

THE STRUCTURE OF A SUBCLASS OF AMENABLE BANACH ALGEBRAS

R. EL HARTI

ABSTRACT. We give sufficient conditions that allow contractible (resp. reflexive amenable) Banach algebras to be finite-dimensional and semisimple algebras. Moreover, we show that any contractible (resp. reflexive amenable) Banach algebra in which every maximal left ideal has a Banach space complement, is indeed a direct sum of finitely many full matrix algebras. Finally, we characterize hermitian *-algebras that are contractible.

1. INTRODUCTION AND NOTATIONS.

The purpose of this paper is to establish the structure of some class of amenable Banach algebras. Let \mathcal{A} be a Banach algebra over the complex field \mathbb{C} . A Banach \mathcal{A} -bimodule \mathcal{X} is a Banach space \mathcal{X} , such that \mathcal{X} is an \mathcal{A} -bimodule algebraically and the operation

$$\mathcal{A} \times \mathcal{X} \times \mathcal{A} \rightarrow \mathcal{X}, \quad (a, x, b) \mapsto axb$$

is jointly continuous, where $\mathcal{A} \times \mathcal{X} \times \mathcal{A}$ carries the cartesian product topology. For each Banach \mathcal{A} -bimodule \mathcal{X} , the dual \mathcal{X}^* is naturally a Banach \mathcal{A} -bimodule with the module actions defined by

$$(a.T)(x) = T(xa), \quad (T.a)(x) = T(ax), \quad \text{for all } a \in \mathcal{A}, T \in \mathcal{X}^* \text{ and } x \in \mathcal{X},$$

where $T(x)$ denotes the evaluation of T at x .

A derivation from \mathcal{A} into a Banach \mathcal{A} -bimodule \mathcal{X} is a linear operator $d : \mathcal{A} \rightarrow \mathcal{X}$ which satisfies $D(ab) = d(a)b + ad(b)$, $\forall a, b \in \mathcal{A}$. Recall that for any $x \in \mathcal{X}$, the mapping $\delta_x : \mathcal{A} \rightarrow \mathcal{X}$ defined by $\delta_x(a) = ax - xa$, $a \in \mathcal{A}$, is a continuous derivation, called an inner derivation. A Banach algebra \mathcal{A} is

2000 *Mathematics Subject Classification.* 46H05, 46H25, 18G55, 46J05, 46J45, 46M20.

Financially supported by CRM, Universitat Autònoma de Barcelona, Spain.

said to be contractible if for every Banach \mathcal{A} -bimodule \mathcal{X} , each continuous derivation from \mathcal{A} into \mathcal{X} is inner. We say that \mathcal{A} is amenable whenever every continuous derivation from \mathcal{A} into \mathcal{X}^* is inner for each Banach \mathcal{A} -bimodule \mathcal{X} . Obviously, every contractible Banach algebra is amenable. The algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices, is contractible algebra and every direct sum of a certain finite family of full matrix algebras is contractible or amenable.

The focus here is to settle the following raised in [?] and [?] and [?] :

QUESTION 1. Is every contractible Banach algebra semisimple ?.

QUESTION 2. Is every contractible Banach algebra finite-dimensional?.

QUESTION 3. Is every reflexive amenable Banach algebra finite-dimensional and semisimple?

Recall that a Banach algebra is called a reflexive Banach algebra if it is reflexive as a Banach space. In this paper, we shall present two situations in which a contractible Banach algebra is finite-dimensional. First, we shall give a partial answer to the questions above where we assume that each maximal left ideal is complemented as a Banach space. This result improves (Proposition IV.4.3, [?],) for contractible Banach algebras and (Corollary 2.3, [?]) for reflexive amenable Banach algebras where the authors suppose only that all of whose primitive ideals have finite-codimensional. In the second time, we will show that a hermitian Banach $*$ -algebra is contractible if, and only if, it is a finite-dimensional semisimple algebra. Finally, we establish some equivalent statements to questions 2 and 3.

2. PRELIMINARIES.

In this second section, we recall some facts about the structure of contractible and amenable Banach algebras. First, consider on \mathcal{A}^{**} the Banach \mathcal{A} -bimodule structure defined by $aT = \eta(a)T$ and $Ta = T\eta(a)$ with $\eta : \mathcal{A} \rightarrow \mathcal{A}^{**}$ the injective linear map. Notice that if a Banach algebra \mathcal{A} has a bounded approximate identity, then its bidual \mathcal{A}^{**} has an identity. It is a fact that a contractible Banach algebra has an identity and an amenable Banach algebra admits bounded right, left, bilateral approximate identities. Of course, a reflexive amenable Banach algebra must to be unital. Let \mathcal{A}

be a Banach algebra over the complex \mathbb{C} . We denote the identity element in a Banach algebra by $1_{\mathcal{A}}$ and we write $\mathcal{A}\hat{\otimes}\mathcal{A}$ for the completed projective tensorial product (see [?]).

For a unital Banach algebra \mathcal{A} , a diagonal of \mathcal{A} is an element $d \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that

$$ad = da, \forall a \in \mathcal{A} \quad \text{and} \quad \pi(d) = 1_{\mathcal{A}}$$

where $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \rightarrow \mathcal{A}$ is the canonical Banach \mathcal{A} -module morphism. For a such Banach algebra \mathcal{A} , a virtual diagonal of \mathcal{A} is an element $d \in \mathcal{A}\hat{\otimes}\mathcal{A}^{**}$ such that

$$ad = da, \forall a \in \mathcal{A} \quad \text{and} \quad \pi^{**}(d) = 1_{\mathcal{A}}$$

where $\pi^{**} : \mathcal{A}\hat{\otimes}(\mathcal{A} \rightarrow \mathcal{A})^{**}$ is the bidual Banach \mathcal{A} -module morphism of π . In the following theorems, we present characterization of contractible (resp. amenable) Banach algebra.

Theorem 2.1. [?] *Let \mathcal{A} be a Banach algebra. The following are equivalent:*

- (1) \mathcal{A} is contractible.
- (2) \mathcal{A} has a diagonal.

Theorem 2.2. [?] *Let \mathcal{A} be a Banach algebra. The following are equivalent:*

- (1) \mathcal{A} is amenable .
- (2) \mathcal{A} has a virtual diagonal.

Next, the following properties state:

Proposition 2.1. *Let \mathcal{A} be a \langle contractible, amenable \rangle Banach algebra. Then:*

If $\theta : \mathcal{A} \rightarrow \mathcal{B}$ is a continuous homomorphism from \mathcal{A} into another Banach algebra \mathcal{B} with dense range, then \mathcal{B} is \langle contractible, amenable \rangle .

In particular, if \mathcal{I} is a closed two-sided ideal of a \langle contractible, amenable \rangle Banach algebra, then \mathcal{A}/\mathcal{I} is a \langle contractible, amenable \rangle too.

Proposition 2.2. *Let \mathcal{A} be a contractible or reflexive amenable Banach algebra and assume that \mathcal{I} is a closed \langle left, two-sided \rangle ideal of \mathcal{A} which has a Banach space complement. Then there exists a closed \langle left, two-sided \rangle ideal \mathcal{J} of \mathcal{A} such that*

$$\mathcal{A} = \mathcal{I} + \mathcal{J}.$$

Notice that for each closed two-sided ideal \mathcal{I} of a reflexive Banach algebra, \mathcal{I} and the quotient \mathcal{A}/\mathcal{I} are reflexive Banach algebra too.

3. MAIN RESULTS.

Theorem 3.1. *Let \mathcal{A} be a contractible or reflexive amenable Banach algebra. Assume that each maximal left ideal of \mathcal{A} is complemented as a Banach space in \mathcal{A} . Then, there are $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} = \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \dots \oplus \mathbb{M}_{n_k}(\mathbb{C})$$

Proof. By Preliminaries, the algebra \mathcal{A} has an identity $1_{\mathcal{A}}$. Let $(\mathcal{M}_i)_{i \in I}$ be a family of all maximal left ideals. Since \mathcal{M}_i is complemented as Banach space for each i , there exists a left ideal \mathcal{J}_i such that $\mathcal{A} = \mathcal{M}_i \oplus \mathcal{J}_i$. Notice that

$$\text{Rad}(\mathcal{A}) = \bigcap_i \mathcal{M}_i$$

is the Jacobson radical of \mathcal{A} and

$$\bigoplus_i \mathcal{J}_i \subseteq \text{Soc}(\mathcal{A}) \quad (*)$$

where $\text{Soc}(\mathcal{A})$ is the socle of the algebra \mathcal{A} , i.e, it is the sum of all minimal left ideals of \mathcal{A} and it coincides with the sum of all minimal right ideals of \mathcal{A} . Recall that every minimal left ideal of \mathcal{A} is of the form $\mathcal{A}e$ where e is a minimal idempotent, i.e, $e^2 = e \neq 0$ and $e\mathcal{A}e = \mathbb{C}e$. On the other hand, for a finite family of minimal idempotents $(e_k)_{k \in K}$, we have

$$\mathcal{A} = \bigoplus_{k \in K} \mathcal{A}e_k \bigoplus \bigcap_{k \in K} \mathcal{A}(1_{\mathcal{A}} - e_k) \quad (**).$$

It follows from (*) and (**) that $\text{Soc}(\mathcal{A})$ is dense in $\mathcal{A}/\text{Rad}(\mathcal{A})$. This shows that $\mathcal{A}/\text{Rad}(\mathcal{A})$ is finite-dimensional. Therefore

$$\mathcal{A} = \text{Rad}(\mathcal{A}) \bigoplus \text{Soc}(\mathcal{A}).$$

If $\text{Rad}(\mathcal{A}) \neq \{0\}$, this would mean that $\text{Rad}(\mathcal{A})$ has an identity which is impossible. So, $\mathcal{A} = \text{Soc}(\mathcal{A})$ and then it is a finite direct sum of certain full matrix algebras.

Corollary 3.1. *Let \mathcal{A} be a contractible or reflexive amenable Banach algebra such that every irreducible representation of \mathcal{A} is finite-dimensional. Then \mathcal{A} is finite-dimensional and semisimple.*

Proof. It is easy to check that every primitive ideal of a Banach algebra is finite-codimensional if, and only if, every its maximal left ideal is finite-codimensional. So, the corollary follows.

Now, we give some equivalent statements to the questions 2 and 3 .

Theorem 3.2. *The following statements are equivalent:*

- (1) *Every \langle contractible, reflexive amenable \rangle Banach algebra is finite-dimensional and semisimple.*
- (2) *Every \langle contractible, reflexive amenable \rangle primitive Banach algebra is finite-dimensional.*
- (3) *Every \langle contractible, reflexive amenable \rangle simple contractible Banach algebra is finite-dimensional.*

Proof. It is clear that $1 \Rightarrow 2$ and $2 \Rightarrow 3$. For $3 \Rightarrow 1$, let \mathcal{A} be a contractible Banach algebra. Let \mathcal{P} be a primitive ideal of \mathcal{A} . Then the algebra \mathcal{A}/\mathcal{P} is a \langle contractible, reflexive amenable \rangle Banach algebra. Put $\mathcal{B} = \mathcal{A}/\mathcal{P}$ and consider some maximal two-sided ideal \mathcal{M} of \mathcal{B} . Since \mathcal{B}/\mathcal{M} is a \langle contractible, reflexive amenable \rangle simple Banach algebra, it is finite-dimensional. There exists then a closed two-sided ideal \mathcal{J} such that $\mathcal{A} = \mathcal{M} \oplus \mathcal{J}$. Recall that in a simple algebra, every nonzero ideal is essential, i.e, it has a nonzero intersection with every nonzero ideal of the algebra. It follows that $\mathcal{M} = 0$ and so, \mathcal{B} is finite dimensional. Using Corollary 3.1, \mathcal{A} must be a finite-dimensional and semisimple algebra. This completes the proof.

Note that to show that every \langle contractible, reflexive amenable \rangle simple contractible Banach algebra is finite-dimensional, one can not use Theorem 3.1, because a such algebra does not contain a maximal left ideal complemented as a Banach space. Indeed, if not, the algebra has to have a non trivial minimal left ideal. Since $Soc(\mathcal{A}) = 0$, this is a contradiction.

Finally, assume that \mathcal{A} is a unital Banach $*$ -algebra which admits at least one state τ . Then there exists a $*$ -representation π_τ of \mathcal{A} on a Hilbert space H_τ , with cyclic vector ζ of norm 1 in H_τ such that $\tau(a) = \langle \pi_\tau(a)\zeta, \zeta \rangle$, for all $a \in \mathcal{A}$.

Theorem 3.3. *A hermitian Banach *-algebra \mathcal{A} is contractible if, and only if, there are $n_1, n_2, \dots, n_k \in \mathbb{N}$ such that*

$$\mathcal{A} \cong \mathbb{M}_{n_1}(\mathbb{C}) \oplus \mathbb{M}_{n_2} \oplus \mathbb{M}_{n_2}(\mathbb{C}) \oplus \mathbb{M}_{n_k}(\mathbb{C}).$$

Proof. Let $T(\mathcal{A})$ be the set of all states of \mathcal{A} and let $R^*(\mathcal{A})$ be the *-radical of \mathcal{A} , i.e, the intersection of the kernels of all *-representations of \mathcal{A} on Hilbert spaces. Since \mathcal{A} is hermitian and has an identity, $T(\mathcal{A}) \neq \emptyset$ and so $R^*(\mathcal{A}) \neq \mathcal{A}$. Put

$$\pi = \bigoplus_{\tau \in T(\mathcal{A})} \pi_{\tau} \quad \text{and} \quad H = \bigoplus_{\tau \in T(\mathcal{A})} H_{\tau}.$$

Then π is a *-representation of \mathcal{A} on H . Consider

$$\|\pi(a)\| = \sup_{\tau \in T(\mathcal{A})} \{\pi_{\tau}(a)\}.$$

Then $\|\cdot\|$ is a C^* -norm on $\pi(\mathcal{A})$. Let \mathcal{B} denote the closure of $(\pi(\mathcal{A}), \|\cdot\|)$. Moreover, $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is continuous mapping into C^* -algebra \mathcal{B} such that $\ker(\pi(\mathcal{A})) = R^*(\mathcal{A})$. As \mathcal{A} is contractible, \mathcal{B} is also contractible. Using [?], the algebra \mathcal{B} has to be finite dimensional. Notice that $\mathcal{A}/R^*(\mathcal{A})$ is isometric with the *-subalgebra $\pi(\mathcal{A})$ of \mathcal{B} . Thus, it follows that $\mathcal{A}/R^*(\mathcal{A})$ finite-dimensional. Since $R^*(\mathcal{A})$ is a finite-codimensional closed two-sided *-ideal, there exists a closed two-sided ideal \mathcal{K} such that

$$\mathcal{A} = R^*(\mathcal{A}) \oplus \mathcal{K}.$$

Next, note that $\|\pi(a)\|^2 = \sup\{\tau(a^*a), \tau \in T(\mathcal{A})\} \leq |a^*a|_{\sigma}$ where $|a|_{\sigma}$ is the spectral radius of $a \in \mathcal{A}$. By Ptak [?], we obtain $\|\pi(a)\|^2 \geq |a|_{\sigma}^2$. So, if $a \in R^*(\mathcal{A})$, then $|a|_{\sigma} = 0$. Therefore every element of $R^*(\mathcal{A})$ is quasinilpotent. Notice that in general $Rad(\mathcal{A}) \subseteq R^*(\mathcal{A})$. Since $R^*(\mathcal{A})$ is a closed two-sided *-ideal, we have $R^*(\mathcal{A}) = Rad(\mathcal{A})$ and so, \mathcal{A} is finite-dimensional and semisimple.

REFERENCES

- [1] R. EL Harti, *Contractible Fréchet algebras*, Proc. Amer. Math. Soc. Article electronically published on October9, (2003).
- [2] J. E. Galé, T. J. Ransford, and M. C. White, *Weakly compact homomorphisms*, Trans. Amer. Math. Soc. 331 (1992), 815-682 4.

- [3] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc, 16 (1955)140 pp.
- [4] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*, Kluwer Academic Press, (1989).
- [5] A. Ya. Helemskii, *Some remarks about ideas and results in topological. homology. I*, In [loy], pp. 203- 238.
- [6] A. Ya. Helemskii, *Banach and Locally convex algebras. I*, Oxford University Press, (1993).
- [7] B. E. Jhonson, *Cohomology in Banach algebras*, Memoire of Amer. Math. Soc, vol 127 (1970).
- [8] V. Ptak, *On the spectral radius in Banach algebras*. Bull. London. Math. Soc. 2(1970), 327- 334.
- [9] Yu. V. Selivanov, *Some questions on the homological classification of Banach algebras*, Author's review of PH.D. dissertation, Izdat. Moskov. Univ. L Moscow 1978.

Rachid EL Harti

University Hassan I

FST de Settat, BP 577

2600 Settat, Morocco

E-mail : elharti@ibnsina.uh1.ac.ma