M = L(\mathbb{R})$, the class of sets constructible from $\mathbb{R}$, where $\mathbb{R}$ is the set of reals in a generic extension of V obtained using the Levy order to collapse an inaccessible cardinal of V to $(\omega_1)^M$.

The following theorem establishes that in a Solovay model, the fields of sets $\mathcal{C}(\vec{m})$ and $\mathcal{PC}(\vec{m})$, defined in the previous sections. include all subsets of the corresponding spaces. Since in a Solovay model all subsets of $\mathbb{N}^{[\infty]}$ are Ramsey, Lemma 2 holds there.

Theorem 8. *([9]) In every Solovay model, for every sequence $\{m_i\}_{i=0}^{\infty}$ of positive integers, every subset of $\mathbb{N}^{\infty}$ is in $\mathcal{C}(\vec{m})$, and every subset of $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ is in $\mathcal{PC}(\vec{m})$.*

The main ingredients of the proof are several well known properties of Solovay models, in particular the fact that in a Solovay model every subset of $\mathbb{N}^{\infty}$ or of $\mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]}$ is the union of $\aleph_1$ analytic sets.

From the fact about $\mathcal{C}(\vec{m})$ follows that in a Solovay model, for every sequence $\{m_i\}_{i=0}^{\infty}$, there is a sequence $\{n_i\}_{i=0}^{\infty}$ such that $(\vec{n}) \to (\vec{m})$. Using $\mathcal{PC}(\vec{m})$ we obtain the parametrized version.

Corollary 4. *In every Solovay model, for every sequence $\{m_i\}_{i=0}^{\infty}$ there exists a sequence $\{n_i\}_{i=0}^{\infty}$ such that for every coloring*

$$c : \prod_{i=0}^{\infty} n_i \times \mathbb{N}^{[\infty]} \to \{0,1\}$$

there exist $H_i \subseteq n_i$, $|H_i| = m_i$ for every i, and an infinite $H \subseteq \mathbb{N}$ such that $(\prod_{i=0}^{\infty} H_i) \times H^{[\infty]}$ is monochromatic.

Notice that this implies that in every Solovay model, for every sequence $\{m_i\}_{i=0}^{\infty}$ and for every coloring

$$c : \mathbb{N}^{\infty} \times \mathbb{N}^{[\infty]} \to \{0,1\}$$

there exist $H_i \subseteq n_i$, $|H_i| = m_i$ for every i, and an infinite $H \subseteq \mathbb{N}$ such that $(\prod_{i=0}^{\infty} H_i) \times H^{[\infty]}$ is monochromatic.

The consistency of this parametrized partition property can be used to show that the property $\omega \to (\omega)^{\omega}$ is not implied by the polarized partition property

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}.$$

This is achieved starting from a Solovay model $L(\mathbb{R})$, and adding a generic selective ultrafilter U to obtain the generic extension $L(\mathbb{R})[U]$. The parametrized partition property in $L(\mathbb{R})$ is then used to show that this generic extension satisfies the polarized partition property

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}.$$

Since in the presence of non principal ultrafilters on $\mathbb{N}$ there are non Ramsey subsets of $\mathbb{N}^{[\infty]}$, the property $\omega \to (\omega)^\omega$ does not hold in the generic extension $L(\mathbb{R})[U]$. Therefore, the property

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

does not imply $\omega \to (\omega)^\omega$.

We end with a question related to the relationship between the polarized partition property

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$$

and the property $\omega \to (\text{perfect})^\omega$ of Section 3.

Question 7. *It is clear that $\omega \to (\textit{perfect})^\omega$ follows from*

$$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ \vdots \end{pmatrix},$$

since every infinite product of pairs is perfect. Are these two partition properties equivalent? Both are consistent with the existence of ultrafilters on $\mathbb{N}$, but perhaps there is a stronger consequence of the Axiom of Choice which can be used to distinguish between them.

10. Determinacy and partitions

Given a set $\mathcal{X} \subseteq \mathbb{N}^\infty$, consider the game $G_\mathcal{X}$ played by two players, I and II, who alternate playing natural numbers. I plays n_0, then II plays n_1, I plays n_2, etc., forming an infinite sequence $x = \langle n_0, n_1, n_2, \dots \rangle$. Player I wins the game if $x \in \mathcal{X}$. A winning strategy for player I is a function $\sigma : \mathbb{N}^{<\infty} \to \mathbb{N}$ such that any run of the game $G_\mathcal{X}$ in which I's moves are determined using σ produces an element of $\mathcal{X}$. Analogously, we can define winning strategy for II. A set $X \subseteq \mathbb{N}^\infty$ is determined if one of the players has a winning strategy for the game $G_\mathcal{X}$. The Axiom of Determinacy is the statement "all subsets of $\mathbb{N}^\infty$ are determined". The axiom of choice implies that there are non-determined sets, in fact, if neither $\mathcal{X}$ nor $\mathbb{N}^\infty \setminus \mathcal{X}$ contain a perfect set, then $\mathcal{X}$ is not determined. Nevertheless, all Borel sets are determined [25].

There are some unresolved questions regarding the connections between the partition properties we have considered here and the Axiom of Determinacy (AD).

AD implies that every uncountable subset of $\mathbb{N}^\infty$ contains a perfect set and that every subset of $\mathbb{N}^\infty$ has the Baire property. Therefore, AD implies $\omega \to (\text{perfect})^\omega$ and $\omega \to ((\omega))^\omega$. It is unknown if $\omega \to (\omega)^\omega$ follows from AD. Prikry [32] showed that $\omega \to (\omega)^\omega$ is a consequence of $AD_\mathbb{R}$, determinacy of subsets of $\mathbb{R}^\infty$. The property $\omega \to (\omega)^\omega$ is also a consequence of AD *and* $V = L(\mathbb{R})$ [26].

Question 8. *Is $\omega \to (\omega)^\omega$ a consequence of AD? Is the partition property* $\begin{pmatrix}\omega\\ \omega\\ \vdots\end{pmatrix} \to \begin{pmatrix}2\\ 2\\ \vdots\end{pmatrix}$ *a consequence of AD?*

References

[1] Bernstein, F., Zur Theorie der Trigonometrischen Reiche. *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig Mathematisch-Physiche Klasse* 60 (1908) 325-338.

[2] Brendle, J., L. Halbeisen and B. Löwe, Silver measurability and its relations to other regularity properties, (preprint).

[3] Di Prisco, C. A., Partition properties and perfect sets. *Notas de Lógica Matemática*, Universidad Nacional del Sur, Bahía Blanca, Argentina, 38 (1993)119-127.

[4] Di Prisco, C. A. and J. Henle, Partitions of products. *Journal of Symbolic Logic* 58 (1993), 860-871.

[5] Di Prisco, C. A. and J. Henle, Partitions of the reals and choice. In: Models, algebras and proofs (X. Caicedo and C. H. Montenegro, Eds.) Marcel Dekker, Inc. 1999, 13-23.

[6] Di Prisco, C. A. and J. Henle, Doughnuts, floating ordinals, square brackets, and ultraflitters. *Journal of Symbolic Logic* 65 (2000) 461-473.

[7] Di Prisco, C. A., J. Llopis and S. Todorcevic, Borel partitions of products of finite sets and the Ackermann function. *Journal of Combinatorial Theory*, Series A, 93 (2001) 333-349.

[8] Di Prisco, C. A., J. Llopis y S. Todorcevic, Parametrized partitions of products of finite sets. *Combinatorica* (to appear).

[9] Di Prisco, C.A. and S. Todorcevic, Souslin partitions of products of finite sets. *Advances in Mathematics*, 176 (2003) 145-173.

[10] Ellentuck, E., A new proof that analytic sets are Ramsey. *Journal of Symbolic Logic* 39 (1974), 256-290.

[11] Erdös, P. and R. Rado, Combinatorial theorems on classification of subsets of a given set. *Proceedings of the London mathematical Society* 3 (1952) 417-439.

[12] Erdös, P. and R. Rado, A partition calculus in set theory, *Bulletin of the Amer. Math. Soc.* 62 (1956), 427-489. (Abstract 68T-368).

[13] Galvin, F., A generalization of Ramsey's Theorem. *Notices of the American Mathematical Society* 15 (1968) 548.

[14] Galvin, F. and K. Prikry, Borel sets and Ramsey's theorem, *Journal of Symbolic Logic* 38 (1973), 193-198.

[15] Graham, R. L., B. L. Rotschild and J. H. Spencer, Ramsey Theory. John Wiley and sons, 1990.

[16] Halbeisen, L., Making doughnuts of Cohen reals. *Mathematical Logic Quarterly* 49 (2003) 173–178.

[17] Halpern, J.D. and H. Läuchli, A partition theorem, *Transactions of the American Mathematical Society* 124 (1966)360-367.

[18] Halpern, J.D. and A. Levy, The Boolean prime ideal theorem does not imply the axiom of choice. In "Axiomatic Set Theory". Proc. Symp. Pure Mathematics 13 I (D. Scott, Ed.) pp 83-134. American Mathematical Society, 1971.

[19] Henle, J., The consistency of one fixed omega, *Journal of Symbolic Logic* 60 (1995), 172-177.

[20] Jech, T., Set Theory. Springer Verlag, 1997.

[21] Kanamori, A., The higher infinite, Springer Verlag, 1997.

[22] Laver, R., Products of infinitely many perfect trees, *Journal of the London Mathematical Society* (2), 29(1984) 385-396.

[23] Llopis, J., A note in polarized partitions, *Notas de Lógica Matemática*, Universidad Nacional del Sur, Bahía Blanca, Argentina, 39 (1994), 89-94.

[24] Llopis, J. and S. Todorcevic, Borel partitions on products of finite sets, *Acta Científica Venezolana* 47 (1996), 85-88.

[25] D. A. Martin, Borel Determinacy. *Annals of Mathematics* 102 (1975) 2363-371.

[26] Martin, D. A. and J. R. Steel, The extent of scales in $L(\mathbb{R})$, in Cabal Seminar 79-81 (A. S. Kechris, D. A. Martin and Y. Moschovakis, Eds.). Lecture Notes in Mathematics 1019, 1983

[27] Mathias, A. R. D., A remark on rare filters, in: Infinite and finite sets (A. Hajnal. R. Rado and Vera Sós, Eds.). Colloquia Mathematica Societatis Janos Bolyai 10, , North Holland, 1975.

[28] Mathias, A.R.D., Happy Families. *Annals of Pure and Applied Logic* 12 (1977) 59-111.

[29] G. Moran, G. and D. Strauss, Countable partitions of product spaces, *Mathematika* 27 (1980) 213-224.

[30] Nash-Williams, C.St. J. A., On well quasi-ordering transfinite sequences. *Proceedings of the Cambridge Philosophical Society* 61 (1965) 33-39.

[31] Nešetřil, J. and V. Rödl, Eds., Mathematics of Ramsey Theory. Springer Verlag, 1990.

[32] Prikry, K.L., Determinateness and partitions. *Proceedings of the American Mathematical Society* 54 (1976)303-306.

[33] Prömel, H. J. and B. Voigt, Recent results in partition (Ramsey) theory for finite lattices. Dicrete Mathematics 35 (1981) 185-198.

[34] Ramsey, F. P., On a problem of formal logic. *Proc. of the London Mathematical Society, Ser. 2* 30, Part 4 (1928) 338–384.

[35] Shelah, S., Can you take Solovay's inaccessible away? *Israel Journal of mathematics* 48 (1984) 1-47.

[36] Silver, J., Every analytic set is Ramsey, *Journal of Symbolic Logic* 35 (1970), 60-64.
[37] Solovay, R., A model of set theory where every set of reals is Lebesgue measurable. *Annals of Mathematics* 92 (1970) 1-56.
[38] Todorcevic, S., A new quantitative analysis of some basic principles in the theory of functions of a real variable. *Bull. de l'Acad. Serbe des Sci.* vol. CXXII, No, 26 (2001), 133-144.
[39] Todorcevic, S., Introduction to Ramsey Spaces (in preparation).

C.A. Di Prisco
Instituto Venezolano de Investigaciones Científicas,
Caracas, Venezuela