Algebraic mapping-class groups of orientable surfaces with boundaries

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Abstract

Let $S_{g,b,p}$ denote a surface which is connected, orientable, has genus g, has b boundary components, and has p punctures. Let $\Sigma_{g,b,p}$ denote the fundamental group of $S_{g,b,p}$. Let $\mathrm{Out}_{g,b,p}$ denote the algebraic mapping-class group of $S_{g,b,p}$.

We study the exact sequence

$$1 \to \check{\Sigma}_{g,b,p} \to \mathrm{Out}_{g,b\perp 1,p} \to \mathrm{Out}_{g,b,p} \to 1$$

that arises from filling in the interior of a boundary component of $S_{g,b+1,p}$. Here $\mathrm{Out}_{g,b\perp 1,p}$ is a subgroup of index $b+1$ in $\mathrm{Out}_{g,b+1,p}$. If (g,b,p) is $(0,0,0)$ or $(0,0,1)$, then $\check{\Sigma}_{g,b,p}$ is trivial. If (g,b,p) is $(0,0,2)$ or $(1,0,0)$, then $\check{\Sigma}_{g,b,p}$ is infinite cyclic. In all other cases, $\check{\Sigma}_{g,b,p}$ is the fundamental group of the unit-tangent bundle of $S_{g,b,p}$, a certain central extension of $\Sigma_{g,b,p}$.

We give a description of the conjugation action of $\mathrm{Out}_{g,b\perp 1,p}$ on $\check{\Sigma}_{g,b,p}$ in terms of the following three ingredients: the natural action of $\mathrm{Out}_{g,b\perp 1,p}$ on $\Sigma_{g,b+1,p}$; the natural homomorphism $\Sigma_{g,b+1,p} \to \check{\Sigma}_{g,b,p}$; the twisting-number map $\Sigma_{g,b+1,p} \to \mathbb{Z}$.

We apply our results to verify Matsumoto's simplification of Wajnryb's presentation of $\mathrm{Out}^{+}_{g,0,0}$.

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1 Topological background

Let $(g, b, p) \in \mathbb{N}^3$ be fixed throughout the article.

Let $S_{g,b,p}$ denote a connected, orientable surface which has genus g, and b boundary components, and p punctures.

Let α be a self-homeomorphism of $S_{g,b,p}$, and let $\partial\alpha$ denote the induced self-homeomorphism of the boundary $\partial S_{g,b,p}$. We shall now describe restrictions we wish to impose on α. If α is orientation-preserving, we want $\partial\alpha$ to act as the identity on each component of $\partial S_{g,b,p}$ that is carried to itself. But we want to allow certain orientation-reversing maps and maps that permute boundary components; this will give a better interplay between algebra and topology. We consider a referential S^1, say the complex numbers of modulus 1, and endow S^1 with an orientation-reversing map of order two, say complex conjugation. Together with the identity map, this gives a group of two distinguished self-homeomorphisms of S^1. We further specify a homeomorphism between each component of $\partial S_{g,b,p}$ and S^1, and thus obtain a b-to-1 map $\partial S_{g,b,p} \to S^1$. Then α is *admissible* if $\partial\alpha$ forms part of a commuting square in which two of the sides are the specified b-to-1 map $\partial S_{g,b,p} \to S^1$, and the remaining side is one of our two distinguished self-homeomorphisms of S^1. Thus, if $b \geqslant 1$, we allow exactly $2b!$ possibilities for $\partial\alpha$. To avoid awkward exceptions, we think of the empty set as having two self-maps, one of which is orientation-preserving and the other is orientation-reversing. This peculiar sort of convention will be useful throughout, in algebraic, as well as topological, contexts.

The surface $S_{g,b,p}$ is formed by deleting b open discs and p points from an orientable surface of genus g. A point is a degenerate closed disc. For our purposes, it is often convenient to view the $b + p$ discs as being distinguished, rather than deleted. From this viewpoint, we are looking at self-homeomorphisms of the orientable surface of genus g which act on the $b+p$ discs in one of $2b!p!$ possible ways. Thus, for example, we shall speak of the permutation of the set of punctures induced by a self-homeomorphism of $S_{g,b,p}$.

The group of all admissible self-homeomorphisms of $S_{g,b,p}$ contains a normal subgroup consisting of those self-homeomorphisms which can be homotoped to the identity map through self-homeomorphisms which act as the identity on the boundary and on the set of punctures. The resulting quotient group is called the *mapping-class group* of $S_{g,b,p}$, denoted $\mathcal{MC}_{g,b,p}$. Thus the elements of $\mathcal{MC}_{g,b,p}$ are isotopy classes of admissible self-homeomorphisms of $S_{g,b,p}$.

Let $\mathcal{MC}^+_{g,b,p}$ denote the index-two subgroup of $\mathcal{MC}_{g,b,p}$ consisting of isotopy classes of orientation-preserving admissible self-homeomorphisms.

We can choose an orientation-reversing element of order two in $\mathcal{MC}_{g,b,p}$, and express $\mathcal{MC}_{g,b,p}$ as a semidirect product $\mathcal{MC}_{g,b,p} \simeq \mathcal{MC}^+_{g,b,p} \rtimes C_2$; we write C_2 to denote a cyclic, multiplicative group, of order two.

Terminology is not standardized in this area; for example, some authors call $\mathcal{MC}^+_{g,0,b}$ the mapping class group of $S_{g,b,0}$, and call $\mathcal{MC}_{g,0,b}$ the extended mapping class group of $S_{g,b,0}$.

Let us return to $S_{g,b,p}$. Recall that the $b+p$ discs can be viewed as either distinguished or deleted. We pass to a different surface if one of the discs is, in one viewpoint, made undistinguished, and, in the other viewpoint, filled in. We shall call this process *elimination* of a puncture or boundary component. Another well-behaved process is that of undistinguishing, or filling in, all but one point of an open disc. We call this process *converting* a boundary component to a puncture. Converting a boundary component to a puncture can also be viewed as contracting a distinguished open disc, and its boundary, to a distinguished point. Notice that converting a boundary component to a puncture and then eliminating the puncture is equivalent to eliminating the boundary component; this simple observation will be used frequently.

Each of the three processes mentioned above determines a surjective partial homomorphism of mapping-class groups, with domain of finite index, namely the stabilizer of the disc involved. Moreover, the kernel can be described explicitly; see, for example, [7, Sections 2.8 and 6.3]. If we eliminate a boundary component of $S_{g,b+1,p}$, we get an exact sequence of groups

$$1 \to \check{\Sigma}_{g,b,p} \to \mathcal{MC}_{g,b\perp 1,p} \to \mathcal{MC}_{g,b,p} \to 1.$$

Here the notation $b \perp 1$ indicates that the domain of the partial homomorphism $\mathcal{MC}_{g,b+1,p} \rightsquigarrow \mathcal{MC}_{g,b,p}$ consists of isotopy classes of admissible self-homeomorphisms which carry the $b+1$st boundary component to itself. If (g,b,p) is $(0,0,0)$ or $(0,0,1)$, then $\check{\Sigma}_{g,b,p}$ is trivial. If (g,b,p) is $(0,0,2)$ or $(1,0,0)$, then $\check{\Sigma}_{g,b,p}$ is infinite cyclic. In all other cases, $\check{\Sigma}_{g,b,p}$ is the fundamental group of the unit-tangent bundle of $S_{g,b,p}$, a certain central extension of $\Sigma_{g,b,p}$.

Let us briefly discuss the case $(b,p) = (0,0)$, which is mentioned in [7, Section 6.3]. We have a surjective map $\phi_g\colon \mathcal{MC}_{g,1,0} \to \mathcal{MC}_{g,0,0}$. The kernel, $\check{\Sigma}_{g,0,0}$, is also the kernel of the induced map $\phi_g^+\colon \mathcal{MC}^+_{g,1,0} \to \mathcal{MC}^+_{g,0,0}$. For $g \in \{0,1\}$, these maps are well understood: ϕ_0 is an isomorphism of groups of order two, and ϕ_1^+ is equivalent to the quotient map from the braid group on three strings to the braid group on three strings modulo the square of its infinite cyclic center. Now suppose that $g \geqslant 2$. Here, $\check{\Sigma}_{g,0,0}$ is an extension of an infinite cyclic (central) subgroup by $\Sigma_{g,0,0}$. Wajnryb [19] showed that, as a normal subgroup of $\mathcal{MC}^+_{g,1,0}$, $\check{\Sigma}_{g,0,0}$ is generated by a single element.

The main part of Wajnryb's article gives a presentation of $\mathcal{MC}^+_{g,1,0}$, and the foregoing step is then used to show that adding a single relator yields a presentation of $\mathcal{MC}^+_{g,0,0}$. (See [20] for an elementary, self-contained version, in which the length of the added relator drops from $32g^2 + 8g - 26$ to $8g^2+6$.) Matsumoto [12] gave a relator of length $8g^2-10g+2$ using a simple description in terms of the center of an Artin group that is involved in the presentation. His proof that his word is suitable as a relator uses the theory of miniversal deformations; Labruère-Paris [8, Proposition 2.12(i)] have given a topological approach. (The complementary part of Matsumoto's article gives an aesthetic presentation of $\mathcal{MC}^+_{g,1,0}$ in terms of Artin groups and their centers; the proof uses Wajnryb's presentation and computer verifications, and the only values of g involved are 2 and 3. It may be possible to obtain these results more directly using the approach of Benvenuti [2].)

In this article, we present these facts from an algebraic viewpoint. In Sections 2–3 and Theorem 6.6, we give an algebraic description of $\mathcal{MC}_{g,b,p}$ by adjusting a well-known algebraic description of $\mathcal{MC}_{g,0,b+p}$; this adjustment is closely related to a similar definition given by Levitt [10, Section 4]. In Sections 4–10, we give algebraic verifications of some isomorphisms, exact sequences, and semidirect product decompositions, which arise from eliminating boundaries and punctures. In Section 11, we give an interesting description of the action of $\mathcal{MC}_{g,b\perp 1,p}$ on the normal subgroup $\check{\Sigma}_{g,b,p}$. We then turn to applications. Sections 12–13 briefly review Artin diagrams. Now let $g \geqslant 2$. In Section 14, we use the algebraic setting to give a brute-force proof that Matsumoto's word is suitable as a relator in a presentation of $\mathcal{MC}^+_{g,0,0}$. In Sections 15–16, we use many of our results to construct a partial embedding $\mathcal{MC}_{g-1,1,1} \rightsquigarrow \mathcal{MC}_{g,1,0}$ that sheds some light on Matsumoto's word.

2 Surface groups and related groups

2.1 Notation. For group elements x,y, we write $\overline{x} = x^{-1}$, $x^y = \overline{y}xy$, $[x, y] = \overline{x}\,\overline{y}xy$, and $[x]$ for the conjugacy class of x, usually thought of as a cyclic word in a specified generating set.

All actions will be on the right, generally written as exponents, and compositions are to be read from left to right. We write $x^{-\alpha} = \overline{x}^{\alpha}$. □

Recall that $(g, b, p) \in \mathbb{N}^3$.

2.2 Definitions. We define $\chi_{g,b,p} = 2 - 2g - b - p$ and

$$X_{g,b,p} = \{x_i, y_i \mid 1 \leqslant i \leqslant g\} \;\cup\; \{z_j \mid 1 \leqslant j \leqslant b\} \;\cup\; \{t_k \mid 1 \leqslant k \leqslant p\}.$$

This set will be involved in the presentations of the surface groups. The symbols z_j and t_k correspond to simple closed curves which start and finish at the base point of the surface and isolate the jth boundary component and the kth puncture, respectively.

Where we wish to blur the distinction between boundary components and punctures, it will be useful to set

$$z_{b+k} = t_{p+1-k} \ (1 \leqslant k \leqslant p), \quad t_{p+j} = z_{b+1-j}, \ (1 \leqslant j \leqslant b), \tag{1}$$

and then we write

$$X_{g,b+p} = \{x_i, y_i \mid 1 \leqslant i \leqslant g\} \ \cup \ \{z_j \mid 1 \leqslant j \leqslant b+p\} = X_{g,b,p}.$$

In the free group on $X_{g,b,p}$, we set

$$w_{g,b,p} = \prod_{i=1}^{g}[x_i, y_i] \cdot \prod_{j=1}^{b} z_j \cdot \prod_{k=p}^{1} t_k,$$

and also write

$$w_{g,b+p} = \prod_{i=1}^{g}[x_i, y_i] \cdot \prod_{j=1}^{b+p} z_j = w_{g,b,p}.$$

We define the (g,b,p)-*surface group* as

$$\Sigma_{g,b,p} = \langle X_{g,b,p} \mid w_{g,b,p} \rangle,$$

and also write

$$\Sigma_{g,b+p} = \langle X_{g,b+p} \mid w_{g,b+p} \rangle = \Sigma_{g,b,p}. \quad \square$$

2.3 Remarks. Notice that $\Sigma_{g,b,p}$ is trivial if and only if $2g+b+p \leqslant 1$, that is $(g,b,p) = (0,0,0)$, $(0,1,0)$ or $(0,0,1)$. These three values correspond to $S_{g,b,p}$ being a sphere, a closed disc, and an open disc, respectively. We call these three values the *trivial cases*. The remaining values of (g,b,p) will be called the *nontrivial cases*.

Similarly, $\Sigma_{g,b,p}$ is abelian if and only if $2g+b+p \leqslant 2$. As well as the trivial cases, we have $2g+b+p = 2$, that is $(g,b,p) = (1,0,0)$, $(0,2,0)$, $(0,1,1)$ or $(0,0,2)$. These four values correspond to $S_{g,b,p}$ being a torus, a closed annulus, a punctured closed disc, and a punctured open disc (or open annulus), respectively.

If $2g+b+p \geqslant 3$, then $\Sigma_{g,b,p}$ has trivial center.

Denoting the center of $\Sigma_{g,b,p}$ by Ctr, we have

$$\Sigma_{g,b,p}/\mathrm{Ctr} = \begin{cases} 1 & \text{if} \quad 2g+b+p \leqslant 2, \\ \Sigma_{g,b,p} & \text{if} \quad 2g+b+p \neq 2. \end{cases}$$

If $b+p \geqslant 1$, then $w_{g,b,p}$ can be extended to a basis of the free group on $X_{g,b,p}$; here, $\Sigma_{g,b,p}$ is a free group of rank $2g+b+p-1 = 1-\chi_{g,b,p}$. □

2.4 Definitions. In the tensor-ring of the abelianization of $\Sigma_{g,b+p}$, let $\Omega_{g,b+p}$ denote the (infinite cyclic) additive subgroup generated by

$$\begin{cases} 1 & \text{if} \quad (g,b+p) \in \{(0,0),(0,1)\}, \\ \sum_{i=1}^{g}(x_i \otimes y_i - y_i \otimes x_i) & \text{if} \quad b+p = 0, g \geqslant 1, \\ \sum_{j=1}^{b+p} z_j & \text{if} \quad b+p \geqslant 1, (g,b+p) \neq (0,1). \end{cases}$$

There is an isomorphism

$$\text{sign}\colon \operatorname{Aut}(\Omega_{g,b+p}) \xrightarrow{\sim} \{1,-1\},$$

and any $\alpha \in \operatorname{Aut}(\Omega_{g,b+p})$ acts on $\Omega_{g,b+p}$ by multiplication by $\text{sign}(\alpha)$. □

We now define some groups closely related to $\Sigma_{g,b,p}$, and later they will be seen to arise naturally.

2.5 Definition. We write

$$\tilde{\Sigma}_{g,b,p} = \begin{cases} 1 & \text{if} \quad (g,b,p) = (1,0,0) \text{ or } (0,0,2), \\ \Sigma_{g,b,p} & \text{otherwise.} \end{cases}$$

We view $\tilde{\Sigma}_{g,b,p}$ as being given as a quotient group of $\Sigma_{g,b,p}$.

Observe that $(g,b+p) = (0,2)$ is the only case where "$\tilde{\Sigma}_{g,b+p}$" is not defined. □

2.6 Definition. Let σ be a new symbol, and let $\check{\Sigma}_{g,b,p}$ denote the group presented with generating set $X_{g,b,p} \cup \{\sigma\}$ and relations saying the following:

(i) σ is central;

(ii) $w_{g,b,p} = \sigma^{\chi_{g,b,p}}$ (or $[w_{g,b,p}] = \sigma^{2-2g-b-p}$ where $w_{g,b,p}$ can be written as a cyclic word because σ is central);

(iii) $\sigma = 1$ (or $\check{\Sigma}_{g,b,p} = \{1\}$) if (g,b,p) is $(0,0,0)$ or $(0,0,1)$;

(iv) $X_{g,b,p} = \{1\}$ if (g,b,p) is $(1,0,0)$ or $(0,0,2)$.

Here "$\check{\Sigma}_{g,b+p}$" is not defined for $(g,b+p) = (0,1)$ or $(0,2)$.

We view $\tilde{\Sigma}_{g,b,p}$ as being given as a quotient group of $\check{\Sigma}_{g,b,p}$, namely $\check{\Sigma}_{g,b,p}/\langle\sigma\rangle = \tilde{\Sigma}_{g,b,p}$. □

2.7 Remarks. (i) We summarize the seven cases where $\Sigma_{g,b,p}$ is abelian:

$\Sigma_{0,0,0}=1$	$\tilde{\Sigma}_{0,0,0}=1$	$\check{\Sigma}_{0,0,0}=1$
$\Sigma_{0,0,1}=1$	$\tilde{\Sigma}_{0,0,1}=1$	$\check{\Sigma}_{0,0,1}=1$
$\Sigma_{0,1,0}=1$	$\tilde{\Sigma}_{0,1,0}=1$	$\check{\Sigma}_{0,1,0}=\langle\sigma \mid \ \rangle$
$\Sigma_{0,0,2}=\langle t_1 \mid \ \rangle$	$\tilde{\Sigma}_{0,0,2}=1$	$\check{\Sigma}_{0,0,2}=\langle\sigma \mid \ \rangle$
$\Sigma_{0,1,1}=\langle z_1 \mid \ \rangle$	$\tilde{\Sigma}_{0,1,1}=\langle z_1 \mid \ \rangle$	$\check{\Sigma}_{0,1,1}=\langle z_1,\sigma \mid [z_1,\sigma]=1\rangle$
$\Sigma_{0,2,0}=\langle z_1 \mid \ \rangle$	$\tilde{\Sigma}_{0,2,0}=\langle z_1 \mid \ \rangle$	$\check{\Sigma}_{0,2,0}=\langle z_1,\sigma \mid [z_1,\sigma]=1\rangle$
$\Sigma_{1,0,0}=\langle x_1,y_1 \mid [x_1,y_1]=1\rangle$	$\tilde{\Sigma}_{1,0,0}=1$	$\check{\Sigma}_{1,0,0}=\langle\sigma \mid \ \rangle$.

(ii) If $g \geqslant 2$, then $\check{\Sigma}_{g,0,0}$ contains the universal central extension of $\Sigma_{g,0,0}$ as a subgroup of index $2g-2=-\chi_{g,0,0}$.

(iii) Suppose $b+p \geqslant 1$, so $\tilde{\Sigma}_{g,b,p}$ is a free group. We have $\check{\Sigma}_{0,0,1} = \tilde{\Sigma}_{0,0,1} = 1$. If $(g,b,p) \neq (0,0,1)$, then $\check{\Sigma}_{g,b,p} \simeq \tilde{\Sigma}_{g,b,p} \times \langle\sigma \mid \ \rangle$. □

2.8 Definitions. We now introduce a set of new symbols, $E_b = \{e_j \mid 1 \leqslant j \leqslant b\}$. Intuitively, e_j corresponds to a simple curve in $S_{g,b,p}$ joining the base point of the surface to the basepoint of the jth boundary component. Formally, the e_j are elements of a groupoid having $\Sigma_{g,b,p}$ as basepoint group. However, we shall treat the e_j group theoretically to avoid dealing with groupoids.

Let

$$\Sigma_{g,b,p} * E_b = \langle X_{g,b,p} \cup E_b \mid w_{g,b,p}\rangle = \Sigma_{g,b,p} * \langle E_b \mid \ \rangle.$$

We define

$$\mathrm{Aut}_{g,b,p} = \mathrm{Aut}(\Sigma_{g,b,p} * E_b,\ \Sigma_{g,b,p},\ \{z_j^{\pm e_j}\}_{j=1}^{b},\ \{[t_k]^{\pm 1}\}_{k=1}^{p},\ \Omega_{g,b+p}), \quad (2)$$

as follows.

In the nontrivial cases, (2) is understood to mean that $\mathrm{Aut}_{g,b,p}$ is the group consisting of those automorphisms of $\Sigma_{g,b,p} * E_b$ which map each of the following three sets onto itself:

$$\Sigma_{g,b,p}, \quad \{z_j^{e_j}, \bar{z}_j^{e_j} \mid 1 \leqslant j \leqslant b\}, \quad \{[t_k],[\bar{t}_k] \mid 1 \leqslant k \leqslant p\}.$$

It can be shown that $\mathrm{Aut}_{g,b,p}$ then acts on the (infinite cyclic) group $\Omega_{g,b+p}$ introduced in Definitions 2.4.

In the trivial cases, (2) is understood to mean that $\mathrm{Aut}_{g,b,p}$ is $\mathrm{Aut}(\Omega_{g,b+p})$, acting trivially on $\Sigma_{g,b,p} * E_b$.

In all cases, $\mathrm{Aut}_{g,b,p}$ acts on $\Sigma_{g,b,p} * E_b$ and on $\Omega_{g,b+p}$. Moreover, an element of $\mathrm{Aut}_{g,b,p}$ is completely specified by its actions on $\Sigma_{g,b,p} * E_b$ and

$\Omega_{g,p+b}$; the former is determinative in the nontrivial cases and the latter in the trivial cases.

We define

$$\text{sign}\colon \operatorname{Aut}_{g,b,p} \to \{+1,-1\}$$

to be the surjective composite $\operatorname{Aut}_{g,b,p} \to \operatorname{Aut}(\Omega_{g,b+p}) \xrightarrow{\sim} \{+1,-1\}$. The preimage of $+1$ is denoted $\operatorname{Aut}^{+}_{g,b,p}$, and its elements are said to be *positive*. The preimage of -1 is denoted $\operatorname{Aut}^{-}_{g,b,p}$, and its elements are said to be *negative*. □

We shall be working with $\operatorname{Aut}_{g,b,p}$ throughout the article. Notation similar to that in the right-hand side of (2) will be used later, and the correct interpretation should always be obvious.

2.9 Remark. Suppose α is an element of $\operatorname{Aut}^{+}_{g,b,p}$ which sends $z_j^{e_j}$ to $z_{j'}^{e_{j'}}$, for some $1 \leqslant j, j' \leqslant b$.

Clearly, α carries $[z_j]$ to $[z_{j'}]$.

Writing $A_j = e_j^{\alpha}$ and $Z_j = z_j^{\alpha}$, we have $Z_j^{A_j} = z_{j'}^{e_{j'}}$ and

$$z_j^{\alpha} = Z_j = z_{j'}^{e_{j'}\overline{A}_j}.$$

If α fixes z_j, then

$$e_j^{\alpha} = A_j \in \langle z_j \rangle e_j = e_j \langle z_j^{e_j} \rangle,$$

since $j' = j$ and, in the nontrivial cases, the centralizer of z_j in the free group $\Sigma_{g,b,p} * E_b$ is $\langle z_j \rangle$.

We shall use these properties throughout. □

We now describe some special elements of $\operatorname{Aut}_{g,b,p}$.

2.10 Definitions. For $1 \leqslant i \leqslant g$, we define α_i, $\beta_i \in \operatorname{Aut}^{+}_{g,b,p}$ by

$$\alpha_i\colon \begin{cases} x_i \mapsto \overline{y}_i x_i, & \\ w \mapsto w & \text{for all } w \in E_b \cup X_{g,b,p} - \{x_i\}, \end{cases}$$

$$\beta_i\colon \begin{cases} y_i \mapsto x_i y_i, & \\ w \mapsto w & \text{for all } w \in E_b \cup X_{g,b,p} - \{y_i\}. \end{cases}$$

For $1 \leqslant i \leqslant g-1$, we define $\gamma_i \in \operatorname{Aut}^{+}_{g,b,p}$ by

$$\gamma_i\colon \begin{cases} x_i \mapsto \overline{x}_{i+1} y_{i+1} x_{i+1} \overline{y}_i x_i, \\ y_i \mapsto \overline{x}_{i+1} y_{i+1} x_{i+1} y_i \overline{x}_{i+1} \overline{y}_{i+1} x_{i+1} = y_i^{\overline{y}_{i+1}^{x_{i+1}}}, \\ x_{i+1} \mapsto x_{i+1} y_i \overline{x}_{i+1} \overline{y}_{i+1} x_{i+1}, \\ w \mapsto w \quad \text{for all } w \in E_b \cup X_{g,b,p} - \{x_i, y_i, x_{i+1}\}. \end{cases}$$

If $g \geqslant 1$ and $b+p \geqslant 1$, we define γ_0, $\gamma_g \in \mathrm{Aut}^+_{g,b,p}$ by

$$\gamma_0\colon \begin{cases} x_1 \mapsto x_1 z_{b+p}\overline{x}_1\overline{y}_1 x_1, \\ z_{b+p} \mapsto \overline{x}_1 y_1 x_1 z_{b+p}\overline{x}_1\overline{y}_1 x_1 = z_{b+p}^{\overline{y}_1^{x_1}}, \\ e_{b+p} \mapsto \overline{x}_1 y_1 x_1 \overline{z}_{b+p} e_{b+p} \quad \text{if} \quad p=0, \\ w \mapsto w \quad \text{for all } w \in (X_{g,b,p} \cup E_b) - \{x_1, z_{b+p}, e_{b+p}\}. \end{cases}$$

$$\gamma_g\colon \begin{cases} x_g \mapsto \overline{z}_1\overline{y}_g x_g, \\ y_g \mapsto \overline{z}_1 y_g z_1 = y_g^{z_1}, \\ z_1 \mapsto \overline{z}_1\overline{y}_g z_1 y_g z_1 = z_1^{y_g z_1}, \\ e_1 \mapsto \overline{z}_1\overline{y}_g e_1 \quad \text{if} \quad b \geqslant 1, \\ w \mapsto w \quad \text{for all } w \in (X_{g,b,p} \cup E_b) - \{x_g, y_g, z_1, e_1\}. \end{cases}$$

We call the α_i, β_i and γ_i (distinguished) *Dehn twist automorphisms.*

For $1 \leqslant j < b$, we define $\tau_j \in \mathrm{Aut}^+_{g,b,p}$ by

$$\tau_j : \begin{cases} z_j \mapsto z_{j+1}, \quad e_j \mapsto e_{j+1}, \\ z_{j+1} \mapsto z_j^{z_{j+1}}, \quad e_{j+1} \mapsto \overline{z}_{j+1}\,\overline{z}_j e_j, \\ w \quad \mapsto w \qquad \text{for all } w \in (X_{g,b,p} \cup E_b) - \{z_j, e_j, z_{j+1}, e_{j+1}\}. \end{cases}$$

For $1 \leqslant k < p$, we define $\mu_k \in \mathrm{Aut}^+_{g,b,p}$ by

$$\mu_k : \begin{cases} t_k \quad \mapsto t_{k+1}, \\ t_{k+1} \mapsto t_k^{t_{k+1}}, \\ w \quad \mapsto w \qquad \text{for all } w \in E_b \cup X_{g,b,p} - \{t_k, t_{k+1}\}. \end{cases}$$

We call the τ_j and μ_k *braid automorphisms.*

For $1 \leqslant j \leqslant b$, we define $\sigma_j \in \mathrm{Aut}^+_{g,b,p}$ by

$$\sigma_j : \begin{cases} e_j \mapsto \overline{z}_j e_j = e_j \overline{z}_j^{e_j}, \\ w \mapsto w \qquad \text{for all } w \in X_{g,b,p} \cup E_b - \{e_j\}. \end{cases}$$

We call σ_j a *boundary Dehn twist automorphism.*

Recall (1), and set

$$w_{g,j} = \prod_{i=1}^{g}[x_i, y_i] \cdot \prod_{j'=1}^{j} z_{j'}, \quad 0 \leqslant j \leqslant b+p.$$

Even in the trivial cases, we define an order-two element ζ of $\mathrm{Aut}^{-}_{g,b,p}$ which acts on $\Sigma_{g,b+p} * E_b$ by

$$\zeta : \begin{cases} x_i \mapsto y_{g-i}, \quad y_i \mapsto x_{g-i}, & i = 1, \dots, g, \\ z_j \mapsto \overline{z}_j^{\,\overline{w}_{g,j-1}}, & j = 1, \dots, b+p, \\ e_j \mapsto w_{g,j-1} e_j, & j = 1, \dots, b. \end{cases}$$

It is not difficult to show that $\mathrm{Aut}_{g,b,p} = \mathrm{Aut}^{+}_{g,b,p} \rtimes \langle \zeta \rangle$. □

3 The algebraic mapping-class group

3.1 Definition. Consider an element w of $\Sigma_{g,b,p}$. There is an associated element $\widetilde{w}$ of $\mathrm{Aut}^{+}_{g,b,p}$ which (right) conjugates each element of $X_{g,b,p}$ by w, and left multiplies each element of E_b by w^{-1}. It can be shown that we have a homomorphism

$$\Sigma_{g,b,p} \to \mathrm{Aut}_{g,b,p}, \quad w \mapsto \widetilde{w}. \tag{3}$$

Moreover the image, as quotient of $\Sigma_{g,b,p}$, is isomorphic to the group $\widetilde{\Sigma}_{g,b,p}$, introduced in Definition 2.5. The image of (3) will again be denoted $\widetilde{\Sigma}_{g,b,p}$.

It is not difficult to see that $\widetilde{\Sigma}_{g,b,p}$ is normal in $\mathrm{Aut}_{g,b,p}$, and that the action of $\mathrm{Aut}_{g,b,p}$ on $\widetilde{\Sigma}_{g,b,p}$ by conjugation agrees with the action of $\mathrm{Aut}_{g,b,p}$ on $\widetilde{\Sigma}_{g,b,p}$ induced from the action on $\Sigma_{g,b,p}$.

The quotient group will be denoted $\mathrm{Out}_{g,b,p} = \mathrm{Aut}_{g,b,p} / \widetilde{\Sigma}_{g,b,p}$, and the quotient map by

$$\mathrm{Aut}_{g,b,p} \to \mathrm{Out}_{g,b,p}, \quad \phi \mapsto \breve{\phi}.$$

We call $\mathrm{Out}_{g,b,p}$ the *algebraic* (g,b,p)*-mapping-class group.*

We define $\mathrm{Out}^{+}_{g,b,p} = \mathrm{Aut}^{+}_{g,b,p} / \widetilde{\Sigma}_{g,b,p}$, a subgroup of index two in $\mathrm{Out}_{g,b,p}$. □

3.2 Remark. There is a natural map $\mathcal{MC}_{g,b,p} \to \mathrm{Out}_{g,b,p}$ which carries $\mathcal{MC}^{+}_{g,b,p}$ to $\mathrm{Out}^{+}_{g,b,p}$. In Theorem 6.6, we shall see that these maps are isomorphisms. For $b = 0$ this was already known, and the case $b \geqslant 1$ will quickly be reduced to the case $b = 0$. See also Remarks 8.3(ii).

Notice that $\mathrm{Out}_{g,b,p} = \mathrm{Out}^{+}_{g,b,p} \rtimes \langle \breve{\zeta} \rangle$. Recall that a similar result holds for the topological mapping-class groups. □

3.3 Notation. Let Sym_b denote the group of permutations of $\{j \mid 1 \leqslant j \leqslant b\}$.

We have a natural homomorphism $\mathrm{Aut}_{g,b,p} \to \mathrm{Sym}_b \times \mathrm{Sym}_p$ arising from the action of $\mathrm{Aut}_{g,b,p}$ on $\{z_j^{\pm e_j}\}_{j=1}^{b} \cup \{[t_k]^{\pm 1}\}_{k=1}^{p}$. It is not difficult to show that this homomorphism is surjective.

Suppose that $(b_1, \ldots, b_m)$ and $(p_1, \ldots, p_n)$ are sequences of positive integers which sum to b and p, respectively. These determine an embedding

$$\prod_{j=1}^{m} \mathrm{Sym}_{b_j} \times \prod_{k=1}^{n} \mathrm{Sym}_{p_k} \to \mathrm{Sym}_b \times \mathrm{Sym}_p,$$

and we denote the image by $\mathrm{Sym}_{b_1 \perp \ldots \perp b_m} \times \mathrm{Sym}_{p_1 \perp \ldots \perp p_n}$. The preimage of the latter in $\mathrm{Aut}_{g,b,p}$ will be denoted

$$\mathrm{Aut}_{g, b_1 \perp \ldots \perp b_m, p_1 \perp \ldots \perp p_n}\,.$$

In the case where the b_j and p_k are all 1, we write $\mathrm{Aut}_{g,1^{\perp b},1^{\perp p}}$.

We leave it to the reader to define notation similar to the foregoing with Aut replaced by Out, Aut^+, and Out^+.

We remark that $\mathrm{Aut}_{g,b,p}$ acts on $\Omega_{g,b+p} \times \{[z_j]^{\pm 1}\}_{j=1}^{b+p}$. Here the kernel of the action is $\mathrm{Aut}^+_{g,1^{\perp b},1^{\perp p}}$, and the image of the action is isomorphic to $C_2 \times \mathrm{Sym}_{b \perp p}$. A similar statement holds for $\mathrm{Out}_{g,b,p}$. We call $\mathrm{Out}^+_{g,1^{\perp b},1^{\perp p}}$ the *pure* algebraic (g, b, p)-mapping-class group.

For $b \geqslant 1$, we write $\mathrm{Aut}_{g,b-1\perp\hat{1},p}$ to denote the subgroup of $\mathrm{Aut}_{g,b,p}$ consisting of the elements of $\mathrm{Aut}_{g,b,p}$ which fix e_b, and, hence fix $z_b^{\pm 1}$. For $p \geqslant 1$, we define $\mathrm{Aut}_{g,b,p-1\perp\hat{1}}$ to be the subgroup of $\mathrm{Aut}_{g,b,p}$ consisting of the elements of $\mathrm{Aut}_{g,b,p}$ which fix $t_p^{\pm 1}$. We shall give a simpler description of these subgroups in Remarks 5.2, and we shall see the following isomorphisms in Sections 4 and 7.

$$\begin{array}{lclcl}
\mathrm{Aut}_{g,b,p\perp\hat{1}} & \simeq & \mathrm{Aut}_{g,b\perp\hat{1},p} & \xrightarrow{\sim} & \mathrm{Out}_{g,b\perp 1,p}\,. \\[2ex]
\mathrm{Aut}_{g,b,p\perp\hat{1}}\,/\langle \widetilde{t}_{p+1}\rangle & \simeq & \mathrm{Aut}_{g,b\perp\hat{1},p}\,/\langle \sigma_{b+1}\,\tilde{z}_{b+1}^{-1}\rangle & \xrightarrow{\sim} & \mathrm{Out}_{g,b\perp 1,p}\,/\langle \breve{\sigma}_{b+1}\rangle. \\
\parallel & & & & \\
\mathrm{Aut}_{g,b,p\perp\hat{1}}\,/\langle \widetilde{t}_{p+1}\rangle & \xrightarrow{\sim} & \mathrm{Out}_{g,b,p\perp 1}\,. & & \\
\parallel & & & & \\
\mathrm{Aut}_{g,b,p\perp\hat{1}}\,/\langle \widetilde{t}_{p+1}\rangle & \xrightarrow{\sim} & \mathrm{Aut}_{g,b,p}\,. & &
\end{array}$$

□

4 Isomorphisms

Recall that, for $b \geqslant 1$, Definitions 2.10 give the boundary twist automorphism $\sigma_b \in \mathrm{Aut}_{g,b,p}$.

4.1 Proposition. *If $b \geqslant 1$, then $\Sigma_{g,b,p}$ is free of rank $2g+b+p-1$, there is a semidirect product decomposition*

$$\mathrm{Aut}_{g,b-1\perp 1,p} \;=\; \mathrm{Aut}_{g,b-1\perp \hat{1},p} \ltimes \Sigma_{g,b,p},$$

and there are isomorphisms

$$\mathrm{Out}_{g,b-1\perp 1,p} \xrightarrow{\sim} \mathrm{Aut}_{g,b-1\perp\hat{1},p} \xrightarrow{\sim} \mathrm{Aut}_{g,b-1,p\perp\hat{1}}$$

which carry $\breve{\sigma}_b$ first to $\sigma_b\tilde{z}_b^{-1}$ and then to $\tilde{t}_{p+1}^{-1}$.

Proof. Consider any $\alpha \in \mathrm{Aut}_{g,b-1\perp 1,p}$. Then α fixes $z_b^{e_b}$. By Remark 2.9, there is a unique $s_\alpha \in \Sigma_{g,b,p}$ such that $e_b^\alpha = s_\alpha e_b$. Now $\alpha\tilde{s}_\alpha$ fixes e_b. Moreover, $\tilde{\Sigma}_{g,b,p} = \Sigma_{g,b,p}$, since $b \geqslant 1$. Now the semidirect product decomposition is clear.

It follows that $\mathrm{Aut}_{g,b-1\perp\hat{1},p} \to \mathrm{Out}_{g,b-1\perp 1,p}$, $\alpha \mapsto \breve{\alpha}$, is an isomorphism. It is clear that $\sigma_b\tilde{z}_b^{-1}$ is mapped to $\breve{\sigma}_b$.

To see the second isomorphism, we note that

$$\begin{aligned}\mathrm{Aut}_{g,b-1\perp\hat{1},p} &= \mathrm{Aut}(\Sigma_{g,b,p} * E_b, \Sigma_{g,b,p}, \{z_j^{\pm e_j}\}_{j=1}^{b}, e_b, \{[t_k]^{\pm 1}\}_{k=1}^{p}, \Omega_{g,b+p}) \\ &= \mathrm{Aut}(\Sigma_{g,b,p} * E_b, \Sigma_{g,b,p}, \{z_j^{\pm e_j}\}_{j=1}^{b-1}, z_b^{\pm 1}, e_b, \{[t_k]^{\pm 1}\}_{k=1}^{p}, \Omega_{g,b+p}).\end{aligned}$$

The latter can be identified with

$$\mathrm{Aut}(\Sigma_{g,b,p} * E_{b-1}, \Sigma_{g,b,p}, \{z_j^{\pm e_j}\}_{j=1}^{b-1}, z_b^{\pm 1}, \{[t_k]^{\pm 1}\}_{k=1}^{p}, \Omega_{g,b+p}),$$

since there is a bijection between the group of all automorphisms of $\Sigma_{g,b,p} * E_{b-1}$ and the group of automorphisms of

$$\Sigma_{g,b,p} * E_b = (\Sigma_{g,b,p} * E_{b-1}) * \langle e_b \rangle$$

which fix e_b and map $\Sigma_{g,b,p} * E_{b-1}$ to itself. Under this bijection $\sigma_b\tilde{z}_b^{-1}$ corresponds to $\tilde{z}_b^{-1}$. On applying the relabelling $z_b = t_{p+1}$, we get $\mathrm{Aut}_{g,b-1,p\perp\hat{1}}$, and this justifies the second isomorphism. □

By applying the braid automorphisms, we can obtain similar results with any other boundary component fixed, although more notation would be required to state them.

Thus, if $b \geqslant 1$, then the index b subgroup $\mathrm{Out}_{g,b-1\perp 1,p}$ of $\mathrm{Out}_{g,b,p}$ is isomorphic to an automorphism group. This corresponds to contracting the simple curve e_b and using the base point of the bth boundary component as the base point of the surface.

If $p \geqslant 1$, then we can use the pth puncture as the base point of the surface and get the following.

4.2 Proposition. *If* $p \geqslant 1$, *then* $\mathrm{Out}_{g,b,p-1\perp 1} \simeq \mathrm{Aut}_{g,b,p-1\perp\hat{1}} / \langle \tilde{t}_p \rangle$.

Proof.
$$\begin{aligned}\mathrm{Aut}_{g,b,p-1\perp\hat{1}} / \langle \tilde{t}_p \rangle &= \mathrm{Aut}_{g,b,p-1\perp\hat{1}} / (\mathrm{Aut}_{g,b,p-1\perp\hat{1}} \cap \tilde{\Sigma}_{g,b,p}) \\ &\simeq (\mathrm{Aut}_{g,b,p-1\perp\hat{1}} \cdot \tilde{\Sigma}_{g,b,p}) / \tilde{\Sigma}_{g,b,p} \\ &= \mathrm{Aut}_{g,b,p-1\perp 1} / \tilde{\Sigma}_{g,b,p} \\ &= \mathrm{Out}_{g,b,p-1\perp 1}\,.\end{aligned}$$
□

5 A minor simplification

The following will be useful.

5.1 Lemma. *Let* G *be an arbitrary group, and let* α *be an endomorphism of the free product* $\Sigma_{g,b+p} * G$. *In a natural way,* α *induces a well-defined action on the set of all* $\Sigma_{g,b+p} * G$*-conjugacy classes. If* $b + p \geqslant 1$, *and* α *permutes* $\{[z_j]^{\pm 1}\}_{j=1}^{b+p}$, *then, on* $\Sigma_{g,b+p}$, α *acts as the composition of an automorphism of* $\Sigma_{g,b+p}$ *followed by conjugation by an element* c *of* $\Sigma_{g,b+p} * G$. *In this event, either* $\Sigma_{g,b+p}$ *is trivial, or, for each* $j \in \{1, \ldots, b+p\}$, *the coset* $\Sigma_{g,b+p}c$ *is uniquely determined by* $z_j^\alpha \in \Sigma_{g,b+p}^c$.

Proof. It suffices to prove the first part, since the last sentence then follows as in Remark 2.9.

We recall ζ and the braid automorphisms of Definitions 2.10. It is then clear that, by precomposing α with a suitable sequence of endomorphisms which act as automorphisms on $\Sigma_{g,b+p}$, we may assume that α fixes each element of $\{[z_j]\}_{j=1}^{b+p}$. By postcomposing α with conjugation by a suitable element of $\Sigma_{g,b+p} * G$, we may assume that α fixes z_{b+p}. Thus we can write

$$x_i^\alpha = X_i, \qquad y_i^\alpha = Y_i, \qquad z_j^\alpha = z_j^{C_j}, \qquad C_{b+p} = 1,$$
$$\overline{z}_{b+p} = \prod_{i=1}^{g}[x_i, y_i] \cdot \prod_{j=1}^{b+p-1} z_j, \qquad \overline{z}_{b+p}^\alpha = \prod_{i=1}^{g}[X_i, Y_i] \cdot \prod_{j=1}^{b+p-1} z_j^{C_j}.$$

Let H denote $\Sigma_{g,b+p}$ viewed as a free group with distinguished basis

$$\{x_i, y_i \mid 1 \leqslant i \leqslant g\} \ \cup\ \{z_j \mid 1 \leqslant j \leqslant b+p-1\}, \tag{4}$$

and let $F = H * G$. Now

$$H^\alpha = \langle X_i, Y_i, z_j^{C_j} \mid 1 \leqslant i \leqslant g,\ 1 \leqslant j \leqslant b+p-1 \rangle,$$

and we shall show that H^α contains (4), and hence contains H.

For any (right) derivation $\partial\colon F \to \mathbb{Z}[F]$, we have

$$\begin{aligned}
\overline{z}^{\partial}_{b+p} &= \sum_{i=1}^{g} (x_i^{\partial} \cdot y_i \cdot (1 - \overline{y}_i^{x_i y_i}) \cdot \prod_{i'=i+1}^{g} [x_{i'}, y_{i'}] \cdot \prod_{j=1}^{b+p-1} z_j) \\
&\quad + \sum_{i=1}^{g} (y_i^{\partial} \cdot (1 - x_i^{y_i}) \cdot \prod_{i'=i+1}^{g} [x_{i'}, y_{i'}] \cdot \prod_{j=1}^{b+p-1} z_j) \\
&\quad + \sum_{j=1}^{b+p-1} (z_j^{\partial} \cdot \prod_{j'=j+1}^{b+p-1} z_{j'}), \\
\overline{z}^{\alpha\partial}_{b+p} &= \sum_{i=1}^{g} (X_i^{\partial} \cdot Y_i \cdot (1 - \overline{Y}_i^{X_i Y_i}) \cdot \prod_{i'=i+1}^{g} [X_{i'}, Y_{i'}] \cdot \prod_{j=1}^{b+p-1} z_j^{C_j}) \\
&\quad + \sum_{i=1}^{g} (Y_i^{\partial} \cdot (1 - X_i^{Y_i}) \cdot \prod_{i'=i+1}^{g} [X_{i'}, Y_{i'}] \cdot \prod_{j=1}^{b+p-1} z_j^{C_j}) \\
&\quad + \sum_{j=1}^{b+p-1} (C_j^{\partial} \cdot (1 - z_j^{C_j}) \cdot \prod_{j'=j+1}^{b+p-1} z_{j'}^{C_{j'}}) \\
&\quad + \sum_{j=1}^{b+p-1} (z_j^{\partial} \cdot C_j \cdot \prod_{j'=j+1}^{b+p-1} z_{j'}^{C_{j'}}).
\end{aligned}$$

On examining the image of the latter under the map

$$\mathbb{Z}[F] \to \mathbb{Z}[F/H^{\alpha}], \quad r \mapsto r \cdot H^{\alpha},$$

we see $\overline{z}^{\alpha\partial}_{b+p} \cdot H^{\alpha} = \sum_{j=1}^{b+p-1} z_j^{\partial} \cdot C_j \cdot H^{\alpha}$. Since α fixes z_{b+p}, we have

$$\overline{z}^{\partial}_{b+p} \cdot H^{\alpha} = \sum_{j=1}^{b+p-1} z_j^{\partial} \cdot C_j \cdot H^{\alpha}. \tag{5}$$

For each element x of the distinguished free basis of H, we have a specified free product decomposition $F = \langle x \rangle * Q_x$ for a well-defined group Q_x, and hence we have the Fox derivative $\frac{\partial}{x\partial}\colon F \to \mathbb{Z}[F]$ uniquely determined by the fact that it sends x to 1 and vanishes on Q_x. (This can be described in terms of a decomposition of the augmentation ideal of $\mathbb{Z}[F]$ as a direct sum with one of the summands being $(x-1)\mathbb{Z}[F] \simeq \mathbb{Z}[F]$. It can also be described in terms of paths in the Bass-Serre tree for the expression of F as an HNN extension with vertex group Q_x and trivial edge group.)

For $1 \leqslant j \leqslant b+p-1$, taking $\partial = \frac{\partial}{z_j \partial}$ in (5), we get

$$\prod_{j'=j+1}^{b+p-1} z_{j'} \cdot H^\alpha = C_j \cdot H^\alpha,$$

so C_j lies in $\prod_{j'=j+1}^{b+p-1} z_{j'} \cdot H^\alpha$. Since $z_j^{C_j} \in H^\alpha$, we can inductively show that C_j and z_j lie in H^α for $b+p-1 \geqslant j \geqslant 1$.

For $1 \leqslant i \leqslant g$, taking $\partial = \frac{\partial}{x_i \partial}$ and then $\partial = \frac{\partial}{y_i \partial}$, in (5), we get

$$y_i \cdot (1 - \overline{y}_i^{x_i y_i}) \cdot \prod_{i'=i+1}^{g} [x_{i'}, y_{i'}] \cdot \prod_{j=1}^{b+p-1} z_j \cdot H^\alpha = 0,$$

$$(1 - x_i^{y_i}) \cdot \prod_{i'=i+1}^{g} [x_{i'}, y_{i'}] \cdot \prod_{j=1}^{b+p-1} z_j \cdot H^\alpha = 0,$$

respectively, so $\overline{y}_i^{x_i y_i}$ and $x_i^{y_i} = x_i^{x_i y_i}$, lie in $\prod_{i'=i+1}^{g} [x_{i'}, y_{i'}] \cdot H^\alpha$. By induction, $x_i y_i$, x_i and y_i lie in H^α for $g \geqslant i \geqslant 1$.

Thus $H^\alpha \geqslant H$. But H is a free factor of F, and hence of $H^\alpha \geqslant H$. Any free factor complementary to H in H^α can be collapsed to the trivial group, giving a surjective map $H^\alpha \xrightarrow{\text{collapse}} H$. The composition $H \xrightarrow{\alpha} H^\alpha \xrightarrow{\text{collapse}} H$ is a surjective endomorphism of the finitely generated free group H. By a theorem of Nielsen's, the endomorphism is bijective. Hence, both (surjective) factors are bijective. Thus, $H^\alpha \xrightarrow{\text{collapse}} H$ is injective, and this forces the complementary free factor to be trivial. Hence, $H^\alpha \xrightarrow{\text{collapse}} H$ is the identity map, and α acts as an automorphism on H. □

5.2 Remarks. Recall from Notation 3.3 that, for $p \geqslant 1$,

$$\mathrm{Aut}_{g,b,p-1\perp\hat{1}} = \mathrm{Aut}(\Sigma_{g,b,p} * E_b, \Sigma_{g,b,p}, \{z_j^{\pm e_j}\}_{j=1}^{b}, \{[t_k]^{\pm 1}\}_{k=1}^{p-1}, t_p^{\pm 1}, \Omega_{g,b+p}).$$

Notice that the restraint that $\Sigma_{g,b,p}$ be mapped to itself is redundant where $t_p^{\pm 1}$ is fixed, by Lemma 5.1. Thus

$$\mathrm{Aut}_{g,b,p-1\perp\hat{1}} = \mathrm{Aut}(\Sigma_{g,b,p} * E_b, \{z_j^{\pm e_j}\}_{j=1}^{b}, \{[t_k]^{\pm 1}\}_{k=1}^{p-1}, t_p^{\pm 1}, \Omega_{g,b+p}).$$

Similarly, for $b \geqslant 1$, an analogous remark holds for $\mathrm{Aut}_{g,b-1\perp\hat{1},p}$, since this is isomorphic to $\mathrm{Aut}_{g,b-1,p\perp\hat{1}}$, by Proposition 4.1. □

6 Converting a boundary to a puncture

In this section we construct short exact sequences of groups which correspond to converting boundary components to punctures.

6.1 Definition. Let $b \geqslant 1$.

There is a natural embedding

$$\Sigma_{g,b-1,p+1} * E_{b-1} \to \Sigma_{g,b,p} * E_b$$

which sends t_{p+1} to z_b, and we may view it as an inclusion because we understand $t_{p+1} = z_b$. Even in the trivial cases, pullback along this inclusion, or restriction, induces a unique sign-preserving map $\mathrm{Aut}_{g,b-1\perp 1,p} \to \mathrm{Aut}_{g,b-1,p\perp 1}$. The latter map carries $\widetilde{\widetilde{\Sigma}}_{g,b,p}$ to $\widetilde{\widetilde{\Sigma}}_{g,b-1,p+1}$, and hence determines a map

$$\mathrm{elim}(e_b)\colon\ \mathrm{Out}_{g,b-1\perp 1,p} \to \mathrm{Out}_{g,b-1,p\perp 1}\,.$$

It corresponds to converting the bth boundary component into a new puncture, the $p+1$st puncture.

By Proposition 4.1, we have isomorphisms

$$\mathrm{Out}_{g,b-1\perp 1,p} \simeq \mathrm{Aut}_{g,b-1\perp \hat{1},p} \simeq \mathrm{Aut}_{g,b-1,p\perp \hat{1}},$$

and quotient isomorphisms

$$\mathrm{Out}_{g,b-1\perp 1,p} /\langle \breve{\sigma}_b\rangle \simeq \mathrm{Aut}_{g,b-1\perp \hat{1},p} /\langle \sigma_b \widetilde{\widetilde{z}}_b^{\,-1}\rangle \simeq \mathrm{Aut}_{g,b-1,p\perp \hat{1}} /\langle \widetilde{\widetilde{t}}_{p+1}\rangle.$$

The latter group is isomorphic to $\mathrm{Out}_{g,b-1,p\perp 1}$, by Proposition 4.2. It is not difficult to see that we have a factorization of $\mathrm{elim}(e_b)$ as

$$\begin{aligned} \mathrm{Out}_{g,b-1\perp 1,p} &\xrightarrow{\sim} \mathrm{Aut}_{g,b-1\perp \hat{1},p} \\ &\xrightarrow{\sim} \mathrm{Aut}_{g,b-1,p\perp \hat{1}} \\ &\to \mathrm{Aut}_{g,b-1,p\perp \hat{1}} /\langle \widetilde{\widetilde{t}}_{p+1}\rangle \\ &\xrightarrow{\sim} \mathrm{Out}_{g,b-1,p\perp 1}\,. \end{aligned}$$

□

We make the resulting exact sequence explicit.

6.2 Proposition. *If $b \geqslant 1$, then there is an exact sequence*

$$1 \to \langle \breve{\sigma}_b\rangle \to \mathrm{Out}_{g,b-1\perp 1,p} \xrightarrow{\mathrm{elim}(e_b)} \mathrm{Out}_{g,b-1,p\perp 1} \to 1.$$

□

6.3 Notation. It is easy to check the following in $\mathrm{Aut}_{g,b,p}$.

$$\begin{array}{lll} \sigma_1 = 1 & \text{if} & (g,b,p) = (0,1,0); \\ \sigma_1 = \widetilde{z}_1 & \text{if} & (g,b,p) = (0,1,1); \\ \sigma_1 = \widetilde{z}_1\sigma_2 & \text{if} & (g,b,p) = (0,2,0). \end{array}$$

Let $B_{g,b,p} = \langle \sigma_j \mid 1 \leqslant j \leqslant b \rangle \leqslant \mathrm{Aut}_{g,b,p}$, and

$$\breve{B}_{g,b,p} = \langle \breve{\sigma}_j \mid 1 \leqslant j \leqslant b \rangle \leqslant \mathrm{Out}_{g,b,p}\,.$$

It is then not difficult to show that $B_{g,b,p}$ is presented as the abelian group with the given generators together with the relation $\sigma_1 = 1$ if $(g,b,p) = (0,1,0)$. Also, $\breve{B}_{g,b,p}$ is presented as the abelian group with the given generators together with the following relations:

$$\begin{array}{lll} \breve{\sigma}_1 = 1 & \text{if} & (g,b,p) = (0,1,0) \quad \text{or} \quad (0,1,1); \\ \breve{\sigma}_1 = \breve{\sigma}_2 & \text{if} & (g,b,p) = (0,2,0). \end{array}$$

For $1 \leqslant j \leqslant b$, σ_j can be thought of as right multiplying e_j by $\overline{z}_j^{e_j}$, and it follows that σ_j commutes with any element of $\mathrm{Aut}_{g,b,p}$ which fixes $z_j^{e_j}$. One can check that $B_{g,b,p}$ is normal in $\mathrm{Aut}_{g,b,p}$, and that, in the nontrivial cases, its centralizer is $\mathrm{Aut}^+_{g,1^{\perp b},p}$, a subgroup of index $2 \times b!$. Similar statements hold when Aut is replaced with Out. □

Repeated applications of Proposition 6.2, and extending by Sym_b, give the following.

6.4 Proposition. *There is an exact sequence*

$$1 \to \breve{B}_{g,b,p} \to \mathrm{Out}_{g,b,p} \xrightarrow{\mathrm{elim}(e_1,.,e_b)} \mathrm{Out}_{g,0,b\perp p} \to 1.$$

□

6.5 Remark. It is not difficult to show that there is also an exact sequence

$$1 \to B_{g,b,p} \to \mathrm{Aut}_{g,b,p} \to \mathrm{Aut}_{g,0,b\perp p} \to 1.$$

□

We next observe that the algebraic mapping-class group agrees with the usual mapping-class group. See also Remarks 8.3(ii).

6.6 Theorem. *The natural map* $\mathcal{MC}_{g,b,p} \to \mathrm{Out}_{g,b,p}$ *is an isomorphism.*

Proof. The Dehn twists around the boundary components determine an exact sequence

$$1 \to \breve{B}_{g,b,p} \to \mathcal{MC}_{g,b,p} \to \mathcal{MC}_{g,0,b+p} \to 1;$$

see [15, Theorem 4.1]. We claim that the natural map $\mathcal{MC}_{g,0,b+p} \to \mathrm{Out}_{g,0,b+p}$ is an isomorphism. In the trivial cases, this holds by our artificial definitions. In the nontrivial cases, this isomorphism is a deep, classic result; see [11] and [7, Theorem 2.9.A] for two different proofs. Now the desired result follows from Proposition 6.4 and the five lemma. □

7 More isomorphisms

7.1 Theorem. *If* $p \geqslant 1$, *then* $\mathrm{Aut}_{g,b,p-1\perp\hat{1}} / \langle \tilde{t}_p \rangle \simeq \mathrm{Aut}_{g,b,p-1}$.

Proof. Let us denote by $\mathrm{prelim}(t_p)$ the composite

$$\mathrm{Out}_{g,b,p-1\perp 1} \xrightarrow{\sim} \mathrm{Aut}_{g,b,p-1\perp\hat{1}} / \langle \tilde{t}_p \rangle \to \mathrm{Aut}_{g,b,p-1},$$

where it is understood that signs are preserved. It has long been known to topologists that when $b = 0$ this is an isomorphism; the history and references, together with an algebraic proof using [13], can be found in [5]. Hence

$$\mathrm{prelim}(t_{b+p})\colon \mathrm{Out}_{g,0,b+p-1\perp 1} \to \mathrm{Aut}_{g,0,b+p-1}$$

is an isomorphism, and restricting gives another isomorphism

$$\mathrm{prelim}(t_{b+p})\colon \mathrm{Out}_{g,0,b\perp p-1\perp 1} \xrightarrow{\sim} \mathrm{Aut}_{g,0,b\perp p-1}\,. \tag{6}$$

Notice that $\mathrm{prelim}(t_p)\colon \mathrm{Out}_{g,b,p-1\perp 1} \to \mathrm{Aut}_{g,b,p-1}$ carries the subgroup $\breve{B}_{g,b,p}$ of $\mathrm{Out}_{g,b,p-1\perp 1}$ onto the subgroup $B_{g,b,p-1}$ of $\mathrm{Aut}_{g,b,p-1}$; we claim it does so bijectively. This is clear if $B_{g,b,p-1}$ has rank b, so it remains to consider the case $(g, b, p-1) = (0, 1, 0)$, and here both groups are trivial, so the claim holds.

Now $\mathrm{prelim}(t_p)$ induces a map

$$\mathrm{Out}_{g,b,p-1\perp 1} / \breve{B}_{g,b,p} \to \mathrm{Aut}_{g,b,p-1} / B_{g,b,p-1},$$

which is an isomorphism, because it is equivalent to the isomorphism (6) by Proposition 6.4 and Remark 6.5.

Hence $\mathrm{prelim}(t_p)$ is itself an isomorphism. □

By Proposition 4.2, $\mathrm{Out}_{g,b,p-1\perp 1} \simeq \mathrm{Aut}_{g,b,p-1\perp\hat{1}} / \langle \tilde{t}_p \rangle$, so we have the following.

7.2 Corollary. *If* $p \geqslant 1$, *then* $\mathrm{Out}_{g,b,p-1\perp 1} \simeq \mathrm{Aut}_{g,b,p-1}$. □

Expressed another way this says $\mathrm{Aut}_{g,b,p} \simeq \mathrm{Out}_{g,b,p\perp 1}$.

8 Eliminating a puncture

In this section, we construct a short exact sequences of groups which corresponds to eliminating a puncture, and obtain partial splittings in some cases.

8.1 Definitions. Suppose that $p \geqslant 1$.

View $\Sigma_{g,b,p-1} * E_b$ as the quotient of $\Sigma_{g,b,p} * E_b$ by the normal subgroup generated by t_p.

There is induced a map

$$\mathrm{mod}(t_p)\colon\ \mathrm{Aut}_{g,b,p-1\perp 1} \to \mathrm{Aut}_{g,b,p-1}$$

which assigns, to any $\alpha \in \mathrm{Aut}_{g,b,p-1\perp 1}$, the element with the same sign and the induced action on the quotient group $\Sigma_{g,b,p-1} * E_b$ of $\Sigma_{g,b,p} * E_b$.

Now $\mathrm{mod}(t_p)$ carries $\tilde{\tilde{\Sigma}}_{g,b,p}$ to $\tilde{\tilde{\Sigma}}_{g,b,p-1}$, so induces a map

$$\mathrm{elim}(t_p)\colon\ \mathrm{Out}_{g,b,p-1\perp 1} \to \mathrm{Out}_{g,b,p-1}\,.$$

It is not difficult to see that we have a factorization of $\mathrm{elim}(t_p)$ as

$$\mathrm{Out}_{g,b,p-1\perp 1} \xrightarrow{\sim} \mathrm{Aut}_{g,b,p-1\perp \hat{1}}/\langle \tilde{t}_p\rangle \xrightarrow{\sim} \mathrm{Aut}_{g,b,p-1} \to \mathrm{Out}_{g,b,p-1}\,.$$

□

Since the kernel of $\mathrm{Aut}_{g,b,p-1} \to \mathrm{Out}_{g,b,p-1}$ is $\tilde{\tilde{\Sigma}}_{g,b,p-1}$, we get the following.

8.2 Corollary. *If $p \geqslant 1$, then there is an exact sequence*

$$1 \to \tilde{\tilde{\Sigma}}_{g,b,p-1} \to \mathrm{Out}_{g,b,p-1\perp 1} \xrightarrow{\mathrm{elim}(t_p)} \mathrm{Out}_{g,b,p-1} \to 1.$$

□

8.3 Remarks. (i) We shall describe the copy of $\tilde{\tilde{\Sigma}}_{g,b,p-1}$ in $\mathrm{Out}_{g,b,p-1\perp 1}$ explicitly in Remarks 9.6(iii).

(ii) We leave it as an exercise to show that $\mathrm{Out}_{g,b,p}$ has a descending subnormal series in which the sequence of isomorphism classes of factor groups is given by

$$\mathrm{Sym}_{b\perp p},\ \mathrm{Out}_{g,0,0},\ \tilde{\tilde{\Sigma}}_{g,0,0},\ \tilde{\tilde{\Sigma}}_{g,0,1},\ \ldots,\ \tilde{\tilde{\Sigma}}_{g,0,b+p-2},\ \tilde{\tilde{\Sigma}}_{g,0,b+p-1},\ \breve{B}_{g,b,p}.$$

A corresponding result is known for $\mathcal{M}_{g,b,p}$, and this gives another way to verify that $\mathcal{M}_{g,b,p} \simeq \mathrm{Out}_{g,b,p}$. □

By combining Propositions 4.1 and Theorem 7.1, we get a finite-index splitting for the exact sequence in Corollary 8.2, in some cases.

8.4 Corollary. *If $b \geqslant 1$ and $p \geqslant 1$, then $\Sigma_{g,b,p-1}$ is free of rank $2g+b+p-2$, and*

$$\begin{aligned} \mathrm{Out}_{g,b-1\perp 1,p-1\perp 1} &\simeq \mathrm{Aut}_{g,b-1\perp 1,p-1} \\ &= \mathrm{Aut}_{g,b-1|\hat{1},p-1} \ltimes \Sigma_{g,b,p-1} \\ &\simeq \mathrm{Out}_{g,b-1\perp 1,p-1} \ltimes \Sigma_{g,b,p-1}. \end{aligned}$$

□

We shall give another description of the splitting map

$$\mathrm{Out}_{g,b-1\perp 1,p-1} \to \mathrm{Out}_{g,b-1\perp 1,p-1\perp 1} \tag{7}$$

in Remarks 10.3.

The splitting can be iterated.

8.5 Corollary. *If $b \geqslant 1$, then*

$$\mathrm{Out}_{g,b-1\perp 1,1^{\perp p}} \simeq \mathrm{Out}_{g,b-1\perp 1,0} \ltimes \Sigma_{g,b,0} \ltimes \Sigma_{g,b,1} \ltimes \cdots \ltimes \Sigma_{g,b,p-1},$$

where $\Sigma_{g,b,i}$ is free of rank $2g+b+i-1$, for $i = 0, \ldots, p-1$. □

Here we use the convention that left parentheses are understood to accumulate on the left; for example, $A \ltimes B \ltimes C \ltimes D$ denotes $((A \ltimes B) \ltimes C) \ltimes D$.

9 Eliminating a boundary component

In this section, we construct a short exact sequence of groups which corresponds to eliminating a boundary component.

9.1 Definition. Suppose that $b \geqslant 1$.

In Definitions 6.1 and 8.1, we introduced the restriction and quotient maps

$$\begin{aligned} \mathrm{elim}(e_b)\colon\ \mathrm{Out}_{g,b-1\perp 1,p} &\to \mathrm{Out}_{g,b-1,p\perp 1}, \\ \mathrm{elim}(t_{p+1})\colon\ \mathrm{Out}_{g,b-1,p\perp 1} &\to \mathrm{Out}_{g,b-1,p}, \end{aligned}$$

respectively. Here $t_{p+1} = z_b$, and we define

$$\mathrm{elim}(z_b, e_b) = \mathrm{elim}(e_b) \cdot \mathrm{elim}(t_{p+1})\colon\ \mathrm{Out}_{g,b-1\perp 1,p} \to \mathrm{Out}_{g,b-1,p}\,.$$

□

This is a surjective map and we shall want to describe the kernel. Notice that $\mathrm{elim}(e_b)$ has kernel $\langle \breve{\sigma}_b \rangle$, and $\mathrm{elim}(t_{p+1})$ has kernel $\widetilde{\Sigma}_{g,b-1,p}$, so the kernel of $\mathrm{elim}(z_b, e_b)$ is an extension of $\langle \breve{\sigma}_b \rangle$ by $\widetilde{\Sigma}_{g,b-1,p}$, and we shall find that it is $\check{\Sigma}_{g,b-1,p}$. For convenience, we recall our factorizations of $\mathrm{elim}(e_b)$ and $\mathrm{elim}(t_{p+1})$ through automorphism groups, and deduce a corresponding factorization for $\mathrm{elim}(z_b, e_b)$.

$$\begin{array}{llll}
\mathrm{Out}_{g,b-1\perp 1,p} & \tilde{\to} & \mathrm{Aut}_{g,b-1\perp\hat{1},p} & \text{fix } e_b \\
 & \tilde{\to} & \mathrm{Aut}_{g,b-1,p\perp\hat{1}} & \text{ignore } e_b\text{, write } z_b = t_{p+1} \\
 & \to & \mathrm{Aut}_{g,b-1,p\perp\hat{1}} /\langle \widetilde{t}_{p+1} \rangle & \text{mod } \langle \widetilde{t}_{p+1} \rangle \\
 & \tilde{\to} & \mathrm{Out}_{g,b-1,p\perp 1} & \text{natural.}
\end{array}$$

$$\begin{array}{llll}
\mathrm{Out}_{g,b-1,p\perp 1} & \tilde{\to} & \mathrm{Aut}_{g,b-1,p\perp\hat{1}} /\langle \widetilde{t}_{p+1} \rangle & \text{fix } t_{p+1} \\
 & \tilde{\to} & \mathrm{Aut}_{g,b-1,p} & \text{kill } t_{p+1} \\
 & \to & \mathrm{Aut}_{g,b-1,p} /\widetilde{\Sigma}_{g,b-1,p} & \text{mod } \widetilde{\Sigma}_{g,b-1,p}. \\
 & \tilde{\to} & \mathrm{Out}_{g,b-1,p} & \text{natural.}
\end{array}$$

$$\begin{array}{llll}
\mathrm{Out}_{g,b-1\perp 1,p} & \tilde{\to} & \mathrm{Aut}_{g,b-1\perp\hat{1},p} & \text{fix } e_b \\
 & \tilde{\to} & \mathrm{Aut}_{g,b-1,p\perp\hat{1}} & \text{ignore } e_b\text{, write } z_b = t_{p+1} \\
 & \to & \mathrm{Aut}_{g,b-1,p\perp\hat{1}} /\langle \widetilde{t}_{p+1} \rangle & \text{mod } \langle \widetilde{t}_{p+1} \rangle \\
 & \tilde{\to} & \mathrm{Aut}_{g,b-1,p} & \text{kill } t_{p+1} \\
 & \to & \mathrm{Aut}_{g,b-1,p} /\widetilde{\Sigma}_{g,b-1,p} & \text{mod } \widetilde{\Sigma}_{g,b-1,p}. \\
 & \tilde{\to} & \mathrm{Out}_{g,b-1,p} & \text{natural.}
\end{array}$$

We now describe elements of $\mathrm{Aut}_{g,b-1\perp 1,p}$ which vanish in $\mathrm{Aut}_{g,b-1,p}$ if we ignore e_b and kill z_b. Moreover, if we use the above factorization, that is, first fixing e_b, we find these elements map to generators of $\widetilde{\Sigma}_{g,b-1,p}$ in $\mathrm{Aut}_{g,b-1,p}$.

9.2 Definitions. Suppose that $b \geqslant 1$.

We want to construct a homomorphism

$$\Sigma_{g,b,p} \to \mathrm{Aut}_{g,b,p}, \quad x \mapsto \hat{x}. \tag{8}$$

Recall that $\Sigma_{g,b,p}$ is free on $X_{g,b-1,p}$. We begin by defining, for each $x \in X_{g,b-1,p}$, an endomorphism $\hat{x}$ of the free group on $X_{g,b,p} \cup E_b$, namely, for $p \geqslant k \geqslant 1$, $1 \leqslant i \leqslant g$, $1 \leqslant j \leqslant b-1$, we define the following.

$$\hat{t}_k\colon \begin{cases} w \mapsto w^{[\bar{t}_k, \bar{z}_b]} & \text{for} \quad w = t_p, ., t_{k+1}, \\ t_k \mapsto t_k^{\bar{z}_b}, & \\ w \mapsto w & \text{for} \quad w = t_{k-1}, ., t_1, x_1, ., y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\bar{t}_k \bar{z}_b}, \quad e_b \mapsto z_b t_k e_b. & \end{cases}$$

$$\widehat{x}_i\colon \begin{cases} w \mapsto w^{[z_b,\overline{x}_i]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ x_i \mapsto x_i^{z_b\overline{x}_i}, \quad y_i \mapsto x_i\overline{z}_b\,\overline{x}_i y_i, & \\ w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{x}_i}, \quad e_b \mapsto x_i\overline{z}_b e_b. & \end{cases}$$

$$\widehat{y}_i\colon \begin{cases} w \mapsto w^{[\overline{y}_i,\overline{z}_b]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ x_i \mapsto z_b x_i[\overline{y}_i, \overline{z}_b], \quad y_i \mapsto y_i^{\overline{z}_b}, & \\ w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{y}_i\overline{z}_b}, \quad e_b \mapsto z_b y_i e_b. & \end{cases}$$

$$\widehat{z}_j\colon \begin{cases} w \mapsto w^{[\overline{z}_j,\overline{z}_b]} & \text{for} \quad w = t_p, ., t_1, x_1, ., y_g, z_1, ., z_{j-1}, \\ w \mapsto [\overline{z}_b, \overline{z}_j]w & \text{for} \quad w = e_1, ., e_{j-1}, \\ z_j \mapsto z_j^{\overline{z}_b}, \quad e_j \mapsto z_b e_j, & \\ w \mapsto w & \text{for} \quad w = z_{j+1}, e_{j+1}, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{z}_j\overline{z}_b}, \quad e_b \mapsto z_b z_j e_b. & \end{cases}$$

To facilitate the definitions, we postpone, to Lemma 9.3, the verification that, for each $x \in X_{g,b-1,p}$, the above $\widehat{x}$ determines an element of $\mathrm{Aut}^+_{g,b,p}$, again denoted $\widehat{x}$, and hence we have a homomorphism (8). There is then an induced homomorphism

$$\Sigma_{g,b,p} \to \mathrm{Out}_{g,b,p}, \quad x \mapsto \breve{x}.$$

It is easy to check that the composite

$$\mathrm{Out}_{g,b-1\perp 1,p} \xrightarrow{\sim} \mathrm{Aut}_{g,b-1\perp\hat{1},p} \to \mathrm{Aut}_{g,b-1,p}$$

carries $\breve{x}$ to $\widetilde{x}$, for each $x \in X_{g,b-1,p}$ Here one considers the unique $s_x \in \Sigma_{g,b,p}$ such that $e_b^{\widehat{x}} = s_x e_b$, and calculates the action of $\widehat{x}\,\widetilde{s}_x$ modulo the normal subgroup generated by z_b and e_b. □

9.3 Lemma. *Let $b \geqslant 1$ and $x \in X_{g,b-1,p}$. Then $\widehat{x}$ induces an element of $\mathrm{Aut}^+_{g,b-1\perp 1,p}$.*

Proof. We observe the following.

$$\widehat{t}_k\colon \begin{cases} w \mapsto w & \text{for} \quad w = t_{k-1}, ., t_1, x_1, ., y_g, z_1, ., z_{b-1}, \\ z_b \mapsto z_b^{\bar{t}_k \overline{z}_b} = z_b[\bar{t}_k, \overline{z}_b], & \\ w \mapsto w^{[\bar{t}_k, \overline{z}_b]} & \text{for} \quad w = t_p, ., t_{k+1}, \\ t_k \mapsto t_k^{\overline{z}_b} = [\overline{z}_b, \bar{t}_k]t_k. & \end{cases}$$

$$\widehat{x}_i\colon \begin{cases} w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, ., z_{b-1}, \\ z_b \mapsto z_b^{\overline{x}_i} = z_b[z_b, \overline{x}_i], & \\ w \mapsto w^{[z_b, \overline{x}_i]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ [x_i, y_i] \mapsto [\overline{x}_i, z_b][x_i, y_i]. & \end{cases}$$

$$\widehat{y}_i\colon \begin{cases} w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, ., z_{b-1}, \\ z_b \mapsto z_b^{\overline{y}_i \overline{z}_b} = z_b[\overline{y}_i, \overline{z}_b], & \\ w \mapsto w^{[\overline{y}_i, \overline{z}_b]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ [x_i, y_i] \mapsto [\overline{z}_b, \overline{y}_i][x_i, y_i]. & \end{cases}$$

$$\widehat{z}_j\colon \begin{cases} w \mapsto w & \text{for} \quad w = z_{j+1}, ., z_{b-1}, \\ z_b \mapsto z_b^{\overline{z}_j \overline{z}_b} = z_b[\overline{z}_j, \overline{z}_b], & \\ w \mapsto w^{[\overline{z}_j, \overline{z}_b]} & \text{for} \quad w = t_p, ., t_1, x_1, ., y_g, z_1, ., z_{j-1}, \\ z_j \mapsto z_j^{\overline{z}_b} = [\overline{z}_b, \overline{z}_j]z_j. & \end{cases}$$

It can now be seen that $\widehat{x}$ fixes some $\Sigma_{g,b,p}$-conjugate of $w_{g,b,p}$. Hence $\widehat{x}$ induces an endomorphism of $\Sigma_{g,b,p}$, and hence an endomorphism $\widehat{x}$ of $\Sigma_{g,b,p} * E_b$. Moreover $\widehat{x}$ fixes the $z_j^{e_j}$ and the $[t_k]$. One can check that this $\widehat{x}$ is an automorphism either by straightforward calculation, or by applying Lemma 5.1. It is now clear that we have an element $\widehat{x}$ of $\mathrm{Aut}^+_{g,1^{\perp b},1^{\perp p}} \leqslant \mathrm{Aut}_{g,b-1\perp 1,p}$. $\square$

9.4 Lemma. *If* $b \geqslant 1$*, the kernel of* $\mathrm{elim}(z_b, e_b)\colon \mathrm{Out}_{g,b-1\perp 1,p} \to \mathrm{Out}_{g,b-1,p}$ *is presented with the generating set* $\{\breve{\sigma}_b, \breve{x} \mid x \in X_{g,b-1,p}\} \subseteq \mathrm{Out}_{g,b,p}$ *and the relations of* $\breve{\Sigma}_{g,b-1,p}$.

Proof. We have seen that $\mathrm{Ker}(\mathrm{elim}(z_b, e_b))$ is an extension of $\langle\breve{\sigma}_b\rangle$ by $\widetilde{\Sigma}_{g,b-1,p}$, and it is straightforward to see that the given set generates $\mathrm{Ker}(\mathrm{elim}(z_b, e_b))$.

Notice that the given generators lie in $\mathrm{Out}^+_{g,1^{\perp b},1^{\perp p}}$, and therefore commute with $\breve{\sigma}_b$.

From its presentation, we see that $\breve{\Sigma}_{g,b-1,p}$ too is an extension of $\langle\breve{\sigma}_b\rangle$ by $\widetilde{\Sigma}_{g,b-1,p}$. By the five lemma, it suffices to check that the given generators of $\mathrm{Ker}(\mathrm{elim}(z_b, e_b))$ satisfy all the relations of $\breve{\Sigma}_{g,b-1,p}$.

We compute

$$\widehat{x}_i^{-1}\colon \begin{cases} w \mapsto w^{[x_i,\overline{z}_b]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ x_i \mapsto x_i^{\overline{z}_b}, \quad y_i \mapsto z_b y_i, & \\ w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{x_i \overline{z}_b}, \quad e_b \mapsto z_b \overline{x}_i e_b. & \end{cases}$$

$$\widehat{y}_i^{-1}\colon \begin{cases} w \mapsto w^{[z_b,y_i]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ x_i \mapsto \overline{y}_i \overline{z}_b y_i x_i [z_b, y_i], \quad y_i \mapsto y_i^{z_b y_i}, & \\ w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{y_i}, \quad e_b \mapsto \overline{y}_i \overline{z}_b e_b. & \end{cases}$$

If $v = [x_i, y_i]$, then

$$\widehat{v}\colon \begin{cases} w \mapsto w^{[\overline{v},\overline{z}_b]} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_{i-1}, y_{i-1}, \\ w \mapsto w^{\overline{z}_b} & \text{for} \quad w = x_i, y_i, \\ w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{v}\,\overline{z}_b}, \quad e_b \mapsto z_b v z_b e_b. & \end{cases}$$

If $v = \prod_{k'=p}^{k} t_{k'}$, then

$$\widehat{v}\colon \begin{cases} w \mapsto w^{\overline{z}_b} & \text{for} \quad w = t_p, ., t_k, \\ w \mapsto w & \text{for} \quad w = t_{k-1}, ., t_1, x_1, y_1, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{v}\,\overline{z}_b}, \quad e_b \mapsto z_b v z_b^{p-k} e_b. & \end{cases}$$

If $v = \prod_{k=p}^{1} t_k \cdot \prod_{i'=1}^{i} [x_{i'}, y_{i'}]$, then

$$\widehat{v}\colon \begin{cases} w \mapsto w^{\overline{z}_b} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_i, y_i, \\ w \mapsto w & \text{for} \quad w = x_{i+1}, y_{i+1}, ., x_g, y_g, z_1, e_1, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{v}\,\overline{z}_b}, \quad e_b \mapsto z_b v z_b^{p+2i-1} e_b. & \end{cases}$$

If $v = \prod_{k=p}^{1} t_k \cdot \prod_{i=1}^{g} [x_i, y_i] \cdot \prod_{j'=1}^{j} z_{j'}$, then

$$\widehat{v}\colon \begin{cases} w \mapsto w^{\overline{z}_b} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_g, y_g, z_1, ., z_j, \\ w \mapsto z_b w & \text{for} \quad w = e_1, ., e_j, \\ w \mapsto w & \text{for} \quad w = z_{j+1}, e_{j+1}, ., z_{b-1}, e_{b-1}, \\ z_b \mapsto z_b^{\overline{v}\,\overline{z}_b}, \quad e_b \mapsto z_b v z_b^{p+2g+j-1} e_b. & \end{cases}$$

If $v = \prod_{k=p}^{1} t_k \cdot \prod_{i=1}^{g}[x_i, y_i] \cdot \prod_{j=1}^{b-1} z_j$, then $[v] = [w_{g,b-1,p}]$, $[z_b v] = [w_{g,b,p}]$, and

$$\widehat{v}\colon \begin{cases} w \mapsto w^{\overline{z}_b} & \text{for} \quad w = t_p, ., t_1, x_1, y_1, ., x_g, y_g, z_1, ., z_{b-1}, \\ w \mapsto z_b w & \text{for} \quad w = e_1, ., e_{b-1}, \\ z_b \mapsto z_b^{\overline{v}\,\overline{z}_b}, \quad e_b \mapsto z_b v z_b^{2g+b+p-2} e_b. \end{cases}$$

The latter gives rise to $\widetilde{z}_b^{-1} \sigma_b^{3-2g-b-p}$ in $\mathrm{Aut}_{g,b,p}$, and to $\breve{\sigma}_b^{3-2g-b-p}$ in $\mathrm{Out}_{g,b,p}$. Since $3 - 2g - b - p = \chi_{g,b-1,p}$, we see that we have verified one of the relations.

The other relations are straightforward to check. □

9.5 Theorem. *If $b \geqslant 1$, then there is an exact sequence*

$$1 \to \breve{\Sigma}_{g,b-1,p} \to \mathrm{Out}_{g,b-1\perp 1,p} \xrightarrow{\mathrm{elim}(z_b, e_b)} \mathrm{Out}_{g,b-1,p} \to 1. \tag{9}$$

□

The image of $\breve{\Sigma}_{g,b-1,p}$ in $\mathrm{Out}_{g,b-1\perp 1,p}$ will again be denoted $\breve{\Sigma}_{g,b-1,p}$.

9.6 Remarks. (i) In a standard way, (9) gives an action of $\mathrm{Out}_{g,b-1\perp 1,p}$ on $\breve{\Sigma}_{g,b-1,p}$, and an outer action of $\mathrm{Out}_{g,b-1,p}$ on $\breve{\Sigma}_{g,b-1,p}$, that is, (9) gives homomorphisms

$$\mathrm{Out}_{g,b-1\perp 1,p} \to \mathrm{Aut}(\breve{\Sigma}_{g,b-1,p}) \quad \text{and} \quad \mathrm{Out}_{g,b-1,p} \to \mathrm{Out}(\breve{\Sigma}_{g,b-1,p}).$$

We shall discuss the action of $\mathrm{Out}_{g,b-1\perp 1,p}$ on $\breve{\Sigma}_{g,b-1,p}$ in Section 11.

(ii) Étienne Ghys has pointed out the following to us.

In the case where $(b,p) = (1,0)$ and $g \geqslant 2$, the exact sequence (9) corresponds to the exact sequence

$$1 \to \pi_1(TS_{g,0,0}) \to \mathcal{MC}_{g,1,0} \to \mathcal{MC}_{g,0,0} \to 1,$$

where $TS_{g,0,0}$ denotes the unit-tangent bundle over $S_{g,0,0}$. See, for example, [7, Section 6.3], which is based on [14].

Here, the resulting outer action of $\mathcal{MC}_{g,0,0}$ on $\pi_1(TS_{g,0,0})$ is related to classical geometric constructions. Thus, if we give $S_{g,0,0}$ a hyperbolic metric, then the universal covering space, $\widetilde{S}_{g,0,0}$, has a circle at infinity, $\partial\widetilde{S}_{g,0,0}$. The set of ordered triples of distinct points of $\partial\widetilde{S}_{g,0,0}$, modulo the action of $\Sigma_{g,0,0}$, gives $TS_{g,0,0}$. Now, given an element of $\mathcal{MC}_{g,0,0}$, we can choose a representative self-homeomorphism of $S_{g,0,0}$, lift this to a self-homeomorphism of $\widetilde{S}_{g,0,0}$, and extend it continuously to a self-homeomorphism of

$\partial\widetilde{S}_{g,0,0}$. Now we get a self-homeomorphism of the set of ordered triples of distinct points of $\partial\widetilde{S}_{g,0,0}$, and, on dividing out by the action of $\Sigma_{g,0,0}$, we get a self-homeomorphism of $TS_{g,0,0}$, and hence an outer automorphism of $\pi_1(TS_{g,0,0})$. See, for example, [18, p. 31]. □

(iii) If $b \geqslant 1$, then, by Proposition 6.2 and Theorem 9.5, there is a copy of $\widetilde{\Sigma}_{g,b-1,p} \simeq \check{\Sigma}_{g,b-1,p}/\langle\check{\sigma}_b\rangle$ in $\mathrm{Out}_{g,b-1\perp 1,p}/\langle\check{\sigma}_b\rangle \simeq \mathrm{Out}_{g,b-1,p\perp 1}$. Here, for each $x \in X_{g,b-1,p}$, to see the image of $\check{x}$ in $\mathrm{Out}_{g,b-1,p\perp 1}$, we choose the representative of $\check{x}$ which fixes e_b, then ignore e_b and write z_b as t_{p+1}. We now have an explicit description of the generators of the copy of $\widetilde{\Sigma}_{g,b-1,p}$ in $\mathrm{Out}_{g,b-1,p\perp 1}$. Replacing (b,p) with $(b+1,p-1)$, we get an explicit description of the copy of $\widetilde{\Sigma}_{g,b,p-1}$ in $\mathrm{Out}_{g,b,p-1\perp 1}$, given by Corollary 8.2, for $p \geqslant 1$. □

10 Another semidirect product decomposition

For $b \geqslant 2$, we shall define a map

$$\mathrm{pinch}(z_{b-1})\colon\ \mathrm{Out}_{g,b-2\perp 1,p} \to \mathrm{Out}_{g,b-2\perp 1\perp 1,p}$$

which corresponds to identifying two disjoint closed subintervals of the $(b-1)$st boundary component, an identification which does not behave well with respect to the referential S^1s.

10.1 Definitions. Suppose that $b \geqslant 2$.

There is a unique endomorphism of the free group on $X_{g,b,p} \cup E_b$ which is the identity on

$$(X_{g,b,p} \cup E_b) - \{z_{b-1}\}$$

and sends z_{b-1} to $z_{b-1}z_b$. Notice that this is an automorphism which sends $w_{g,b-1,p}$ to $w_{g,b,p}$, and hence induces an isomorphism

$$\Sigma_{g,b-1,p} * E_{b-1} * \langle z_b, e_b \mid \quad \rangle \xrightarrow{\sim} \Sigma_{g,b,p} * E_b. \tag{10}$$

There is a natural sign-preserving map

$$\mathrm{mod}(z_b, e_b)\colon\ \mathrm{Aut}_{g,b-1\perp\hat{1},p} \to \mathrm{Aut}_{g,b-1,p}$$

corresponding to considering the action modulo the normal subgroup generated by e_b and z_b. This induces a map on index $b-1$ subgroups

$$\mathrm{Aut}_{g,b-2\perp 1\perp\hat{1},p} \to \mathrm{Aut}_{g,b-2\perp 1,p},$$

and, by abuse of notation, we also call this map $\mathrm{mod}(z_b, e_b)$.

We shall construct a map

$$\mathrm{Aut}_{g,b-2\perp\hat{1},p} \to \mathrm{Aut}_{g,b-2\perp 1\perp\hat{1},p}, \quad \alpha \mapsto \alpha'',$$

such that the composite

$$\mathrm{Aut}_{g,b-2\perp\hat{1},p} \to \mathrm{Aut}_{g,b-2\perp 1\perp\hat{1},p} \xrightarrow{\mathrm{mod}(z_b,e_b)} \mathrm{Aut}_{g,b-2\perp 1,p}$$

is the inclusion map.

Consider any $\alpha \in \mathrm{Aut}_{g,b-2\perp\hat{1},p}$.

If α fixes z_{b-1}, we extend α to an automorphism α' of

$$\Sigma_{g,b-1,p} * E_{b-1} * \langle z_b, e_b \mid \quad \rangle$$

so as to fix z_b and e_b. Then, via the isomorphism (10), α' induces an automorphism α'' of $\Sigma_{g,b,p} * E_b$. Notice that α'' fixes $z_{b-1}z_b$, z_b, e_b, and hence also z_{b-1}, $z_{b-1}^{e_{b-1}}$. It is clear that $\mathrm{mod}(z_b, e_b)$ carries α'' to α.

If α inverts z_{b-1}, we restrict α to $\Sigma_{g,b-1,p} * E_{b-2}$ and extend this restricted map to a map α' which inverts z_b, fixes e_b, and sends e_{b-1} to $\overline{z}_b e_{b-1}$. Then, via the isomorphism (10), α' induces an automorphism α'' of $\Sigma_{g,b,p} * E_b$. Notice that α'' inverts $z_{b-1}z_b$ and z_b, fixes e_b, and sends e_{b-1} to $\overline{z}_b e_{b-1}$. Thus α'' sends z_{b-1} to $\overline{z}_{b-1}^{z_b}$, and inverts $z_{b-1}^{e_{b-1}}$. Again, it is clear that $\mathrm{mod}(z_b, e_b)$ carries α'' to α.

In all cases, α'' fixes

$$\{(z_{b-1}, e_{b-1}, z_b, e_b), (\overline{z}_{b-1}^{z_b}, \overline{z}_b e_{b-1}, \overline{z}_b, e_b)\}.$$

It is not difficult to show that $\alpha \mapsto \alpha''$ is a homomorphism.

Also, $\sigma_{b-1}\tilde{z}_{b-1}^{-1}$ is mapped to τ_{b-1}^2, which can be expressed as

$$\sigma_{b-1}\hat{z}_{b-1}^{-1}[\tilde{z}_{b-1}, \tilde{z}_b].$$

By Proposition 4.1, $\mathrm{Aut}_{g,b-2\perp\hat{1},p} \simeq \mathrm{Out}_{g,b-2\perp 1,p}$ and

$$\mathrm{Aut}_{g,b-2\perp 1\perp\hat{1},p} \simeq \mathrm{Out}_{g,b-2\perp 1\perp 1,p},$$

so we get a homomorphism

$$\mathrm{pinch}(z_{b-1})\colon \mathrm{Out}_{g,b-2\perp 1,p} \to \mathrm{Out}_{g,b-2\perp 1\perp 1,p} \tag{11}$$

which is a left inverse of (the index $b-1$ restriction)

$$\mathrm{elim}(z_b, e_b)\colon \mathrm{Out}_{g,b-2\perp 1\perp 1,p} \to \mathrm{Out}_{g,b-2\perp 1,p}\,. \tag{12}$$

Notice that $\mathrm{pinch}(z_{b-1})$ maps $\breve{\sigma}_{b-1}$ to $\breve{\tau}_{b-1}^2 = \breve{\sigma}_{b-1}\breve{z}_{b-1}^{-1}$, or, with extra notation, maps $\breve{\sigma}_{b-1}^{(g,b-1,p)}$ to $\breve{\sigma}_{b-1}^{(g,b,p)}\breve{z}_{b-1}^{-1}$. □

We have constructed a finite-index splitting of the exact sequence in Theorem 9.5.

10.2 Theorem. *If $b \geqslant 2$, then*

$$\mathrm{Out}_{g,b-2\perp 1\perp 1,p} \simeq \mathrm{Out}_{g,b-2\perp 1,p} \ltimes \check{\Sigma}_{g,b-1,p}.$$

Here

$\breve{\sigma}_j$ *corresponds to* $\breve{\sigma}_j \ltimes 1$, *for* $1 \leqslant j \leqslant b-2$;

$\breve{\sigma}_{b-1}^{(g,b,p)}$ *corresponds to* $\breve{\sigma}_{b-1}^{(g,b-1,p)} \ltimes \check{z}_{b-1}$;

$\breve{\sigma}_b$ *corresponds to* $1 \ltimes \breve{\sigma}_b$. □

10.3 Remarks. (i) We shall describe the action of $\mathrm{Out}_{g,b-2\perp 1\perp 1,p}$ on $\check{\Sigma}_{g,b-1,p}$ in Section 11. The action of $\mathrm{Out}_{g,b-2\perp 1,p}$ on $\check{\Sigma}_{g,b-1,p}$ is then by pullback along (11).

(ii) We have seen that (11), the map $\mathrm{pinch}(z_{b-1})$, is a left inverse of (12), (an index $b-1$ restriction of) the map $\mathrm{elim}(z_b, e_b)$. The factorization

$$\mathrm{elim}(z_b, e_b) = \mathrm{elim}(e_b) \cdot \mathrm{elim}(t_{p+1})$$

passes to the index $b-1$ subgroups. Thus $\mathrm{pinch}(z_{b-1})$ followed by (an index $b-1$ restriction)

$$\mathrm{elim}(e_b)\colon\ \mathrm{Out}_{g,b-2\perp 1\perp 1,p} \to \mathrm{Out}_{g,b-2\perp 1,p\perp 1},$$

is a left inverse of (an index $b-1$ restriction)

$$\mathrm{elim}(t_{p+1})\colon\ \mathrm{Out}_{g,b-2\perp 1,p\perp 1} \to \mathrm{Out}_{g,b-2\perp 1,p}\,.$$

This corresponds to pinching the $(b-1)$st boundary component, and then eliminating the bth boundary component in two steps, first converting it to a puncture, and then eliminating the puncture, to get back the original surface.

If we replace p with $p-1$, and b with $b+1$, we get the map (7). □

We can iterate Theorem 10.2.

10.4 Corollary. *If $b \geqslant 1$, then*

$$\mathrm{Out}_{g,1^{\perp b},p} \simeq \mathrm{Out}_{g,1,p} \ltimes \check{\Sigma}_{g,1,p} \ltimes \check{\Sigma}_{g,2,p} \cdots \ltimes \check{\Sigma}_{g,b-1,p}.$$

□

10.5 Corollary. *If* $b \geqslant 1$*, then*

$$\begin{aligned}&\mathrm{Out}_{g,1^{\perp b},1^{\perp p}}\\ &\simeq \mathrm{Out}_{g,1^{\perp b},0} \ltimes \Sigma_{g,b,0} \ltimes \Sigma_{g,b,1} \ltimes \cdots \ltimes \Sigma_{g,b,p-1}\\ &\simeq \mathrm{Out}_{g,1,0} \ltimes \check{\Sigma}_{g,1,0} \ltimes \check{\Sigma}_{g,2,0} \ltimes \cdots \ltimes \check{\Sigma}_{g,b-1,0} \ltimes \Sigma_{g,b,0} \ltimes \Sigma_{g,b,1} \ltimes \cdots \ltimes \Sigma_{g,b,p-1}.\end{aligned}$$

Proof. This follows from Corollary 8.5 and Corollary 10.4. Alternatively, since $b + p \geqslant 1$,

$$\mathrm{Out}_{g,1^{\perp b+p},0} \simeq \mathrm{Out}_{g,1,0} \ltimes \check{\Sigma}_{g,1,0} \ltimes \check{\Sigma}_{g,2,0} \ltimes \cdots \ltimes \check{\Sigma}_{g,b+p-1,0}.$$

Since $\langle \check{\sigma}_j \mid b+1 \leqslant j \leqslant b+p \rangle$ lies in the normal subgroup

$$\check{\Sigma}_{g,b,0} \ltimes \check{\Sigma}_{g,b+1,0} \ltimes \cdots \ltimes \check{\Sigma}_{g,b+p-1,0},$$

we get the result. □

11 Description of an action

11.1 Remarks. Let $b \geqslant 1$.

By Theorem 9.5, $\check{\Sigma}_{g,b-1,p}$ is a normal subgroup of $\mathrm{Out}_{g,b-1\perp 1,p}$; we want to describe the resulting action by conjugation.

Since $\check{\sigma}_b$ is fixed by positive elements and inverted by negative elements, it remains to describe the action on the other generators of $\check{\Sigma}_{g,b-1,p}$.

Recall that $\Sigma_{g,b,p}$ is free on $X_{g,b-1,p}$, and has a distinguished element, z_b.

By Proposition 4.1, $\mathrm{Out}_{g,b-1\perp 1,p} \simeq \mathrm{Aut}_{g,b-1\perp\hat{1},p}$, and the latter acts on $\Sigma_{g,b,p}$, fixing z_b.

Definitions 9.2 give a homomorphism $\Sigma_{g,b,p} \to \check{\Sigma}_{g,b-1,p}$, $x \mapsto \check{x}$; for example, $\check{z}_b = \check{\sigma}_b^{-\chi_{g,b-1,p}}$.

Consider any $v \in \Sigma_{g,b,p}$ and any $\alpha \in \mathrm{Aut}_{g,b-1\perp\hat{1},p}$. Thus v, $v^\alpha \in \Sigma_{g,b,p}$ and $\check{v}$, $\breve{v^\alpha} \in \check{\Sigma}_{g,b-1,p}$. Since $\check{\alpha}$ normalizes $\check{\Sigma}_{g,b-1,p}$, we see that $\check{v}^{\check{\alpha}}$ lies in $\check{\Sigma}_{g,b-1,p}$. Moreover, under the composite

$$\mathrm{Out}_{g,b-1\perp 1,p} \to \mathrm{Out}_{g,b-1,p\perp 1} \xrightarrow{\sim} \mathrm{Aut}_{g,b-1,p},$$

both $\breve{v^\alpha}$ and $\check{v}^{\check{\alpha}}$ map to $\widetilde{v^\alpha} = \tilde{v}^{\,\alpha}$. Since the kernel of this composite is generated by $\check{\sigma}_b$, we see that $\check{v}^{\check{\alpha}} = \breve{v^\alpha}\check{\sigma}_b^n$ for some integer n. To describe the action, it remains to describe n in terms of v and α. □

11.2 Definition. Suppose that $b \geqslant 1$.

We now define the *twisting-number map* $\mathrm{tw}\colon \Sigma_{g,b,p} \to \mathbb{Z}$.

For each $v \in \Sigma_{g,b,p}$, define the *twisting number* of v, $\mathrm{tw}(v)$, as follows. Recall that $\Sigma_{g,b,p}$ is a free group, with basis $X_{g,b-1,p}$. When v is expressed as a reduced word in this basis, $\mathrm{tw}(v)$ sums the number of occurrences in v of subwords of the form

$$t_k, \quad \overline{x}_i, \quad y_i, \quad z_j, \quad x_i \cdots y_i, \quad y_i \cdots \overline{x}_i, \quad \overline{y}_i \cdots x_i, \quad \overline{x}_i \cdots \overline{y}_i,$$

and subtracts the number of occurrences in v of subwords of the form

$$\overline{t}_k, \quad x_i, \quad \overline{y}_i, \quad \overline{z}_j, \quad \overline{y}_i \cdots \overline{x}_i, \quad x_i \cdots \overline{y}_i, \quad \overline{x}_i \cdots y_i, \quad y_i \cdots x_i;$$

here $x_i \cdots y_i$ represents all words that begin with x_i and end with y_i. For example, $\mathrm{tw}([x_i, y_i]) = \mathrm{tw}(\overline{x}_i\overline{y}_i x_i y_i) = 0 + 0 + 1 - 1 + 1 + 1 = 2$ and $\mathrm{tw}(\overline{z}_b) = p + 2g + b - 1 = 2 - \chi_{g,b-1,p}$. □

11.3 Remarks. (i) For a reduced word w in a given basis X of a free group F, let $f_w\colon F \to \mathbb{Z}$ be the map which assigns, to each reduced word v in $X \cup X^{-1}$, the number of occurrences of w as a subword of v, minus the number of occurrences of $\overline{w}$ as a subword of v. An interesting history of the use of such maps, especially in the study of $\Sigma_{g,0,1}$, can be found in [1, Section 1.1].

The twisting-number map is a sum of infinitely many such maps.

(ii) One can express the twisting-number map as the result of applying the endomorphism

$$\sum_{k=1}^{p} \frac{\partial}{t_k\partial} + \sum_{i=1}^{g}\Big(\frac{\partial}{y_i\partial} - \frac{\partial}{x_i\partial} + \frac{\partial^2}{x_i\partial \cdot y_i\partial} - \frac{\partial^2}{y_i\partial \cdot x_i\partial}\Big) + \sum_{j=1}^{b-1} \frac{\partial}{z_j\partial}$$

of $\mathbb{Z}[\Sigma_{g,b,p}]$, and then applying the augmentation map $\mathbb{Z}[\Sigma_{g,b,p}] \to \mathbb{Z}$. □

We can now describe the action of $\mathrm{Out}_{g,b-1\perp 1,p}$ on $\check{\Sigma}_{g,b-1,p}$.

11.4 Theorem. *Suppose that $b \geqslant 1$. Let $v \in \Sigma_{g,b,p}$ and $\alpha \in \mathrm{Aut}_{g,b-1\perp\hat{1},p}$. Then*

$$\hat{v}^{\,\alpha} = \widehat{v^{\alpha}}\, \sigma_b^{\mathrm{tw}(v^{\alpha}) - \mathrm{sign}(\alpha)\,\mathrm{tw}(v)} \tag{13}$$

in $\mathrm{Aut}_{g,b,p}$. *Hence* $\check{v}^{\,\check{\alpha}} = \widecheck{v^{\alpha}}\, \check{\sigma}_b^{\mathrm{tw}(v^{\alpha}) - \mathrm{sign}(\alpha)\,\mathrm{tw}(v)}$ *in* $\mathrm{Out}_{g,b,p}$.

Proof. Let $\beta = (\hat{v}^{\alpha})^{-1}\, \widehat{v^{\alpha}}\, \sigma_b^{\mathrm{tw}(v^{\alpha}) - \mathrm{sign}(\alpha)\,\mathrm{tw}(v)}$. It follows from Remarks 11.1 that we can write $\beta = \sigma_b^m \widetilde{w}^{-1}$ for some $m \in \mathbb{Z}$ and some $w \in \Sigma_{g,b,p}$. We want to show that $m = 0$ and $w = 1$.

Now $e_b^{\beta} = w z_b^m e_b$ and, if $b \geqslant 2$, then $e_{b-1}^{\beta} = w e_{b-1}$.

In this paragraph we consider consequences of making z_b central in $\Sigma_{g,b,p} * E_b$. Notice that $\mathrm{Aut}_{g,b-1\perp 1,p}$ acts on this quotient group, and, in particular, so does $\widehat{x}$ for each $x \in X_{g,b-1,p}$. Explicitly, for $p \geqslant k \geqslant 1$, $1 \leqslant i \leqslant g$, $1 \leqslant j \leqslant b-1$, if z_b is made central then we can write the following.

$$\widehat{t}_k\colon \begin{cases} w \mapsto w & \text{for} \quad w \in X_{g,b,p} \cup E_{b-1}, \\ e_b \mapsto z_b t_k e_b. \end{cases}$$

$$\widehat{x}_i\colon \begin{cases} w \mapsto w & \text{for} \quad w \in E_{b-1} \cup X_{g,b,p} - \{y_i\}, \\ y_i \mapsto \overline{z}_b y_i, \\ e_b \mapsto x_i \overline{z}_b e_b. \end{cases}$$

$$\widehat{y}_i\colon \begin{cases} w \mapsto w & \text{for} \quad w \in E_{b-1} \cup X_{g,b,p} - \{x_i\}, \\ x_i \mapsto z_b x_i, \\ e_b \mapsto z_b y_i e_b. \end{cases}$$

$$\widehat{z}_j\colon \begin{cases} w \mapsto w & \text{for} \quad w \in X_{g,b,p} \cup E_{b-1} - \{e_j\}, \\ e_j \mapsto z_b e_j, \\ e_b \mapsto z_b z_j e_b. \end{cases}$$

It is straightforward to check that $e_b^{\widehat{v}} = v z_b^{\mathrm{tw}(v)} e_b$. Hence $e_b^{\widehat{v^\alpha}} = v^\alpha z_b^{\mathrm{tw}(v^\alpha)} e_b$, and

$$e_b^{\widehat{v}^\alpha} = e_b^{\overline{\alpha}\widehat{v}\alpha} = e_b^{\widehat{v}\alpha} = (v z_b^{\mathrm{tw}(v)} e_b)^\alpha = v^\alpha z_b^{\mathrm{sign}(\alpha)\,\mathrm{tw}(v)} e_b.$$

It follows that β fixes e_b.

In summary, $w z_b^m$ is annihilated if z_b is made central.

Let us first consider the special case where $b \geqslant 2$, and α fixes

$$\{(z_{b-1}, e_{b-1}, z_b, e_b), (\overline{z}_{b-1}^{z_b}, \overline{z}_b e_{b-1}, \overline{z}_b, e_b)\},$$

and v does not involve z_{b-1}, that is, $v \in \langle X_{g,b-2,p}\rangle$. It follows that $e_{b-1}^\beta = e_{b-1}$. But $e_{b-1}^\beta = w e_{b-1}$, so $w = 1$. Thus, making z_b central annihilates z_b^m. Since $b \geqslant 2$, z_b is a primitive element of the free group $\Sigma_{g,b,p} * E_b$. It follows that $m = 0$. Thus this special case of the theorem holds.

If we now apply $\mathrm{elim}(z_{b-1}, e_{b-1})$ to the foregoing special case, we get an $\alpha' \in \mathrm{Aut}_{g,b-2\perp\hat{1},p}$ and a $v' \in \Sigma_{g,b-1,p}$ for which the conclusion of the theorem holds. By considering $\mathrm{pinch}(z_b)\colon \mathrm{Out}_{g,b-1\perp 1,p} \to \mathrm{Out}_{g,b-1\perp 1\perp 1,p}$, we see that the general case arises in this way. $\square$

11.5 Remark. It is not difficult to check that (13) can be used to define an action of $\mathrm{Out}_{g,b-1\perp 1,p}$ on $\Sigma_{g,b,p} \times \langle \sigma_b \rangle$. In checking a condition of the form $\widehat{vw}^{\alpha} = \hat{v}^{\alpha}\hat{w}^{\alpha}$, it is useful to know that $\langle v^{\alpha}, w^{\alpha}\rangle = \mathrm{sign}(\alpha)\langle v, w\rangle$, where $\langle v, w\rangle$ denotes the result of applying the augmentation homomorphism to

$$\sum_{i=1}^{g}\left(\frac{v\partial}{x_i\partial}\frac{w\partial}{y_i\partial} - \frac{v\partial}{y_i\partial}\frac{w\partial}{x_i\partial}\right).$$

Notice that $\langle -, -\rangle$ can be given via the usual symplectic product on the abelianization of $\Sigma_{g,b,p}$. □

We can now calculate with some of the special automorphisms of Definitions 2.10.

11.6 Lemma. *If* $b \geqslant 1$ *and* $g \geqslant 1$, *then the following hold in* $\mathrm{Aut}_{g,b,p}$.

$$\begin{aligned}
\hat{x}_i^{\alpha_i} &= \widehat{x_i^{\alpha_i}} &&= \hat{y}_i^{-1}\,\hat{x}_i, \ \text{if} \quad 1 \leqslant i \leqslant g.\\
\hat{y}_i^{\beta_i} &= \widehat{y_i^{\beta_i}} &&= \hat{x}_i\,\hat{y}_i, \ \text{if} \quad 1 \leqslant i \leqslant g.\\
\hat{x}_1^{\gamma_0} &= \widehat{x_1^{\gamma_0}}\,\sigma_b &&= \hat{x}_1\,\hat{t}_p\,\hat{x}_1^{-1}\,\hat{y}_1^{-1}\,\hat{x}_1\,\sigma_b, \ \text{if} \quad p \geqslant 1.\\
\hat{x}_i^{\gamma_i} &= \widehat{x_i^{\gamma_i}}\,\sigma_b^{-1} &&= \hat{x}_{i+1}^{-1}\,\hat{y}_{i+1}\,\hat{x}_{i+1}\,\hat{y}_i^{-1}\,\hat{x}_i\,\sigma_b^{-1}, \ \text{if} \quad 1 \leqslant i \leqslant g-1.\\
\hat{y}_i^{\gamma_i} &= \widehat{y_i^{\gamma_i}} &&= \hat{x}_{i+1}^{-1}\,\hat{y}_{i+1}\,\hat{x}_{i+1}\,\hat{y}_i\,\hat{x}_{i+1}^{-1}\,\hat{y}_{i+1}^{-1}\,\hat{x}_{i+1}, \ \text{if}\ 1 \leqslant i \leqslant g-1.\\
\hat{x}_{i+1}^{\gamma_i} &= \widehat{x_{i+1}^{\gamma_i}}\,\sigma_b &&= \hat{x}_{i+1}\,\hat{y}_i\,\hat{x}_{i+1}^{-1}\,\hat{y}_{i+1}^{-1}\,\hat{x}_{i+1}\sigma_b, \ \text{if} \quad 1 \leqslant i \leqslant g-1.\\
\hat{x}_g^{\gamma_g} &= \widehat{x_g^{\gamma_g}}\,\sigma_b^{-1} &&= \hat{z}_1^{-1}\,\hat{y}_g^{-1}\,\hat{x}_g\,\sigma_b^{-1}, \ \text{if} \quad b \geqslant 2.
\end{aligned}$$

Proof. We compute

$$\begin{aligned}
\mathrm{tw}(x_i^{\alpha_i}) &= \mathrm{tw}(\overline{y}_i x_i) = -(1) + (-1) + 1 = -1,\\
\mathrm{tw}(y_i^{\beta_i}) &= \mathrm{tw}(x_i y_i) = (-1) + (1) + 1 = 1,\\
\mathrm{tw}(x_1^{\gamma_0}) &= \mathrm{tw}(x_1 t_p \overline{x}_1 \overline{y}_1 x_1),\\
&= (-1) + 1 - (-1) - 1 + (-1) + 1 + (-1) + 1 = 0,\\
\mathrm{tw}(x_i^{\gamma_i}) &= \mathrm{tw}(\overline{x}_{i+1} y_{i+1} x_{i+1} \overline{y}_i x_i)\\
&= -(-1) + (1) + (-1) - (1) + (-1) - 1 - 1 + 1 = -2,\\
\mathrm{tw}(y_i^{\gamma_i}) &= \mathrm{tw}(\overline{x}_{i+1} y_{i+1} x_{i+1} y_i \overline{x}_{i+1} \overline{y}_{i+1} x_{i+1}) = \mathrm{tw}(y_i),\\
\mathrm{tw}(x_{i+1}^{\gamma_i}) &= \mathrm{tw}(x_{i+1} y_i \overline{x}_{i+1} \overline{y}_{i+1} x_{i+1})\\
&= (-1) + (1) - (-1) - (1) + (-1) - 1 + 1 + 1 = 0.\\
\mathrm{tw}(x_g^{\gamma_g}) &= \mathrm{tw}(\overline{z}_1 \overline{y}_g x_g) = -(1) - (1) + (-1) + 1 = -2,
\end{aligned}$$

Hence

$$\begin{aligned}
&\operatorname{tw}(x_i^{\alpha_i}) - \operatorname{tw}(x_i) = (-1) - (-1) = 0,\\
&\operatorname{tw}(y_i^{\beta_i}) - \operatorname{tw}(y_i) = 1 - 1 = 0,\\
&\operatorname{tw}(x_1^{\gamma_0}) - \operatorname{tw}(x_1) = (0) - (-1) = 1,\\
&\operatorname{tw}(x_i^{\gamma_i}) - \operatorname{tw}(x_i) = (-2) - (-1) = -1,\\
&\operatorname{tw}(y_i^{\gamma_i}) - \operatorname{tw}(y_i) = 0,\\
&\operatorname{tw}(x_{i+1}^{\gamma_i}) - \operatorname{tw}(x_{i+1}) = (0) - (-1) = 1,\\
&\operatorname{tw}(x_g^{\gamma_g}) - \operatorname{tw}(x_g) = (-2) - (-1) = -1.
\end{aligned}$$

The result now follows. □

11.7 Corollary (Wajnryb). *If $b \geqslant 1$ and $g \geqslant 2$, then the normal closure of $\breve{x}_1$ in $\operatorname{Out}^+_{g,b-1\perp1,p}$ is the subgroup $\check{\Sigma}_{g,b-1,p}$ which occurs in the exact sequence*

$$1 \to \check{\Sigma}_{g,b-1,p} \to \operatorname{Out}_{g,b-1\perp1,p} \to \operatorname{Out}_{g,b-1,p} \to 1.$$

□

11.8 Lemma. *If $b \geqslant 1$ and $g \geqslant 1$, then, in $\operatorname{Aut}_{g,b,0}$,*

$$\gamma_0\alpha_1^{-1} = \hat{x}_1^{-1}\,\hat{y}_1\,\hat{x}_1\colon \begin{cases} x_1 \mapsto y_1x_1z_b\overline{x}_1\overline{y}_1x_1,\\ w \mapsto w \quad \textit{for all} \quad w \in X_{g,b,0} - \{x_1, z_b\},\\ z_b \mapsto z_b^{\overline{y}_1^{x_1}}, \quad e_b \mapsto \overline{x}_1\overline{y}_1x_1\overline{z}_be_b.\end{cases}$$

□

11.9 Corollary (Wajnryb). *If $b \geqslant 1$ and $g \geqslant 2$, then the normal closure of $\breve{\gamma}_0\,\breve{\alpha}_1^{-1} = \breve{x}_1^{-1}\,\breve{y}_1\,\breve{x}_1$ in $\operatorname{Out}^+_{g,b-1\perp1,0}$ is $\check{\Sigma}_{g,b-1,0}$.* □

12 Artin diagrams

12.1 Definitions. A *diagram* (V, n) consists of a set V together with a function n which assigns, to each two-element subset $\{x, y\}$ of V, a value $n_{\{x,y\}} \in \mathbb{N} \cup \{\infty\}$. Such a diagram can be depicted as the complete graph with vertex set V labelled so that the edge joining x and y has label $n_{\{x,y\}}$. One then applies the convention that if $n_{\{x,y\}}$ is "small", then the edge and label are replaced with $n_{\{x,y\}}$ unlabelled edges joining x and y.

An *Artin diagram* in a group G is a diagram (V, n) in which V is a family of elements of G, such that, for each two-element subset $\{x, y\}$ of V, and each $m \in \mathbb{N}$, $(xy)^{m+1} = (yx)^{m+1}$ if $2m = n_{\{x,y\}}$, and $(xy)^m x = (yx)^m y$ if $2m - 1 = n_{\{x,y\}}$. Thus, no conclusion is to be drawn if $n_{\{x,y\}} = \infty$.

For any diagram X there is a universal group G, denoted $\mathrm{Artin}(X)$, such that X is an Artin diagram in G. □

12.2 Example. It is well known that, if $g \geqslant 1$, $b \geqslant 2$ and $p \geqslant 2$, then

$$
\begin{array}{ccccccccccccccc}
 & & \alpha_1 & & \alpha_2 & & & & \alpha_{g-1} & & & & \alpha_g & & \\
 & & | & & | & & & & | & & & & | & & \\
\gamma_0 & — & \beta_1 & — \gamma_1 — & \beta_2 & — \gamma_2 — & \cdots & — & \beta_{g-1} & — & \gamma_{g-1} & — & \beta_g & — & \gamma_g \\
| \infty & & & & & & & & & & & & & & \infty | \\
\mu_1 & — & \mu_2 & — & \cdots & — & \mu_{p-1} & & \tau_{b-1} & — & \cdots & — & \tau_2 & — & \tau_1 \\
 & & & & & & & & & & & & & & \\
 & & & & & & & \sigma_b & \sigma_{b-1} & & \cdots & & \sigma_2 & & \sigma_1
\end{array}
$$

is an Artin diagram in $\mathrm{Aut}_{g,b,p}$. □

12.3 Definitions. For $n \geqslant 1$, the diagram

$$a_1 — a_2 — a_3 — \quad \cdots \quad — a_{n-1} — a_n$$

is denoted A_n. The center of $\mathrm{Artin}(A_n)$ is infinite cyclic with distinguished generator $Z_{A_1} = a_1$ if $n = 1$, and $Z_{A_n} = (\prod_{i=1}^{n} a_i)^{n+1}$ if $n \geqslant 2$.

For $n \geqslant 3$, the diagram

$$
\begin{array}{ccccccc}
 & a_1 & & & & & \\
 & | & & & & & \\
a_2 — & a_3 & — a_4 — & \cdots & — a_{n-1} — & a_n &
\end{array}
$$

is denoted D_n. The center of $\mathrm{Artin}(D_n)$ is infinite cyclic with distinguished generator $Z_{D_n} = (\prod_{i=1}^{n} a_i)^{2n-2}$ if n is odd, and $Z_{D_n} = (\prod_{i=1}^{n} a_i)^{n-1}$ if n is even; see [3]. The diagram

$$
\begin{array}{ccccccccc}
 & a_1 & & & & & & & \\
 & | & & & & & & & \\
a_2 — & a_3 & — a_4 — & \cdots & — a_{n-1} — & a_n & & a_{n+1} & \quad a_{n+2}
\end{array}
$$

is denoted $2A_1 + D_n$. The center of $\mathrm{Artin}(2A_1 + D_n) = \mathrm{Artin}(A_1)^2 \times \mathrm{Artin}(D_n)$ is free abelian of rank three, with distinguished basis $Z_{A_{1,1}} = a_{n+1}$, $Z_{A_{1,2}} = a_{n+2}$ and Z_{D_n}. We shall return to these elements in Section 14.

For any $\phi \in \mathrm{Sym}_n$, replacing $\prod_{i=1}^n a_i$ with $\prod_{i=1}^n a_{i\phi}$ in these expressions gives another expression for each distinguished central element. For $n = 3$, although D_3 and A_3 have different labellings, both Z_{D_3} and Z_{A_3} can be expressed as $(a_1a_2a_3)^4$. $\square$

13 Examples

In this section we describe some illustrative examples taken from topology, some of which do not have algebraic justifications at the time of writing.

13.1 Example. A sphere with p punctures, $S_{0,0,p}$.

By Notation 3.3, $\mathrm{Out}_{0,0,p}$ is an extension of $\mathrm{Out}_{0,0,1^{\perp p}}$ by $C_2 \times \mathrm{Sym}_p$. By Corollary 8.2, $\mathrm{Out}_{0,0,1^{\perp p}}$ has an ascending normal series with sequence of factor groups

$$\widetilde{\Sigma}_{0,0,p-1}, \widetilde{\Sigma}_{0,0,p-2}, \cdots, \widetilde{\Sigma}_{0,0,1}.$$

(Each $\widetilde{\Sigma}_{0,0,i}$ is free, so the normal series gives rise to a decomposition of the form

$$\mathrm{Out}_{0,0,1^{\perp p}} = \widetilde{\Sigma}_{0,0,p-1} \rtimes \widetilde{\Sigma}_{0,0,p-2} \rtimes \cdots \rtimes \widetilde{\Sigma}_{0,0,0},$$

where left parentheses are understood to accumulate on the left. Such an expression is not as informative as the sequence of free factor groups of a normal series.)

If $0 \leqslant i \leqslant 2$, then $\widetilde{\Sigma}_{0,0,i} = 1$, and, if $i \geqslant 3$, then $\widetilde{\Sigma}_{0,0,i} = \Sigma_{0,0,i}$ is free of rank $i - 1$.

If $p \leqslant 3$, then $\mathrm{Out}_{0,0,p} = C_2 \times \mathrm{Sym}_p$.

The group $\mathrm{Out}_{0,0,4}$ is an extension of a free group of rank two by $C_2 \times \mathrm{Sym}_4$, and the latter is isomorphic to $\mathrm{PGL}_2(\mathbb{Z}/2\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z})^2$; in Remarks 13.7(v) and (viii), we shall see that $\mathrm{Out}_{0,0,4} \simeq \mathrm{PGL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z})^2$.

The groups $\mathrm{Out}_{0,0,5}$ and $\mathrm{Out}_{0,0,6}$ have connections with Remarks 13.7(iv) and Example 13.4, respectively. $\square$

13.2 Example. A closed disc with p punctures, $S_{0,1,p}$.

By Proposition 4.1, $\mathrm{Out}_{0,1,p} \simeq \mathrm{Aut}_{0,0,p\perp\hat{1}}$, a group of automorphisms of a free group of rank p.

The subgroup $\mathrm{Out}^+_{0,1,p}$ is the braid group on p strings, and $\mathrm{Out}^+_{0,1,1^{\perp p}}$ is the pure braid group on p strings. By Corollary 8.5, we get the classic decomposition

$$\mathrm{Out}^+_{0,1,1^{\perp p}} \simeq \Sigma_{0,1,1} \ltimes \Sigma_{0,1,2} \ltimes \cdots \ltimes \Sigma_{0,1,p-1},$$

where $\Sigma_{0,1,i}$ is free of rank i.

If $p \geqslant 2$, then

$$\breve{\mu}_1 \text{ --- } \breve{\mu}_2 \text{ --- } \breve{\mu}_3 \text{ --- } \cdots \text{ --- } \breve{\mu}_{p-2} \text{ --- } \breve{\mu}_{p-1}$$

is an Artin diagram in $\mathrm{Out}_{0,1,p}$, and Emil Artin showed that it induces an isomorphism $\mathrm{Artin}(A_{p-1}) \xrightarrow{\sim} \mathrm{Out}^+_{0,1,p}$. Surjectivity was proved algebraically. Injectivity was first proved topologically; algebraic proofs were given by Larue [9] and Shpilrain [17] using results of Dehornoy [4].

The braid group on p strings, $\mathrm{Out}^+_{0,1,p}$, modulo its center, $\langle \breve{\sigma}_1 \rangle$, is isomorphic to $\mathrm{Out}^+_{0,0,p\perp 1}$. This is a trivial group if $p \leqslant 2$, while $\mathrm{Out}^+_{0,0,3\perp 1} \simeq \mathrm{PSL}_2(\mathbb{Z})$, and $\mathrm{Out}^+_{0,0,4\perp 1} \simeq \mathrm{Aut}^+(F_2)$; see Remarks 13.7(viii) and (iv). □

13.3 Notation. Let $\Sigma_{g,b,p^{(2)}}$ denote the group obtained from $\Sigma_{g,b,p}$ by imposing the relations $t_1^2 = t_2^2 = \cdots = t_p^2 = 1$. This is the orbifold group of the surface $S_{g,b,p^{(2)}}$ with p double points, as opposed to punctures. There is little topological difference between a double point and a puncture.

There are natural sign-preserving maps $\mathrm{Aut}_{g,b,p} \to \mathrm{Aut}_{g,b,p^{(2)}}$ and

$$\mathrm{Out}_{g,b,p} \to \mathrm{Out}_{g,b,p^{(2)}}\,.$$

The latter map can be shown to be an isomorphism by using topological arguments. We shall apply this fact only in this section, which is dedicated to examples. □

We now look at the hyperelliptic involution. This is an order-two, orientation-preserving homeomorphism of the surface of genus g induced by a $180°$-rotation of Euclidean three-space about the x-axis, where the surface is embedded so as to meet the axis in $2g+2$ points, and to be invariant under the rotation.

13.4 Example. $S_{g,0,0}$ is a double branched cover of $S_{0,0,(2g+2)^{(2)}}$.

It is not difficult to show that $\Sigma_{0,0,(2g+2)^2}$ has a decomposition of the form $\Sigma_{g,0,0} \rtimes C_2$. Here $\Sigma_{g,0,0}$ is the unique torsion-free, index-two subgroup of $\Sigma_{0,0,(2g+2)^{(2)}}$, so it is a characteristic subgroup. Thus there is a sign-preserving restriction map $\mathrm{Aut}_{0,0,(2g+2)^{(2)}} \to \mathrm{Aut}_{g,0,0}$, and an induced map

$$\mathrm{Aut}_{0,0,(2g+2)^{(2)}}/\Sigma_{g,0,0} \to \mathrm{Out}_{g,0,0} \qquad (14)$$

where we understand that $\Sigma_{g,0,0}$ represents the image of the composite of natural maps

$$\Sigma_{g,0,0} \to \Sigma_{0,0,(2g+2)^{(2)}} \to \mathrm{Aut}_{0,0,(2g+2)^{(2)}}\,.$$

For $g = 0$, $\mathrm{Out}_{0,0,0} = \mathrm{Out}_{0,0,2} = C_2$, and (14) is bijective.

For $g \geqslant 1$, the domain of (14) is an extension of C_2 by $\mathrm{Out}_{0,0,2g+2}$, since there is a short exact sequence

$$1 \to C_2 \to \mathrm{Aut}_{0,0,(2g+2)^{(2)}} / \Sigma_{g,0,0} \to \mathrm{Out}_{0,0,(2g+2)^{(2)}} \to 1.$$

If $g = 1$, then (14) has kernel of order four, and is split surjective; see Remarks 13.7(v) and (viii).

For $g = 2$, a classic result of Birman-Hilden says that (14) is an isomorphism, and hence

$$\mathrm{Aut}_{0,0,6^{(2)}} \simeq \mathrm{Aut}_{2,0,0} \quad \text{and} \quad \mathrm{Out}_{0,0,6} \simeq \mathrm{Out}_{2,0,0} / C_2.$$

If $g \geqslant 3$, then (14) is injective, but not surjective. □

We now remove an invariant disc around one of the skewering points. We note that the hyperelliptic involution is not admissible here, so does not give rise to an element of order 2 in the corresponding mapping-class group.

13.5 Example. $S_{g,1,0}$ is a double branched cover of $S_{0,1,(2g+1)^{(2)}}$.

There is a homomorphism $\Sigma_{g,1,0} * E_1 \to \Sigma_{0,1,(2g+1)^{(2)}} * E_1$ given by

$$\begin{cases} x_i \mapsto t_{2i+1}t_{2i}, \quad 1 \leqslant i \leqslant g, \\ y_i \mapsto t_{2i+1} \prod_{i'=1}^{2i+1} t_{i'}, \quad 1 \leqslant i \leqslant g, \\ z_1 \mapsto z_1^2, \\ e_1 \mapsto e_1. \end{cases}$$

Here $[x_i, y_i]$ is mapped to $\prod_{i'=2i-1}^{1} t_{i'} \prod_{i'=2i}^{2i+1} t_{i'} \prod_{i'=1}^{2i+1} t_{i'}$. It follows that

$$\prod_{i'=1}^{i} [x_{i'}, y_{i'}] \mapsto \prod_{i'=1}^{2i+1} t_{i'} \prod_{i'=1}^{2i+1} t_{i'} = (\prod_{i'=1}^{2i+1} t_{i'})^2,$$

and we see that the map is well-defined. It is not difficult to show that it is injective, and identifies $\Sigma_{g,1,0}$ with a characteristic, index-two subgroup of $\Sigma_{0,1,(2g+1)^{(2)}}$.

This determines a sign-preserving homomorphism

$$\mathrm{Aut}_{0,\hat{1},2g+1} \to \mathrm{Aut}_{g,\hat{1},0},$$

which is equivalent to a homomorphism

$$\mathrm{Out}_{0,1,2g+1} \to \mathrm{Out}_{g,1,0}\,. \tag{15}$$

It can be shown that this is injective. For $g \leqslant 1$, (15) is bijective; see Remarks 13.7(vi) and (viii).

In (15), $\breve{\sigma}_1^2$ $(= \sigma_1^2 \widetilde{z}_1^{-2})$ is mapped to $\breve{\sigma}_1$ $(= \sigma_1 \widetilde{z}_1^{-1})$, and we get an injective quotient map

$$\mathrm{Out}_{0,1,2g+1} /\langle \breve{\sigma}_1^2 \rangle \to \mathrm{Out}_{g,1,0} /\langle \breve{\sigma}_1 \rangle.$$

For $g \geqslant 1$, an extension of C_2 by $\mathrm{Out}_{0,0,2g+1\perp 1}$ embeds in $\mathrm{Out}_{g,0,1}$. It follows, from Example 13.4, that this map remains injective even after composition with

$$\mathrm{elim}(t_1)\colon\ \mathrm{Out}_{g,0,1} \to \mathrm{Out}_{g,0,0}\,.$$

In general, we have the braid group on $2g+1$ strings, $\mathrm{Out}^+_{0,1,2g+1}$, embedded in $\mathrm{Out}^+_{g,1,0}$, and the corresponding map of Artin diagrams is as follows.

$$\begin{array}{ccccccccccccccc}
\breve{\mu}_1 & — & \breve{\mu}_2 & — & \breve{\mu}_3 & — & \breve{\mu}_4 & — & \breve{\mu}_5 & — & \cdots & — & \breve{\mu}_{2g-1} & — & \breve{\mu}_{2g} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\breve{\alpha}_1 & — & \breve{\beta}_1 & — & \breve{\gamma}_1 & — & \breve{\beta}_2 & — & \breve{\gamma}_2 & — & \cdots & — & \breve{\gamma}_{g-1} & — & \breve{\beta}_g & — & \breve{\gamma}_g .
\end{array}$$

The work of many authors culminated in a polished presentation of $\mathrm{Out}^+_{g,1,0}$ in terms of Artin diagrams; see [12]. For $g \leqslant 1$, we get the braid group on $2g+1$ strings. For $g \geqslant 2$, the generating set used is the set of $2g+1$ DLH generators, $\{\breve{\alpha}_i\}_{i=1}^{2} \cup \{\breve{\beta}_i\}_{i=1}^{g} \cup \{\breve{\gamma}_i\}_{i=1}^{g-1}$, which form the following Artin diagram in $\mathrm{Out}^+_{g,1,0}$.

$$\begin{array}{ccccccccccccc}
\breve{\alpha}_1 & & & & \breve{\alpha}_2 & & & & & & & & \\
| & & & & | & & & & & & & & \\
\breve{\beta}_1 & — & \breve{\gamma}_1 & — & \breve{\beta}_2 & — & \breve{\gamma}_2 & — & \cdots & — & \breve{\gamma}_{g-1} & — & \breve{\beta}_g
\end{array} \tag{16}$$

Here $\breve{\alpha}_2$ does not lie in the (faithful) image of the braid group $\mathrm{Out}^+_{0,1,2g+1}$. □

Rather than remove one invariant disc, we can remove a pair of disjoint discs which are interchanged by the rotation.

13.6 Example. $S_{g,2,0}$ is a double branched cover of $S_{0,1,(2g+2)^{(2)}}$.

Here there is a homomorphism $\Sigma_{g,2,0} * E_2 \to \Sigma_{0,1,(2g+2)^{(2)}} * E_1$ given by

$$\begin{cases} x_i \mapsto t_{2i+1}t_{2i}, \quad 1 \leqslant i \leqslant g, \\ y_i \mapsto t_{2i+1} \prod_{i'=1}^{2i+1} t_{i'}, \quad 1 \leqslant i \leqslant g, \\ z_1 \mapsto z_1^{t_{2g+2}}, \quad e_1 \mapsto t_{2g+2}e_1, \\ z_2 \mapsto z_1, \quad e_2 \mapsto e_1. \end{cases}$$

This map is well defined, since

$$\prod_{i=1}^{g}[x_i, y_i]z_1z_2 \mapsto (\prod_{i=1}^{2g+1} t_i)^2(t_{2g+2}z_1)^2 = 1,$$

and it identifies $\Sigma_{g,2,0}$ with a characteristic, index-two subgroup of $\Sigma_{0,1,(2g+2)^{(2)}}$. Notice that $\Sigma_{0,1,(2g+2)^{(2)}}e_1$ has a corresponding partition in two. Here we get a homomorphism $\mathrm{Aut}_{0,1,2g+2} \to \mathrm{Aut}_{g,2,0}$. For example, the image of σ_1 is $\sigma_1\sigma_2$, and the image of $\tilde{t}_{2g+2}$ interchanges e_1 and e_2. Now $\mathrm{Aut}_{0,\hat{1},2g+2}$ is mapped to $\mathrm{Aut}_{g,1\perp\hat{1},0}$, so we get a homomorphism

$$\mathrm{Out}_{0,1,2g+2} \to \mathrm{Out}_{g,1\perp 1,0}\,. \tag{17}$$

Topological arguments can be used to show the injectivity of (17); see [16].

For $g = 0$, (17) is bijective.

For $g \geqslant 1$, since (17) carries $\breve{\sigma}_1 = \sigma_1\tilde{z}_1^{-1}$ to $\breve{\sigma}_1\breve{\sigma}_2 = \sigma_1\sigma_2\tilde{z}_2^{-1}$, composing (17) with $\mathrm{elim}(e_2)$ and with $\mathrm{elim}(e_1, e_2)$ gives embeddings $\mathrm{Out}_{0,1,2g+2} \to \mathrm{Out}_{g,1,1}$ and $\mathrm{Out}_{0,0,2g+2\perp 1} \to \mathrm{Out}_{g,0,1\perp 1}$, respectively. For $g = 1$, these are bijective; see Remarks 13.7(ii) and (iv).

For $g \geqslant 1$, the braid group $\mathrm{Out}^+_{0,1,2g+2}$ embeds in $\mathrm{Out}^+_{g,1,1}$, and the corresponding map of Artin diagrams is as follows.

$$\begin{array}{ccccccccccccccc}
\breve{\mu}_1 & — & \breve{\mu}_2 & — & \breve{\mu}_3 & — & \breve{\mu}_4 & — & \breve{\mu}_5 & — \cdots — & \breve{\mu}_{2g-1} & — & \breve{\mu}_{2g} & — & \breve{\mu}_{2g+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\breve{\alpha}_1 & — & \breve{\beta}_1 & — & \breve{\gamma}_1 & — & \breve{\beta}_2 & — & \breve{\gamma}_2 & — \cdots — & \breve{\gamma}_{g-1} & — & \breve{\beta}_g & — & \breve{\gamma}_g.
\end{array}$$

□

13.7 Remarks. Let us collect together some of the results related to the sphere and the torus, as follows. Let V denote $C_2 \times C_2$. Let F_2 denote a free group of rank two, $F_2 \simeq \Sigma_{1,1} \simeq \Sigma_{0,3}$. The following hold.

(i) $\mathrm{Aut}_{1,0,2} \simeq \mathrm{Aut}_{0,0,4^{(2)}\perp 1}\,.$

(ii) $\mathrm{Out}_{1,1,1} \simeq \mathrm{Out}_{0,1,4}$
$\simeq \mathrm{Aut}_{1,1,0} \simeq \mathrm{Aut}_{1,0,\hat{1}} \ltimes F_2 \simeq \mathrm{Aut}_{0,0,4\perp\hat{1}} \simeq \mathrm{Aut}_{1,0,1\perp\hat{1}}\,.$

(iii) $\mathrm{Out}_{1,0,2} = \mathrm{Out}_{1,0,1\perp 1} \times C_2.$

(iv) $\mathrm{Out}_{1,0,2}\,/C_2 \simeq \mathrm{Out}_{1,0,1\perp 1} \simeq \mathrm{Out}_{0,0,4\perp 1}$
$\simeq \mathrm{Aut}_{0,0,4} \simeq \mathrm{Aut}_{1,0,1} \simeq \mathrm{Aut}(F_2) \simeq \mathrm{Aut}_{0,0,3^{(2)}\perp 1}\,.$

(v) $\mathrm{Out}_{0,0,4} = \mathrm{Out}_{0,0,3\perp 1} \ltimes V.$

(vi) $\mathrm{Out}_{1,1,0} \simeq \mathrm{Out}_{0,1,3}$
$\simeq \mathrm{Aut}_{1,0,\hat{1}} \simeq \mathrm{Aut}_{0,0,3\perp\hat{1}}\,.$

(vii) $\mathrm{Out}_{1,0,1} \simeq \mathrm{Out}_{1,0,0}$
$\simeq \mathrm{Aut}_{1,0,0} \simeq \mathrm{GL}_2(\mathbb{Z})$.

(viii) $\mathrm{Out}_{0,0,4}/V \simeq \mathrm{Out}_{1,0,1}/C_2 \simeq \mathrm{Out}_{0,0,3\perp 1}$
$\simeq \mathrm{PGL}_2(\mathbb{Z}) \simeq \mathrm{Aut}_{0,0,3}$.

The isomorphisms of the positive subgroups are of interest since $\mathrm{Out}^+_{0,1,4}$ (see (ii)) is the braid group on four strings, and passing modulo the center gives $\mathrm{Out}^+_{0,0,4\perp 1}$ (see (iv)). Similarly, $\mathrm{Out}^+_{0,1,3}$ (see (vi)) is the braid group on three strings, and passing modulo the center gives $\mathrm{Out}^+_{0,0,3\perp 1}$ (see (viii)).

Many of the interconnections arise from an action of $V \times C_2$ on $S_{1,0,4}$. Consider the affine action of $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$ on the plane $\mathbb{R}^2$. This induces an action on the punctured plane $\mathbb{R}^2 - \mathbb{Z}^2$. Modulo the action of $(2\mathbb{Z})^2$, this gives an action of $V \rtimes \mathrm{GL}_2(\mathbb{Z})$ on $(\mathbb{R}^2 - \mathbb{Z}^2)/(2\mathbb{Z})^2 = S_{1,0,4}$. This in turn gives a subgroup $V \rtimes \mathrm{GL}_2(\mathbb{Z})$ of $\mathrm{Out}_{1,0,4}$, which we denote by $\mathrm{Out}^\dagger_{1,0,4}$ for the purposes of this digression. The subgroup $V \times C_2$ acts on $S_{1,0,4}$, with the generator of C_2 acting as the hyperelliptic involution fixing each puncture. We find $S_{1,0,4}/C_2 = S_{0,0,4}$, $S_{1,0,4}/V = S_{1,0,1}$ and $S_{1,0,4}/(V \times C_2) = S_{0,0,3^{(2)}}$. The resulting action of V on $S_{1,0,4}/C_2 = S_{0,0,4}$ corresponds to taking an appropriate group of symmetries of the two-skeleton of a regular tetrahedron, and deleting the (four) vertices.

The abelianization of $\Sigma_{0,0,3^{(2)}\perp 1}$ is $V \times C_2$, and we find that $\Sigma_{0,0,3^{(2)}\perp 1}$ is an extension of $\Sigma_{1,0,1}$ by C_2, of $\Sigma_{0,0,4}$ by V, and of $\Sigma_{1,0,4}$ by $V \times C_2$. The subgroups $\Sigma_{1,0,4}$, $\Sigma_{1,0,1}$, and $\Sigma_{0,0,4}$ of $\Sigma_{0,0,3^{(2)}\perp 1}$ are characteristic, or $\mathrm{Aut}_{0,0,3^{(2)}\perp 1}$-invariant, and it can be shown that there are natural identifications

$$\mathrm{Aut}_{0,0,3^{(2)}\perp 1} = \mathrm{Aut}_{1,0,1} = \mathrm{Aut}_{0,0,4} = \mathrm{Aut}^\dagger_{1,0,4},$$

where $\mathrm{Aut}^\dagger_{1,0,4}$ denotes the image of the restriction map $\mathrm{Aut}_{0,0,3^{(2)}\perp 1} \to \mathrm{Aut}_{1,0,4}$. This gives identifications

$$\mathrm{Out}_{0,0,3^{(2)}\perp 1} = \mathrm{Out}_{1,0,1}/C_2 = \mathrm{Out}_{0,0,4}/V = \mathrm{Out}^\dagger_{1,0,4}/(V \times C_2).$$

Let us say a brief word about the proofs of (i)-(viii).

We first consider (iv). Notice that $\Sigma_{1,0,1}$ is free on $\{x_1, y_1\}$, that $\mathrm{Aut}_{1,0,\hat{1}}$ is the group of all automorphisms of $\Sigma_{1,0,1}$ which fix or invert the commutator $[x_1, y_1]$, and that $\mathrm{Aut}_{1,0,1}$ is the group of automorphisms of $\Sigma_{1,0,1}$ which carry the commutator to a conjugate of itself or its inverse. Now, by a result of Nielsen's, $\mathrm{Aut}_{1,0,1}$ is all of $\mathrm{Aut}(\Sigma_{1,0,1}) \simeq \mathrm{Aut}(F_2)$. The isomorphism $\mathrm{Out}^+_{0,0,4\perp 1} \to \mathrm{Aut}^+_{1,0,1}$ is described explicitly in [6], using presentations.

It is a simple matter to verify (i), (ii) and (iii) using (iv).

In (v), V is generated by $\breve{\mu}_1\breve{\mu}_3^{-1}$, $\breve{\mu}_2\breve{\mu}_1\breve{\mu}_3^{-1}\breve{\mu}_2^{-1}$. We leave the proof of (v) as an exercise. Notice the similarity to the classic decomposition $\mathrm{Sym}_4 = \mathrm{Sym}_{3\perp 1} \ltimes V$.

Now (vi), (vii) and (viii) are straightforward. □

14 Matsumoto's relator

Suppose that $g \geqslant 2$. The Artin diagram (16) in $\mathrm{Out}_{g,1,0}$ corresponds to the DLH generators. To find an interesting expression for $\breve{\gamma}_0$ in terms of these generators, we consider the Artin diagram

$$\begin{array}{ccccccccccc} & \breve{\alpha}_1 & \breve{\alpha}_2 & & & & & & & & \\ & & | & & & & & & & & \\ \breve{\gamma}_0 & & \breve{\gamma}_1 - \breve{\beta}_2 & - & \breve{\gamma}_2 & - & \cdots & - & \breve{\gamma}_{g-1} & - & \breve{\beta}_g. \end{array} \tag{18}$$

in $\mathrm{Out}_{g,1,0}$. It determines a homomorphism

$$\mathrm{Artin}(2A_1 + D_{2g-1}) \to \mathrm{Out}_{g,1,0}\,.$$

The distinguished basis of the center of $\mathrm{Artin}(2A_1 + D_{2g-1})$ then distinguishes three elements

$$Z_{A_{1,1}} = \breve{\gamma}_0, \quad Z_{A_{1,2}} = \breve{\alpha}_1, \quad Z_{D_{2g-1}} = (\breve{\alpha}_2 \prod_{i=1}^{g-1} (\breve{\gamma}_i \breve{\beta}_{i+1}))^{4g-4},$$

of $\mathrm{Out}_{g,1,0}$, and we shall find that $Z_{A_{1,1}}\, Z_{A_{1,2}}^{2g-3}\, Z_{D_{2g-1}}^{-1} = 1$.

14.1 Theorem. *If* $g \geqslant 2$, *then* $(\breve{\alpha}_2 \cdot \prod_{i=1}^{g-1} (\breve{\gamma}_i \breve{\beta}_{i+1}))^{4g-4} = \breve{\gamma}_0\ \breve{\alpha}_1^{2g-3}$ *in* $\mathrm{Out}_{g,1,0}$.

Proof. We work in $\mathrm{Aut}_{g,1,0}$. Suppose $2 \leqslant n \leqslant g$. It is straightforward to show

$$\gamma_{n-1}\beta_n\colon \begin{cases} x_{n-1} \mapsto y_n x_n \overline{y}_{n-1} x_{n-1}, \\ y_{n-1} \mapsto y_n x_n y_{n-1} \overline{x}_n \overline{y}_n, \\ x_n \quad\ \mapsto x_n y_{n-1} \overline{x}_n \overline{y}_n, \\ y_n \quad\ \mapsto x_n y_n, \\ w \quad\ \ \mapsto w \quad \text{for all} \quad w \in E_1 \cup X_{g,1,0} - \{x_{n-1}, y_{n-1}, x_n, y_n\}. \end{cases}$$

By induction,

$$\prod_{i=n}^{g}(\gamma_{i-1}\beta_i)\colon \begin{cases} w \quad\mapsto w \quad \text{for all} \quad w \in \{x_i, y_i\}_{i=1}^{n-2} \cup \{z_1, e_1\}, \\ x_{n-1} \mapsto y_g \cdot \prod_{i=g}^{n} x_i \cdot \overline{y}_{n-1} x_{n-1}, \\ y_{n-1} \mapsto y_g \cdot \prod_{i=g}^{n} x_i \cdot y_{n-1} \cdot \prod_{i=n}^{g} \overline{x}_i \cdot \overline{y}_g, \\ x_i \quad\mapsto y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot \overline{y}_i x_i y_{i-1} \cdot \prod_{i'=i}^{g} \overline{x}_{i'} \cdot \overline{y}_g, \quad n \leqslant i \leqslant g, \\ y_i \quad\mapsto y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot \overline{y}_i x_i y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot y_i \cdot \prod_{i'=i+1}^{g} \overline{x}_{i'} \cdot \overline{y}_g, \\ \hfill n \leqslant i \leqslant g. \end{cases}$$

In particular,

$$\prod_{i=2}^{g}(\gamma_{i-1}\beta_i)\colon \begin{cases} x_1 \mapsto y_g \cdot \prod_{i=g}^{2} x_i \cdot \overline{y}_1 x_1, \\ y_1 \mapsto y_g \cdot \prod_{i=g}^{2} x_i \cdot y_1 \cdot \prod_{i=2}^{g} \overline{x}_i \cdot \overline{y}_g, \\ x_i \mapsto y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot \overline{y}_i x_i y_{i-1} \cdot \prod_{i'=i}^{g} \overline{x}_{i'} \cdot \overline{y}_g, \quad 2 \leqslant i \leqslant g, \\ y_i \mapsto y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot \overline{y}_i x_i y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot y_i \cdot \prod_{i'=i+1}^{g} \overline{x}_{i'} \cdot \overline{y}_g, \\ \hfill 2 \leqslant i \leqslant g, \\ z_1 \mapsto z_1, \quad e_1 \mapsto e_1. \end{cases}$$

Now

$$\alpha_2 \cdot \prod_{i=2}^{g}(\gamma_{i-1}\beta_i)\colon \begin{cases} x_1 \mapsto y_g \cdot \prod_{i=g}^{2} x_i \cdot \overline{y}_1 x_1, \\ y_1 \mapsto y_g \cdot \prod_{i=g}^{2} x_i \cdot y_1 \cdot \prod_{i=2}^{g} \overline{x}_i \cdot \overline{y}_g, \\ x_2 \mapsto y_g \cdot \prod_{i=g}^{3} x_i \cdot \overline{y}_2 \cdot \prod_{i=3}^{g} \overline{x}_i \cdot \overline{y}_g y_1 \cdot \prod_{i=2}^{g} \overline{x}_i \cdot \overline{y}_g, \\ x_i \mapsto y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot \overline{y}_i x_i y_{i-1} \cdot \prod_{i'=i}^{g} \overline{x}_{i'} \cdot \overline{y}_g, \quad 3 \leqslant i \leqslant g, \\ y_i \mapsto y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot \overline{y}_i x_i y_g \cdot \prod_{i'=g}^{i+1} x_{i'} \cdot y_i \cdot \prod_{i'=i+1}^{g} \overline{x}_{i'} \cdot \overline{y}_g, \\ \hfill 2 \leqslant i \leqslant g, \\ z_1 \mapsto z_1, \quad e_1 \mapsto e_1, \end{cases}$$

and

$$(\alpha_2 \prod_{i=2}^{g}(\gamma_{i-1}\beta_i))^2\colon \begin{cases} x_1 \mapsto x_g \overline{y}_1 x_1, \\ y_1 \mapsto x_g y_1 \overline{x}_g, \\ x_2 \mapsto x_g \cdot \prod_{i=2}^{g} \overline{x}_i \cdot [y_g, x_g] \cdot \prod_{i=g-1}^{2} x_i \cdot \overline{x}_g, \\ y_2 \mapsto x_g \cdot \prod_{i=2}^{g} \overline{x}_i \cdot \overline{y}_g y_1 \cdot \prod_{i=2}^{g-1} \overline{x}_i \cdot [x_g, y_g] \cdot \prod_{i=g}^{2} x_i \cdot \overline{x}_g, \\ x_i \mapsto x_g x_{i-1} \overline{x}_g, \hfill 3 \leqslant i \leqslant g, \\ y_i \mapsto x_g y_{i-1} \cdot \prod_{i'=i}^{g-1} \overline{x}_{i'} \cdot [x_g, y_g] \cdot \prod_{i'=g}^{i} x_{i'} \cdot \overline{x}_g, \\ \hfill 3 \leqslant i \leqslant g, \\ z_1 \mapsto z_1, \quad e_1 \mapsto e_1. \end{cases}$$

Then

$$(\alpha_2 \prod_{i=2}^{g}(\gamma_{i-1}\beta_i))^2\widetilde{x}_g\colon \begin{cases} x_1 \mapsto \overline{y}_1 x_1 x_g, \\ y_1 \mapsto y_1, \\ x_2 \mapsto \prod_{i=2}^{g}\overline{x}_i \cdot [y_g, x_g] \cdot \prod_{i=g-1}^{2} x_i, \\ y_2 \mapsto \prod_{i=2}^{g}\overline{x}_i \cdot \overline{y}_g y_1 \cdot \prod_{i=2}^{g-1}\overline{x}_i \cdot [x_g, y_g] \cdot \prod_{i=g}^{2} x_i, \\ x_i \mapsto x_{i-1}, & 3 \leqslant i \leqslant g, \\ y_i \mapsto y_{i-1} \cdot \prod_{i'=i}^{g-1}\overline{x}_{i'} \cdot [x_g, y_g] \cdot \prod_{i'=g}^{i} x_{i'}, & 3 \leqslant i \leqslant g, \\ z_1 \mapsto \overline{x}_g z_1 x_g, \quad e_1 \mapsto \overline{x}_g e_1. \end{cases}$$

Let $\phi = \overline{\alpha}_1(\alpha_2 \prod_{i=2}^{g}(\gamma_{i-1}\beta_i))^2\widetilde{x}_g$, so

$$\phi\colon \begin{cases} x_1 \mapsto x_1 x_g, \\ y_1 \mapsto y_1, \\ x_2 \mapsto \prod_{i=2}^{g}\overline{x}_i \cdot [y_g, x_g] \cdot \prod_{i=g-1}^{2} x_i, \\ y_2 \mapsto \prod_{i=2}^{g}\overline{x}_i \cdot \overline{y}_g y_1 \cdot \prod_{i=2}^{g-1}\overline{x}_i \cdot [x_g, y_g] \cdot \prod_{i=g}^{2} x_i, \\ x_i \mapsto x_{i-1}, & 3 \leqslant i \leqslant g, \\ y_i \mapsto y_{i-1} \cdot \prod_{i'=i}^{g-1}\overline{x}_{i'} \cdot [x_g, y_g] \cdot \prod_{i'=g}^{i} x_{i'}, & 3 \leqslant i \leqslant g, \\ z_1 \mapsto z_1^{x_g}, \quad e_1 \mapsto \overline{x}_g e_1. \end{cases}$$

For $1 \leqslant n \leqslant g-1$, let $m = g-1-n$, so $0 \leqslant m \leqslant g-2$ and $\phi^n = \phi^{g-1-m}$. By induction,

$$\phi^n\colon \begin{cases} x_1 \mapsto x_1 \cdot \prod_{i=g}^{m+2} x_i, \\ y_1 \mapsto y_1, \\ \begin{cases} \text{for} \quad 2 \leqslant i \leqslant n+1 \ (= g-m), \\ x_i \mapsto \prod_{i'=2}^{i+m}\overline{x}_{i'} \cdot [y_{i+m}, x_{i+m}] \cdot \prod_{i'=i+m-1}^{2} x_{i'}, \\ y_i \mapsto \prod\limits_{i'=2}^{i+m}\overline{x}_{i'} \cdot \overline{y}_{i+m} y_1 \cdot \prod\limits_{i'=2}^{m+1}\overline{x}_{i'} \cdot \prod\limits_{i'=m+2}^{m+i}[x_{i'}, y_{i'}] \cdot \prod\limits_{i'=i+m}^{2} x_{i'}, \end{cases} \\ \begin{cases} \text{for} \quad n+2 \ (= g-m+1) \leqslant i \leqslant g, \\ x_i \mapsto x_{i-n}, \\ y_i \mapsto y_{i-n} \cdot \prod_{i'=i-n+1}^{g-n}\overline{x}_i \cdot \prod_{i'=g-n+1}^{g}[x_{i'}, y_{i'}] \cdot \prod_{i'=g}^{i-n+1} x_i, \end{cases} \\ z_1 \mapsto z_1^{\prod_{i=g}^{m+2} x_i}, \quad e_1 \mapsto \prod_{i=m+2}^{g}\overline{x}_i \cdot e_1. \end{cases}$$

In particular,

$$\phi^{g-1}\colon \begin{cases} x_1 \mapsto x_1 \cdot \prod_{i=g}^{2} x_i, \\ y_1 \mapsto y_1, \\ \begin{cases} \text{for} \quad 2 \leqslant i \leqslant g, \\ x_i \mapsto \prod_{i'=2}^{i} \overline{x}_{i'} \cdot [y_i, x_i] \cdot \prod_{i'=i-1}^{2} x_{i'}, \\ y_i \mapsto \prod_{i'=2}^{i} \overline{x}_{i'} \cdot \overline{y}_i y_1 \cdot \prod_{i'=2}^{i} [x_{i'}, y_{i'}] \cdot \prod_{i'=i}^{2} x_{i'}, \end{cases} \\ z_1 \mapsto z_1^{\prod_{i=g}^{2} x_i}, \quad e_1 \mapsto \prod_{i=2}^{g} \overline{x}_i \cdot e_1. \end{cases}$$

On squaring the latter map, and recalling the definition of ϕ, we find

$$(\overline{\alpha}_1(\alpha_2 \prod_{i=2}^{g} (\gamma_{i-1}\beta_i))^2 \widetilde{x}_g)^{2g-2}\colon \begin{cases} x_1 \mapsto x_1 \cdot \prod_{i=g}^{2} [y_i, x_i] = x_1 z_1 [x_1, y_1], \\ w \mapsto w^{y_1} \text{ for all } w \in X_{g,1,0} - \{x_1, z_1\}, \\ z_1 \mapsto z_1^{\prod_{i=g}^{2}[y_i,x_i]} = z_1^{[x_1,y_1]}, \\ e_1 \mapsto \prod_{i=2}^{g} [x_i, y_i] \cdot e_1 = [y_1, x_1]\overline{z}_1 e_1. \end{cases} \tag{19}$$

It is straightforward to show that (19) is $\gamma_0 \alpha_1^{-1} \widetilde{y}_1$. Passing to $\mathrm{Out}_{g,1,0}$, we have the desired result. □

By Corollary 11.9, $Z_{A_{1,2}}^{2-2g} Z_{D_{2g-1}} = Z_{A_{1,1}} Z_{A_{1,2}}^{-1} = \breve{\gamma}_0 \breve{\alpha}_1^{-1}$ can be added as a relator to a presentation of $\mathrm{Out}_{g,1,0}$ on the DLH generators to give a presentation of $\mathrm{Out}_{g,0,0}$.

14.2 Theorem (Matsumoto)**.** *If $g \geqslant 2$, then $\breve{\Sigma}_{g,0,0}$ is the normal closure of*

$$\breve{\alpha}_1^{2-2g} (\breve{\alpha}_2 \cdot \prod_{i=1}^{g-1} (\breve{\gamma}_i \breve{\beta}_{i+1}))^{4g-4}$$

in $\mathrm{Out}^{+}_{g,1,0}$. □

15 Four viewpoints

Let $g \geqslant 1$.

In [16], it was shown that the Artin diagram

$$\begin{array}{ccccccccccc} & & \breve{\alpha}_1 & & & & & & & & \\ & & | & & & & & & & & \\ \breve{\gamma}_0 & — & \breve{\beta}_1 & — & \breve{\gamma}_1 & — & \cdots & — & \breve{\gamma}_{g-1} & — & \breve{\beta}_g \end{array} \tag{20}$$

in $\mathrm{Out}_{g,1,1}$ gives rise to an embedding $\mathrm{Artin}(D_{2g+1}) \to \mathrm{Out}_{g,1,1}$.

It is interesting to relate this to Theorem 14.1, which says that the Artin diagram

$$\begin{array}{ccccccccccc} & & \breve{\alpha}_2 & & & & & & & & \\ & & | & & & & & & & & \\ \breve{\gamma}_1 & — & \breve{\beta}_2 & — & \breve{\gamma}_2 & — & \cdots & — & \breve{\gamma}_g & — & \breve{\beta}_{g+1} \end{array} \tag{21}$$

in $\mathrm{Out}_{g+1,1,0}$ has $\breve{\gamma}_0\ \breve{\alpha}_1^{2g-1}$ as distinguished element.

In Remarks 15.1, 15.2, and 15.4, we shall discuss partial isomorphisms

$$\mathrm{Out}_{g,1,1} \leftrightsquigarrow \mathrm{Out}_{g,2,0} \leftrightsquigarrow \mathrm{Out}_{g,3,0} \leftrightsquigarrow \mathrm{Out}_{g+1,1,0}$$

such that the Artin diagrams in $\mathrm{Out}_{g,1,1}$, $\mathrm{Out}_{g,2,0}$ and $\mathrm{Out}_{g,3,0}$ determined by (20), and the Artin diagram in $\mathrm{Out}_{g+1,1,0}$ determined by (21) all correspond. Hence we get a distinguished element in each group, and we shall see that these are, respectively, $\breve{\sigma}_1$, $\breve{\sigma}_1\breve{z}_2 = \breve{\sigma}_1\breve{\sigma}_2^{2g-1}$, $\breve{\tau}_1^2\breve{\sigma}_3^{2g-1}$ and $\breve{\gamma}_0\ \breve{\alpha}_1^{2g-1}$, since we shall see that these correspond under the partial isomorphisms. Thus, calculating the distinguished element in any one case gives it in all four cases.

15.1 Remarks. In Section 16, we shall construct a strange partial isomorphism

$$\mathrm{unelim}(b_2)\colon\ \mathrm{Out}_{g,1,1} \rightsquigarrow \mathrm{Out}_{g,2,0}\,.$$

It corresponds to distinguishing, or removing, an open disk centered at the puncture. It respects (20), and carries $\breve{\sigma}_1$ to $\breve{\sigma}_1\breve{z}_2 = \breve{\sigma}_1\breve{\sigma}_2^{2g-1}$.

This is a partial left inverse of the map $\mathrm{elim}(b_2)\colon\ \mathrm{Out}_{g,1\perp 1,0} \to \mathrm{Out}_{g,1,1}$ which respects (20), and carries $\breve{\sigma}_1$ to $\breve{\sigma}_1$, and $\breve{\sigma}_2$ to 1. □

15.2 Remarks. Essentially as in Definitions 10.1, we can construct a map

$$\mathrm{pinch}(z_1)\colon\ \mathrm{Out}_{g,1\perp 1,0} \to \mathrm{Out}_{g,1\perp 1\perp 1,0}$$

corresponding to pinching the first boundary component to get the first two boundary components, and renumbering the second boundary component as the third boundary component. It is a left inverse of the map

$$\mathrm{elim}(z_2, e_2)\colon\ \mathrm{Out}_{g,1\perp 1\perp 1,0} \to \mathrm{Out}_{g,1\perp 1,0}$$

which corresponds to eliminating the second boundary component and renumbering the third boundary component as the second boundary component. Hence we have a partial isomorphism

$$\mathrm{pinch}(z_1)\colon\ \mathrm{Out}_{g,2,0} \rightsquigarrow \mathrm{Out}_{g,3,0}\,.$$

It respects (20), and carries $\breve{\sigma}_1$ to $\breve{\tau}_1^2$, and $\breve{\sigma}_2$ to $\breve{\sigma}_3$. □

15.3 Example. $g \geqslant 1$, $b \geqslant 4$, $p = 0$.

The interested reader can pinch z_1 a total of $b - 2\,(\geqslant 2)$ times and find that, for the Artin diagram

$$\begin{array}{ccccccccccccccc} & \breve{\alpha}_1 & & & & & & & & & & & & & \\ & | & & & & & & & & & & & & & \\ \breve{\gamma}_0 & — \breve{\beta}_1 & — & \cdots & — & \breve{\gamma}_{g-1} & — & \breve{\beta}_g & \quad & \breve{\tau}_1 & — \breve{\tau}_2 & — & \cdots & — \breve{\tau}_{b-2} & \quad \breve{\sigma}_b \end{array}$$

in $\mathrm{Out}_{g,b,0}$, one has $Z_{D_{2g+1}} Z_{A_{b-2}}^{-1} Z_{A_1}^{1-2g} = 1$, that is,

$$(\breve{\alpha}_1 \cdot \prod_{i=1}^{g} (\breve{\gamma}_{i-1}\breve{\beta}_i))^{4g} = (\prod_{j=1}^{b-2} \breve{\tau}_j)^{b-1}\, \breve{\sigma}_b^{2g-1}.$$

Here $Z_{A_{b-2}}$ corresponds to a Dehn twist around the first $b - 1$ boundary components. □

If $b \geqslant 2$ and we identify two boundary components of $S_{g,b,p}$, respecting the referential S^1, then we get a surface $S_{g+1,b-2,p}$. This operation behaves moderately well with respect to homotopies. We consider a special case.

15.4 Definition. There is a map

$$\Sigma_{g,3,0} * E_3 \to \Sigma_{g+1,1,0} * E_1$$

given by

$$\begin{array}{llll} z_2 \mapsto \overline{x}_1\overline{y}_1 x_1, & e_2 \mapsto 1, & & \\ z_3 \mapsto y_1, & e_3 \mapsto x_1, & & \\ x_i \mapsto x_{i+1}, & y_i \mapsto y_{i+1}, & \text{for} & 1 \leqslant i \leqslant g, \\ z_1 \mapsto z_1, & e_1 \mapsto e_1. & & \end{array}$$

There is then an induced map $\mathrm{Aut}_{g,\hat{1}\perp\hat{1}\perp 1,0} \to \mathrm{Aut}_{g+1,\hat{1},0}$, which is easily seen to be an injective. One can check that this map sends α_i to α_{i+1} and β_i to β_{i+1}, for $1 \leqslant i \leqslant g$, and sends γ_i to γ_{i+1} for $0 \leqslant i \leqslant g - 1$. Also, it sends σ_3 to α_1, and sends $\tau_1^2 \widetilde{z}_2^{-1} \widetilde{z}_1^{-1}$ to

$$\gamma_0 \widetilde{x}_1^{-1} \widetilde{y}_1 \widetilde{x}_1 \widetilde{z}_1^{-1} \colon \begin{cases} x_1 \mapsto z_1 \overline{x}_1 \overline{y}_1 x_1 x_1, \\ w \mapsto w^{\overline{x}_1 y_1 x_1 \overline{z}_1} \quad \text{for all} \quad w \in X_{g,1,0} - \{x_1, z_1\}, \\ z_1 \mapsto z_1, \quad e_1 \mapsto e_1. \end{cases}$$

By Proposition 4.1, this can be viewed as an injective partial map

$$\mathrm{glue}(z_2^{e_2}, z_3^{-e_3}) \colon \mathrm{Out}_{g,3,0} \rightsquigarrow \mathrm{Out}_{g+1,1,0}\,.$$

It transforms (20) to (21) and carries $\breve{\sigma}_3$ to $\breve{\alpha}_0$ and $\breve{\tau}_1^2$ to $\breve{\gamma}_0$. □

16 Converting a puncture to a boundary

We can convert a puncture into a boundary component by removing, or distinguishing, an open disk centered at the puncture. This gives an inclusion of surfaces which behaves badly with respect to homotopies, but we get partial homomorphisms of mapping-class groups.

The following is a consequence of Lemma 11.6.

16.1 Lemma. *If* $b \geqslant 1$ *and* $g \geqslant 1$, *then the following hold in* $\mathrm{Aut}_{g,b,p}$.

$$\begin{aligned}
\widehat{x}_i^{\alpha_i} &= (\widehat{y}_i\sigma_b^{1-i})^{-1}\,\widehat{x}_i\,\sigma_b^{i-1}, \ \textit{if} \quad 1 \leqslant i \leqslant g.\\
(\widehat{y}_i\,\sigma_b^{1-i})^{\beta_i} &= \widehat{x}_i\,(\widehat{y}_i\,\sigma_b^{1-i}), \ \textit{if} \quad 1 \leqslant i \leqslant g.\\
\widehat{x}_i^{\gamma_i} &= \widehat{x}_{i+1}^{-1}\,(\widehat{y}_{i+1}\sigma_b^{-i})\,\widehat{x}_{i+1}\,(\widehat{y}_i\sigma_b^{1-i})^{-1}\,\widehat{x}_i, \ \textit{if} \quad 1 \leqslant i \leqslant g-1.\\
(\widehat{y}_i\,\sigma_b^{1-i})^{\gamma_i} &= \widehat{x}_{i+1}^{-1}\,(\widehat{y}_{i+1}\,\sigma_b^{-i})\,\widehat{x}_{i+1}\,(\widehat{y}_i\,\sigma_b^{1-i})\,\widehat{x}_{i+1}^{-1}\,(\widehat{y}_{i+1}\,\sigma_b^{-i})^{-1}\,\widehat{x}_{i+1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad \textit{if}\ 1 \leqslant i \leqslant g-1.\\
\widehat{x}_{i+1}^{\gamma_i} &= \widehat{x}_{i+1}\,(\widehat{y}_i\,\sigma_b^{1-i})\,\widehat{x}_{i+1}^{-1}\,(\widehat{y}_{i+1}\,\sigma_b^{-i})^{-1}\,\widehat{x}_{i+1}, \ \textit{if} \quad 1 \leqslant i \leqslant g-1.
\end{aligned}$$

□

16.2 Definition. Suppose that $b \geqslant 2$.

We shall construct a partial splitting of the exact sequence in Proposition 6.2. To be more precise, we shall construct a partial splitting of the index $b-1$ exact subsequence

$$1 \to \langle \breve{\sigma}_b \rangle \to \mathrm{Out}_{g,b-2\perp1\perp1,p} \to \mathrm{Out}_{g,b-2\perp1,p\perp1} \to 1. \qquad (22)$$

By Theorem 10.2 and Corollary 8.4, we have semidirect product decompositions

$$\mathrm{Out}_{g,b-2\perp1\perp1,p} \simeq \mathrm{Out}_{g,b-2\perp1,p} \ltimes \breve{\Sigma}_{g,b-1,p}, \qquad (23)$$

$$\mathrm{Out}_{g,b-2\perp1,p\perp1} \simeq \mathrm{Out}_{g,b-2\perp1,p} \ltimes \widetilde{\Sigma}_{g,b-1,p}. \qquad (24)$$

Recall that an element of $\mathrm{Out}_{g,b-2\perp1,p}$ lifts back to a unique representative in $\mathrm{Aut}_{g,b-2\perp\hat{1},p}$, and this acts on $\widetilde{\Sigma}_{g,b-1,p}$. The representative in turn lifts back to a well-defined second representative in $\mathrm{Aut}_{g,b-2\perp1\perp\hat{1},p}$, where $z_{b-1}z_b$ replaces z_{b-1}. This second representative acts on $\Sigma_{g,b,p}$, and hence acts on $\breve{\Sigma}_{g,b-1,p}$ by the formula in Theorem 11.4.

Moreover, (22) arises from (23) and (24) using the identification

$$\breve{\Sigma}_{g,b-1,p}/\langle \breve{\sigma}_b \rangle = \widetilde{\Sigma}_{g,b-1,p}.$$

Since $\widetilde{\Sigma}_{g,b-1,p}$ is free (of rank $2g + b + p - 2$), we can construct an isomorphism

$$\phi\colon \widetilde{\Sigma}_{g,b-1,p} \times \langle \breve{\sigma}_b \rangle \xrightarrow{\sim} \breve{\Sigma}_{g,b-1,p} \tag{25}$$

given by

$$\phi\colon \begin{cases} \widetilde{x}_i \mapsto \breve{x}_i, \quad \widetilde{y}_i \mapsto \breve{y}_i\, \breve{\sigma}_b^{1-i}, \quad \text{for} \quad 1 \leqslant i \leqslant g, \\ \widetilde{z}_j \mapsto \breve{z}_j, \quad \text{for} \quad 1 \leqslant j \leqslant b-2, \\ \widetilde{t}_k \mapsto \breve{t}_k, \quad \text{for} \quad 1 \leqslant k \leqslant p, \\ \breve{\sigma}_b \mapsto \breve{\sigma}_b. \end{cases}$$

Notice that

$$\widetilde{z}_{b-1} = \prod_{j=b-2}^{1} \widetilde{z}_j^{-1} \cdot \prod_{i=g}^{1} [\widetilde{y}_i, \widetilde{x}_i] \cdot \prod_{k=1}^{p} \widetilde{t}_k^{-1}$$

is mapped to

$$\prod_{j=b-2}^{1} \breve{z}_j^{-1} \cdot \prod_{i=g}^{1} [\breve{y}_i, \breve{x}_i] \cdot \prod_{k=1}^{p} \breve{t}_k^{-1} = \breve{z}_{b-1} \breve{z}_b = \breve{z}_{b-1} \breve{\sigma}_b^{1-2g-b-p}.$$

Here $\mathrm{Out}_{g,b-2\perp 1,p}$ acts on both sides of (25); let N denote the subgroup of $\mathrm{Out}_{g,b-2\perp 1,p}$ consisting of those elements which commute with ϕ, so ϕ is a map of N-groups. Then (25) lifts to an isomorphism

$$(N \ltimes \widetilde{\Sigma}_{g,b-1,p}) \times \langle \breve{\sigma}_b \rangle \quad \xrightarrow{\sim} \quad N \ltimes \breve{\Sigma}_{g,b-1,p}$$

We represent the restriction

$$N \ltimes \widetilde{\Sigma}_{g,b-1,p} \to N \ltimes \breve{\Sigma}_{g,b-1,p}$$

as a partial map

$$\mathrm{unelim}(e_b)\colon\ \mathrm{Out}_{g,b-2\perp 1,p\perp 1} \rightsquigarrow \mathrm{Out}_{g,b-2\perp 1\perp 1,p}\,.$$

It is a partial splitting of (22), with domain $N \ltimes \widetilde{\Sigma}_{g,b-1,p}$.

It follows from Lemma 16.1 that N contains $\{\breve{\alpha}_1\} \cup \{\breve{\beta}_i\}_{i=1}^{g} \cup \{\breve{\gamma}_i\}_{i=1}^{g-1}$.

We claim that N also contains $\breve{\sigma}_{b-1}^{(g,b-1,p)}$. The representative of $\breve{\sigma}_{b-1}^{(g,b-1,p)}$ in $\mathrm{Aut}_{g,b-2\perp\hat{1},p}$ is $\sigma_{b-1}\widetilde{z}_{b-1}^{-1}$, and the second representative, in $\mathrm{Aut}_{g,b-2\perp 1\perp\hat{1},p}$, is $\tau_{b-1}^2 \widetilde{z}_b^{-1} \widetilde{z}_{b-1}^{-1}$. The latter acts on each element of

$$\{x_i, y_i\}_{i=1}^{g} \cup \{z_j\}_{j=1}^{b-2} \cup \{t_k\}_{k=1}^{p}$$

as conjugation by $\prod_{k=p}^{1} t_k \cdot \prod_{i=1}^{g}[x_i, y_i] \cdot \prod_{j=1}^{b-2} z_j$. It is not difficult to show that this commutes with ϕ, so N contains $\breve{\sigma}_{b-1}^{(g,b-1,p)}$.

Hence $N \ltimes \widetilde{\Sigma}_{g,b-1,p}$, the domain of unelim$(e_b)$, contains

$$\{\breve{\alpha}_1\} \cup \{\breve{\beta}_i\}_{i=1}^{g} \cup \{\breve{\gamma}_i\}_{i=1}^{g-1},$$

and also contains $\breve{\gamma}_0 = \breve{\alpha}_1 \ltimes \widetilde{x}_1^{-1}\widetilde{y}_1\widetilde{x}_1$, and $\breve{\sigma}_{b-1}^{(g,b,p)} = \breve{\sigma}_{b-1}^{(g,b-1,p)} \ltimes \widetilde{z}_{b-1}$ which are mapped to $\breve{\alpha}_1\breve{x}_1^{-1}\breve{y}_1\breve{x}_1 = \breve{\gamma}_0$ and $\breve{\sigma}_{b-1}^{(g,b-1,p)} \ltimes \breve{z}_{b-1}\breve{z}_b = \breve{\sigma}_{b-1}^{(g,b,p)}\breve{z}_b$, respectively.

If $g \geqslant 2$, then N does not contain $\breve{\alpha}_2$, nor any proper power of $\breve{\alpha}_2$. Thus, if $g \geqslant 2$, then N has infinite index in $\mathrm{Out}_{g,b-2\perp 1,p}$. □

16.3 Example. $g \geqslant 2$, $b = 2$, $p = 0$.

The injective partial map unelim(e_2): $\mathrm{Out}_{g,1,1} \rightsquigarrow \mathrm{Out}_{g,2,0}$ respects (20) and carries $\breve{\sigma}_1$ to $\breve{\sigma}_1\breve{z}_2 = \breve{\sigma}_1\breve{\sigma}_2^{2g-1}$, as desired. □

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