Canonical forms of shift-invariant maps on $[\mathbb{N}]^\infty$

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Abstract

We describe a canonical form for continuous functions $\Phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ that commute with the shift map $X \mapsto X \setminus \{\min X\}$. Then we investigate in which cases such a function Φ satisfies that for every $A \in [\mathbb{N}]^\infty$, there is $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \Phi''[A]^\infty$. This will lead us to solution of Problem 8.3 of [6].

The family $[\mathbb{N}]^\infty$ of infinite sets of non-negative integers is a prototype of a Ramsey space described long ago in papers of Galvin-Prikry [4], Silver [12] and Ellentuck [1]. It is perhaps less known that Nash-Williams [7] proved the first infinite-dimensional version of Ramsey theorem in order to handle the shift graph on $[\mathbb{N}]^\infty$ (or more precisely on $[\mathbb{N}]^{<\mathbb{N}}$). Recall that the shift map $S : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ is defined by $S(A) = A \setminus \{\min A\}$. It is therefore quite natural to investigate how much of the infinite-dimensional Ramsey Theorem is captured by the chromatic properties of the shift graph $([\mathbb{N}]^\infty, S)$. Another motivation for the present note is the study initiated in [3] of the chromatic number theory for Borel coloring of Borel graphs. Note that the Galvin-Prikry Theorem shows that the Borel chromatic number of the shift graph $([\mathbb{N}]^\infty, S)$ is infinite. A problem from [6] asks for a characterization of those Borel subsets of $[\mathbb{N}]^\infty$ on which the shift graph has infinite Borel chromatic number. We shall address this question here by showing that not all infinitely Borel chromatic subgraphs of $([\mathbb{N}]^\infty, S)$ contain subgraphs of the form $[X]^\infty$ for $X \in [\mathbb{N}]^\infty$. We do this by first describing a canonical form of continuous maps $\phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ that commute with the shift map S. Then we show that there are maps $\phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ that commute with S whose ranges do not contain any set of the form $[X]^\infty$, $X \in [\mathbb{N}]^\infty$. It turns out that the canonical form for shift-invariant continuous maps $\phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ is in complexity somewhere between the canonical forms of arbitrary continuous maps of the form $\psi : [\mathbb{N}]^\infty \to \mathbb{N}$ and $\theta : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ described by Pudlak-Rödl [11] and Promel-Voigt [10], respectively. So it is not so surprising that we shall use some of the ideas appearing in these two papers.

Some notation. We denote by $[\mathbb{N}]^\infty$ the set of all infinite subsets of $\mathbb{N}$, the set of natural numbers. $[\mathbb{N}]^\infty$ can be seen as a subspace of the space $2^\mathbb{N}$ equipped with the product topology. Given an infinite $A \subseteq \mathbb{N}$, $[A]^\infty$ denotes the collection of infinite subsets of A. The map $S : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ defined by $S(A) = A \setminus \{\min A\}$ is the *shift map* on $[\mathbb{N}]^\infty$ and the corresponding (directed) graph $([\mathbb{N}]^\infty, S)$ is the *shift graph* on $\mathbb{N}$. We use $[\mathbb{N}]^{<\infty}$ to denote the collection of finite subsets of $\mathbb{N}$. For $s, t \in [\mathbb{N}]^{<\infty}$ and $A \in [\mathbb{N}]^\infty$, $s \sqsubset t$ and $s \sqsubset A$ mean that s is an initial segment of t or, respectively, of A; and we write $n < s$ and $s < t$ as abbreviations of $n < \min s$ and $\max s < \min t$; also, $A/n = \{m \in A : n < m\}$.

If $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, then

$$\mathcal{F}_{[s]} = \{t \in \mathcal{F} : s \sqsubset t\}$$
$$\mathcal{F}_{(s)} = \{t \setminus s : t \in \mathcal{F}_{[s]}\}$$
$$\mathcal{F} \upharpoonright A = \{s \in \mathcal{F} : s \subseteq A\}$$

We write $\mathcal{F}_{[n]}$ and $\mathcal{F}_{(n)}$ when $s = \{n\}$.

1 Uniform families and barriers

The purpose of this section is to gather some standard concepts and results related to Nash-Williams' notion of barrier on an infinite subset of $\mathbb{N}$, which are needed in the rest of the paper.

Definition 1 *([7]) A collection $\mathcal{B} \subseteq [\mathbb{N}]^{<\infty}$ is called a barrier on $\mathbb{N}$ if it is an anti-chain in the partial order given by proper end extension in $\mathcal{B}$, and for every $A \in [\mathbb{N}]^\infty$, there is $s \in \mathcal{B}$ such that $s \sqsubset A$. Analogously, we define barriers on any set A for $A \in [\mathbb{N}]^\infty$, namely, an anti-chain $\mathcal{F}$ with respect to $\sqsubset$ such that every $B \in [A]^\infty$ has an initial segment in $\mathcal{F}$.*

When we say that a collection $\mathcal{B} \subseteq [\mathbb{N}]^{<\infty}$ is a barrier, without explicitly mentioning any set A, it is because it is a barrier on $\mathbb{N}$ or it is a barrier on a set A which is determined by the context. If $\mathcal{B}$ is a barrier and $A \in [\mathbb{N}]^\infty$, we denote by $\iota_\mathcal{B}(A)$ the only initial segment of A which belongs to $\mathcal{B}$.

Definition 2 *([11]) Let α be a countable ordinal. A family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is α-uniform on $A \in [\mathbb{N}]^\infty$ if*

(i) $\alpha = 0$ *and* $\mathcal{F} = \{\emptyset\}$*, or*

(ii) $\alpha = \beta + 1$ *and for every* $n \in \mathbb{N}$*, the collection*

$$\mathcal{F}_{(n)} = \{t : n < t \text{ and } \{n\} \cup t \in \mathcal{F}\}$$

is α-uniform on A/n, or

(iii) α *is a limit ordinal and there is an increasing sequence* $\{\alpha_n : n \in \mathbb{N}\}$ *of ordinals with limit α such that for every $n \in \mathbb{N}$,*

$$\mathcal{F}_{(n)} = \{t : n < t, \{n\} \cup t \in \mathcal{F}\}$$

is α_n-uniform on A/n.

We say that $\mathcal{F}$ is uniform on A if it is α-uniform for some $\alpha < \omega_1$. As before, if the set A is not explicitly mentioned, it is because $A = \mathbb{N}$ or it is determined by the context.

It is easy to verify by induction on the countable ordinals that every uniform family is a barrier.

Definition 3 *Given a barrier $\mathcal{B}$, its rank is the height of the tree*

$$\hat{\mathcal{B}} = \{t : \exists s \in \mathcal{B}(t \sqsubseteq s)\}$$

with the partial order of end extension. Note that the rank of a barrier is always a countable ordinal.

It can be shown by induction on the rank that every barrier on A is a uniform family on some $B \in [A]^\infty$. Moreover, if the rank of $\mathcal{B}$ is α, there is $B \in [A]^\infty$ such that $\mathcal{B}$ is β-uniform on B, for some $\beta \leq \alpha$.

For more information about barriers and related concepts, the reader can consult [3] or [9], from where we take the following standard facts (see also [13]).

Proposition 1 *Let $P(\cdot,\cdot)$ be a property such that for each $n \in \mathbb{N}$ and $X \in [\mathbb{N}]^\infty$ the following holds*

(i) $P(n, X) \Rightarrow P(n, Y)$ *for every* $Y \in [X]^\infty$*, and*

(ii) $\exists Y \in [X]^\infty$ *such that* $P(n, Y)$.

Then, there is $A \in [\mathbb{N}]^\infty$ such that $P(n, A/n)$ holds for every $n \in A$.

Proof. Let $A_0 \in [\mathbb{N}]^\infty$ be such that $P(0, A_0)$, and let $a_0 = \min(A_0)$. Suppose we have defined $A_0, \ldots, A_n$ and $a_0, \ldots, a_n$. Let $A_{n+1} \in [A_n]^\infty$ be such that $P(a_n, A_{n+1})$, and put $a_{n+1} = \min(A_{n+1}/a_n)$. This way we obtain inductively the set $A = \{a_0, a_1, \ldots, \}$ with the desired property. $\Box$

The following three propositions from [11] will be used below.

Proposition 2 *([11]) For every family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and every $X \in [\mathbb{N}]^\infty$, there is a set $Y \in [X]^\infty$ such that $\mathcal{F} \restriction Y = \emptyset$ or $\mathcal{F} \restriction Y$ contains a uniform family (on Y).*

Proof. First notice that if $\mathcal{F}$ is α-uniform, then $\mathcal{F} \restriction A$ is α-uniform for all $A \in [\mathbb{N}]^\infty$. This is verified by induction on the ordinal $\alpha < \omega_1$.

By restricting ourselves to the set of $\subseteq$-minimal elements of $\mathcal{F}$, we can assume that $\mathcal{F}$ is an anti-chain with respect to the ordering given by $\subseteq$.

Let
$$\hat{\mathcal{F}} = \{t : \exists s \in \mathcal{F} \quad t \sqsubseteq s\}.$$

Case 1. If there is an infinite branch $s_1 \sqsubset s_2 \sqsubset \cdots$ in the tree $\hat{\mathcal{F}}$, since $\mathcal{F}$ is an anti-chain, $\mathcal{F} \restriction \bigcup_i s_i = \emptyset$.

Case 2. The tree $(\hat{\mathcal{F}}, \subseteq)$ is well founded (there are no infinite branches). In this case, we work by induction on the height of the tree. Suppose inductively that we have the result for every $\beta < \alpha$, and that $\hat{\mathcal{F}}$ is of height α. Notice that for every $n \in \mathbb{N}$, the height of $\hat{\mathcal{F}}_{(n)} = \{t : n < t, \{n\} \cup t \in \hat{\mathcal{F}}\}$ is less than α.

Consider the property $P(n, X)$ given by "$\mathcal{F}_{(n)}$ is disjoint from $[X]^{<\infty}$ or it includes a family uniform on X". By the observation at the beginning of this proof, P satisfies clause (i) of Proposition 1, and by the inductive hypothesis it satisfies (ii). Therefore, there is a set B such that for every n, $P(n, B/n)$ holds, i.e. for every $n \in B$, $\mathcal{F}_{(n)} \restriction B = \emptyset$ or $\mathcal{F}_{(n)} \restriction B$ is uniform on B/n.

If $\{n \in B : \mathcal{F}_{(n)} \restriction B = \emptyset\}$ is infinite, let Y equal this set. Otherwise, let Y be an infinite subset of B such that for every $n \in Y$, $\mathcal{F}_{(n)} \restriction Y$ is uniform on Y/n, and let α_n be the corresponding ordinal. Clearly, we can find such Y so that this sequence of ordinals is constant or strictly increasing, and in both cases $\mathcal{F} \restriction Y$ is uniform. $\Box$

Corollary 1 *([7]) Let $\mathcal{B} = \mathcal{T}_0 \cup \mathcal{T}_1$ be a partition of a barrier $\mathcal{B}$ into two pieces. Then there is a set A and $i \in \{0, 1\}$ such that $\mathcal{T}_i$ is a barrier on A.*

Proof. Applying Proposition 2 to $\mathcal{T}_0$ we obtain a set A on which $\mathcal{T}_0 \restriction A$ is a barrier or it is empty. In the second case, $\mathcal{B} \restriction A$ is contained in $\mathcal{T}_1$ and therefore $\mathcal{T}_1$ is a barrier on A. $\Box$

Proposition 3 *([11]) Let $\mathcal{B}$ be a barrier on X, and let $h\colon \mathcal{B} \to \mathbb{N}$ be such that $h(s) \notin s$ for every $s \in \mathcal{B}$. Then, there is $Y \in [X]^\infty$ such that $h(s) \notin Y$ for every $s \in \mathcal{B} \restriction Y$.*

Proof. By induction on $\alpha < \omega_1$. If the rank of $\mathcal{B}$ is 0, the result is trivial. Suppose it holds for barriers of rank $< \alpha$ and let $\mathcal{B}$ be of rank α. Pick n_0 arbitrarily, define h_{n_0} on $\mathcal{B}_{(n_0)}$ by $h_{n_0}(t) = h(\{n_0\} \cup t)$. By the inductive hypothesis, there is a set $Y_0 \in [X]^\infty$ such that $h_{n_0}(t) \notin Y_0$ for every $t \in \mathcal{B}_{(n_0)} \restriction Y_0$. Note that for any such t, $h_{n_0}(t) \neq n_0$. Now, let n_1 be the first element of Y_0 above n_0. and repeat the procedure with $\mathcal{B}_{(n_1)}$ and Y_0 to obtain Y_1 such that $h_{n_1}(t) \notin Y_1$ for every $t \in \mathcal{B}_{(n_1)} \restriction Y_1$, and $n_2 \in Y_1$ above n_1. Suppose we have defined Y_{n_i} and $n_{i+1} \in Y_{n_i}$. We apply the inductive hypothesis to $\mathcal{B}_{(n_{i+1})} \restriction Y_{n_i}$ and the function $h_{n_{i+1}}$ defined on $\mathcal{B}_{(n_{i+1})}$ by $h_{n_{i+1}}(t) = h(\{n_{i+1}\} \cup t)$, and obtain $Y_{n_{i+1}}$ such that $h_{n_{i+1}}(t) \notin Y_{n_{i+1}}$ for every $t \in B_{(n_{i+1})} \restriction Y_{n_{i+1}}$, and we put n_{i+2} equal to the first element of $Y_{n_{i+1}}$ above n_{i+1}.

Let $Z = \{n_i : i \in \mathbb{N}\}$. By construction, given $s = \{n_{i_0}, \ldots, n_{i_k}\} \in \mathcal{B}$, $h(s) \notin Z \setminus n_{i_0} = Z \cap [n_{i_0}, \infty)$. But $h(s)$ could belong to $\{n_0, \ldots n_{i_0 - 1}\}$. Using this we define a partition of $\mathcal{B}_{(n_{i_0})}$ into a finite number of pieces, and by Corollary 1, we can assume that $h''\mathcal{B}_{(n_{i_0})} \cap \{n_0, \ldots, n_{i_0-1}\}$ has at most one element. Thus, there is a function $f : Z \to \mathbb{N}$ such that for every $s \in \mathcal{B} \restriction Z$, if $h(s) \in Z$, $h(s) = f(\min s)$. By Ramsey's Theorem, there is an infinite $Y \subseteq Z$ such that $f''Y \cap Y = \emptyset$. The set Y has the required property.
□

Proposition 4 *([11]) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be barriers, and $\phi_1\colon \mathcal{S}_1 \to \mathbb{N}$, $\phi_2\colon \mathcal{S}_2 \to \mathbb{N}$ one to one functions. Then, there is $A \in [\mathbb{N}]^\infty$ such that either*

(i) $\mathcal{S}_1 \restriction A = \mathcal{S}_2 \restriction A$ and $\phi_1(s) = \phi_2(s)$ for every $s \in \mathcal{S}_1 \restriction A$, or

(ii) $\phi_1''\mathcal{S}_1 \restriction A \cap \phi_2''\mathcal{S}_2 \restriction A = \emptyset$.

Proof. Split $\mathcal{S}_1$ in two parts $\mathcal{S}_1 = \mathcal{T}_1 \cup \mathcal{T}_2$ defined as follows:

$\mathcal{T}_1 = \{s \in \mathcal{S}_1 : \exists t \in \mathcal{S}_2 \quad (\phi_1(s) = \phi_2(t))\}$ and $\mathcal{T}_2 = \mathcal{S}_1 \setminus \mathcal{T}_1$.

By Corollary 1 there is a set $X_0 \in [\mathbb{N}]^\infty$ such that $\mathcal{T}_1 \restriction X_0 = \mathcal{S}_1 \restriction X_0$ is a barrier on X_0, or $\mathcal{T}_2 \restriction X_0 = \mathcal{S}_1 \restriction X_0$ is a barrier on X_0.

In the second case, $\phi_1''\mathcal{S}_1 \restriction X_0 \cap \phi_2''\mathcal{S}_2 \restriction X_0 = \emptyset$. So, assume we are in the first case. For every $s \in \mathcal{S}_1 \restriction X_0$ there is a unique $t_s \in \mathcal{S}_2$ with $\phi_2(t_s) = \phi_1(s)$.

Split $\mathcal{S}_2 \restriction X_0$ in a similar way, namely, $\mathcal{S}_2 \restriction X_0 = \mathcal{T}_1' \cup \mathcal{T}_2'$, where $\mathcal{T}_1' = \{t \in \mathcal{S}_2 \restriction X_0 : \exists s \in \mathcal{S}_1 \quad (\phi_1(s) = \phi_2(t))\}$; and $\mathcal{T}_2' = \mathcal{S}_2 \setminus \mathcal{T}_1'$. As before, we can assume that there is $X_1 \in [X_0]^\infty$ such that $\mathcal{S}_2 \upharpoonright X_1 = \mathcal{T}_1' \upharpoonright X_1$ is a barrier on X_1, and for every $t \in \mathcal{S}_2 \upharpoonright X_1$, there is a unique $s_t \in \mathcal{S}_1$ such that $\phi_2(t) = \phi_1(s_t)$.

Now, partition $\mathcal{S}_1 \upharpoonright X_1$ as follows. $\mathcal{S}_1 \upharpoonright X_1 = \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_1 = \{s \in \mathcal{S}_1 \upharpoonright X_1 : s = t_s\}$ and $\mathcal{R}_2 = \{s \in \mathcal{T}_1 : s \neq t_s\}$. Using again Corollary 1, there is $X_2 \in [X_1]^\infty$ such that $\mathcal{S}_1 \upharpoonright X_2 = \mathcal{R}_1 \upharpoonright X_2$ is a barrier on X_2 or $\mathcal{S}_1 \upharpoonright X_2 = \mathcal{R}_2 \upharpoonright X_2$ is a barrier on X_2. In the first case, $\mathcal{S}_1 \upharpoonright X_2 \subseteq \mathcal{S}_2 \upharpoonright X_2$, and by maximality of barriers, we obtain $\mathcal{S}_1 \upharpoonright X_2 = \mathcal{S}_2 \upharpoonright X_2$. Clearly, ϕ_1 and ϕ_2 coincide there, so we get (i) of the statement. Suppose then that the second case occurs. Working with $\mathcal{S}_2$ in a similar way, we can assume that $\mathcal{S}_2 \upharpoonright X_2$ is such that for every $t \in \mathcal{S}_2 \upharpoonright X_2$ we have $t \neq s_t$

Define $h\colon \mathcal{S}_1 \upharpoonright X_2 \to \mathbb{N}$ by picking $h(s) \in t_s \setminus s$ in case $t_s \setminus s$ is non-empty, and otherwise putting $h(s)$ be any arbitrary number not in s. Similarly, define $g\colon \mathcal{S}_2 \upharpoonright X_2 \to \mathbb{N}$ by $h(t)$ is a member of $s_t \setminus t$ if possible, and any number not in t otherwise. By Proposition 3 there is $X_3 \in [X_2]^\infty$ such that for every $s \in \mathcal{S}_1 \upharpoonright X_3$, $h(s) \notin X_3$, and for every $t \in \mathcal{S}_2 \upharpoonright X_3$, $g(t) \notin X_3$. Take $s \in \mathcal{S}_1 \upharpoonright X_3$ and $t \in \mathcal{S}_2 \upharpoonright X_3$. If $h(s) \in t_s$, then $t_s \not\subset X_3$ and therefore $\phi_1(s) \neq \phi_2(t)$. If, on the contrary, $h(s) \notin t_s$, it is because $t_s \subset s$, and in this case, $g(t_s) \in s \setminus t_s$ (notice that $s_{t_s} = s$). But this contradicts that $s \subset X_3$. In conclusion, $\phi_1'' \mathcal{S}_1 \upharpoonright X_3 \cap \phi_2'' \mathcal{S}_2 \upharpoonright X_3 = \emptyset$. $\square$

Definition 4 *A function $\phi : \mathcal{S} \to \mathbb{N}$ is a canonical coloring of $\mathcal{S}$ on X if $\mathcal{S}$ is a barrier on X and there is a barrier $\mathcal{T}$ on X and a mapping $f : \mathcal{S} \to \mathcal{T}$ such that*

- *(a) $f(s) \subseteq s$ for every $s \in \mathcal{S}$,*
- *(b) for every $s, t \in \mathcal{S}$, $\phi(s) = \phi(t)$ if and only if $f(s) = f(t)$.*

  Clause (b) is equivalent to
- *(b') there is a one-to-one function $\psi : \mathcal{T} \to \mathbb{N}$ such that for every $s \in \mathcal{S}$, $\phi(s) = \psi(f(s))$.*

Theorem 1 *[11] For every barrier $\mathcal{S}$ on a set $X \in [\mathbb{N}]^\infty$ and every function $\phi : \mathcal{S} \to \mathbb{N}$, there is $Y \in [X]^\infty$ such that $\phi \upharpoonright (\mathcal{S} \upharpoonright Y)$ is a canonical on Y.*

In the next section we prove a slight extension of this theorem which is of interest for our analysis of shift-invariant continuous functions.

2 Shift-invariant continuous functions

The function $S : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ defined by $S(A) = A \setminus \{\min A\}$, will be called the shift operation on $[\mathbb{N}]^\infty$. The successive iterates of the shift are defined as follows: for every $A \in [\mathbb{N}]^\infty$,

$$S^{(0)}(A) = A, \text{ and}$$
$$S^{(n+1)}(A) = S(S^{(n)}(A)).$$

Definition 5 *Let $\Phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$, we say that Φ is shift-invariant if for every $X \in [\mathbb{N}]^\infty$, $\Phi(S(X)) = S(\Phi(X))$; in other words, if it commutes with the shift operation.*

Any shift-invariant continuous function from $[\mathbb{N}]^\infty$ into $[\mathbb{N}]^\infty$ is determined by a function defined on a barrier taking values in $\mathbb{N}$. We make this more precise.

For every $s \in [\mathbb{N}]^{<\infty}$, let $[s] = \{A \in [\mathbb{N}]^\infty : s \sqsubset A\}$. The collection of sets $\{[s] : s \in [\mathbb{N}]^{<\infty}\}$ is basis for a topology on $[\mathbb{N}]^\infty$, called the metric topology, which is the topology inherited from the product space $2^{\mathbb{N}}$. Each element of the basis is clopen in this topology, and every open subset of $[\mathbb{N}]^\infty$ is the union of a collection of pairwise disjoint basic sets. Moreover, since for every two basic sets are either disjoint or one is contained in the other, for every open set $\mathcal{O}$ we can select a pairwise disjoint family of basic subsets which covers it, namely, formed by the maximal basic sets contained in $\mathcal{O}$. We will consider continuous functions with respect to this topology. We use $[n]$ instead of $[\{n\}]$ to simplify notation.

Proposition 5 *For every shift-invariant continuous function $\Phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$, there is a barrier $\mathcal{B}$ and a function $\phi : \mathcal{B} \to \mathbb{N}$ such that for every $A \in [\mathbb{N}]^\infty$, $\phi(\iota_{\mathcal{B}}(A)) = \min \Phi(A)$. Moreover, for every $A \in [\mathbb{N}]^\infty$, $\Phi(A) = \{\phi(\iota_{\mathcal{B}}(S^{(n)}(A))) : n \in \mathbb{N}\}$.*

Proof. Given a continuous $\Phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$, the pre-image $\Phi^{-1}([n])$, for every $n \in \mathbb{N}$, is an open set. Let $\{[s_i^n] : i \in \mathbb{N}\}$ be a covering of $\Phi^{-1}([n])$ by pairwise disjoint basic neighborhoods. The collection $\{s_i^n : i, n \in \mathbb{N}\}$ of all the finite sets corresponding in this way to all the sets $\Phi^{-1}([n])$ for $n \in \mathbb{N}$ is a barrier which we call $\mathcal{B}(\Phi)$ or simply $\mathcal{B}$ when the Φ is clearly determined by the context.

We define the function $\phi : \mathcal{B}(\Phi) \to \mathbb{N}$ by

$$\phi(s) = n \text{ if and only if } n = \min(\Phi(A))$$

for any (every) $A \in [s]$.

As Φ is shift-invariant, and for each A, the function ϕ determines the first element of $\Phi(A)$, applying ϕ to the initial segments which belong to $\mathcal{B}$ of the consecutive shifts of A, we obtain the elements of $\Phi(A)$, its k-th element in the increasing enumeration is $\phi(\iota_{\mathcal{B}}(S^{(k)}(A)))$. $\square$

We will now restate Theorem 1 in order to include some additional information about the canonization which will be useful to us.

Theorem 2 *Let $\mathcal{B}$ be a barrier on a set X, and $\phi\colon \mathcal{B} \to \mathbb{N}$. We can find a set $Y \in [X]^\infty$ and $\mathcal{T}, f, \psi$ as given by Theorem 1 such that $\mathcal{T}, f, \psi$ canonize ϕ on Y with the following additional property: if $\mathcal{B}_0 = \{s \cap \min f(s) : s \in \mathcal{B}\}$ then, for every $t_0, t_1 \in \mathcal{B}_0 \restriction Y$, and every k above t_0, t_1, if for $i \in \{0,1\}$, s_i is the unique extension of t_i in $\mathcal{B}$ of the form $t_i \cup t'$ for some initial segment t' of Y/k, then $f(s_0) = f(s_1)$.*

Proof. The proof of Theorem 1 given in [11], which goes by induction on ω_1, gives the additional property we desire. We reproduce it here for completeness.

We can assume that $\mathcal{B}$ is uniform on X. For $\mathcal{B}$ of rank 0 the result is trivial, so suppose the result has been proved for every $\beta < \alpha$, and that $\mathcal{B}$ is α-uniform on X. Define for every $n \in \mathbb{N}$,

$$\phi_n(s) = \phi(\{n\} \cup s)$$

for every $s \in \mathcal{B}_{(n)}$. By the inductive hypothesis we can find $\mathcal{T}_n$, f_n and ψ_n, the corresponding family and functions which canonize ϕ_n, such that the property in the statement regarding the family $\mathcal{B}_0$ is satisfied for each n. By a simple diagonalization, we can assume that for each n, the function ϕ_n is canonized by $\mathcal{T}_n$, f_n and ψ_n on (n, ∞). We can also assume that the ranks of the $\mathcal{T}_n$ are all equal or form an increasing sequence.

We will prove that restricting ourselves to some infinite set, for every $n < m$ the following holds,

$$\text{either } \mathcal{T}_n \restriction (m, \infty) = \mathcal{T}_m, \text{ and } \psi_n \restriction \mathcal{T}_m = \psi_m, \text{ or} \tag{1}$$

$$\psi_n''(\mathcal{T}_n) \cap \psi''(\mathcal{T}_m) = \emptyset.$$

For this, it is enough to verify that the property $P(m, Y)$ defined by "for every $n < m$ (1) holds relativized to Y" satisfies the hypothesis of Proposition 1; hypothesis (i) clearly holds, and hypothesis (ii) follows from Proposition 4.

Thus, we may assume that for every $n < m \in X$ (1) holds, and using Ramsey's Theorem (for pairs), we can assume that the same part of the alternative always holds; and we proceed considering two cases.

Case 1. For every $n < m$, $\mathcal{T}_n \restriction (m, \infty) = \mathcal{T}_m$, and $\psi_n \restriction \mathcal{T}_m = \psi_m$.

In this case, define $Y = X \setminus \{n_0\}$ where n_0 is the first element of X, and $\mathcal{T} = \mathcal{T}_{n_0}$, $f(s) = f_n(s \setminus \{n\})$ for $s \in \mathcal{B}$ and $n = \min s$, $\psi = \psi_{n_0}$.

Clearly, $\mathcal{T}$ is a barrier on Y, $f(s) \subseteq s$ for every $s \in \mathcal{B} \restriction Y$, and ψ is one to one. Also,

$$\phi(s) = \phi_{(n)}(s \setminus \{n\}) = \psi_n(f_n(s \setminus \{n\})) = \psi_{n_0}(f_n(s \setminus \{n\})) = \psi(f(s)),$$

for every $s \in \mathcal{B} \restriction Y$ and $n = \min s$.

In this case, the inductive hypothesis gives immediately the additional property we are seeking.

Case 2. For every $n < m$, $\psi''_n(\mathcal{T}_n \restriction (m, \infty)) \cap \psi''(\mathcal{T}_m) = \emptyset$.

In this case, the minimal element of each s plays a role in the canonization. Define

$$\begin{aligned}
&\mathcal{T} = \{\{n\} \cup t : t \in \mathcal{T}_n\, ,\, n \in \mathbb{N}\},\\
&f(s) = \{n\} \cup f_n(s \setminus \{n\}) \text{ for every } s \in \mathcal{B} \text{ with } \min s = n,\\
&\psi(t) = \psi_n(t \setminus \{n\}) \text{ for } n = \min t.
\end{aligned}$$

It is clear that $\mathcal{T}$ is uniform, since $\mathcal{T}_n$ is uniform for each n and we have that the sequence $\{\alpha_n : n \in \mathbb{N}\}$ is constant or strictly increasing. It is also clear that for every s, $f(s) \subseteq S$.

To show that ψ is one-to-one we need some additional work. For every $n < m$ and every $u \subseteq (n, m]$, define a set $\mathcal{T}_{n,(u),m}$ and a mapping $\psi_{n,(u),m}$ as follows

$$v \in \mathcal{T}_{n,(u),m} \text{ if and only if } m < \min v \text{ and } u \cup v \in \mathcal{T}_n;$$

$$\text{and } \psi_{n,(u),m}(v) = \psi_n(u \cup v) \text{ for every } v \in \mathcal{T}_{n,(u),m}.$$

It should be clear that $\mathcal{T}_{n,(u),m}$ is a uniform family. As before, we can find a set X such that for every $n, m \in X$ with $n < m$ and $u \subseteq (n, m] \cap X$, one of the two following conditions holds when we restrict ourselves to X,

$$\mathcal{T}_{n,(u),m} = \mathcal{T}_m \text{ and } \psi_{n,(u),m} = \psi_m, \text{ or} \tag{2}$$

$$\psi''_{n,(u),m}\mathcal{T}_{n,(u),m} \cap \psi''_m\mathcal{T}_m = \emptyset. \tag{3}$$

Given s, $t \in \mathcal{T}$, $s \neq t$, we consider the following cases,

(i) if $\min s = n = \min t$, then $\psi(s) = \psi_n(s \setminus \{n\}) \neq \psi(t \setminus \{n\}) = \psi(t)$, the inequality holds since ψ_n is one to one;

(ii) if $n = \min s < \min t = m$, and $s \setminus \{n\} \subseteq (m, \infty)$, we have that $\psi(s) \neq \psi(t)$ by the assumption of Case 2;

(iii) if $\min s = n < \min t = m$ and $s \cap (n, m]$ is non-empty, let $u = s \cap (n, m]$. $\mathcal{T}_{n,(u),m}$ is uniform on X/m and

$$\mathrm{rank}(\mathcal{T}_{n,(s),m}) < \mathrm{rank}(\mathcal{T}_n) \leq \mathrm{rank}(\mathcal{T}_m)$$

which implies we must have (3). We conclude that ψ is one-to-one.

The functions ψ and f behave as desired since for every $s \in \mathcal{B}$ with $\min s = n$,

$$\phi(s) = \phi_{(n)}(s \setminus \{n\}) = \psi_n(f_n(s \setminus \{n\})) = \psi(\{n\} \cup f_n(s \setminus \{n\}) = \psi(f(s)).$$

To verify that the additional property in the statement of the theorem holds, notice that since $\min s$ belongs to $f(s)$ for every $s \in \mathcal{B} \restriction Y$, $\mathcal{B}_0$ consists only of the empty set and thus, for every k, there is a unique element of $\mathcal{B}$ (extending the empty set) which is an initial segment of Y/k. □

3 Images of shift-invariant continuous functions.

From now on, let $\Phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ be a shift-invariant continuous function, and $\mathcal{B} = \mathcal{B}(\Phi)$. Let $\phi : \mathcal{B} \to \mathbb{N}$ be the function obtained from Φ as in Proposition 5, that is, the function mapping a finite set $s \in \mathcal{B}$ to the first element of $\Phi(A)$ for any A with $s \sqsubset A$. Let Y be a set where the family $\mathcal{T}$ and the functions f, ψ canonize ϕ as given by Theorem 2. We can assume without loss of generality that $Y = \mathbb{N}$.

Lemma 1 *Given $A \in [\mathbb{N}]^\infty$, if there is $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \Phi''[A]^\infty$, then there is $B \in [A]^\infty$ such that the family $\{f(\iota_{\mathcal{B}}(S^{(n)}(B))) : n \in \mathbb{N}\}$ is pairwise disjoint.*

Proof. Let $[X]^\infty \subseteq \Phi''[A]^\infty$. It is clear that there is $\bar{X} \in [X]^\infty$ such that $\{\psi^{-1}(i) : i \in \bar{X}\}$ is pairwise disjoint. Take $B \in [A]^\infty$ such that $\Phi(B) = \bar{X}$. Then, since the function ψ is one-to-one, B is the desired subset of A. □

Proposition 6 *If the barrier $\mathcal{B} = \mathcal{B}(\Phi)$ is of finite rank, then there is a set $A \in [\mathbb{N}]^\infty$ with the property that for every $B \in [A]^\infty$, the set $\{f(\iota_{\mathcal{B}}(S^{(n)}(B)))\}$ is not pairwise disjoint if and only if the rank of the corresponding family $\mathcal{T}$ is > 1.*

Proof. Suppose $\mathcal{B}$ is of finite rank, say, $\mathcal{B} = [\mathbb{N}]^n$. By Corollary 1, there is $A \in [\mathbb{N}]^\infty$ and $i_1, \ldots, i_k < n$ such that for every $s \in \mathcal{B} \restriction A$, if s is written in increasing order as $s = \{s_0, s_1, \ldots, s_{n-1}\}$ then $f(s) = \{s_{i_1}, \ldots, s_{i_k}\}$ (see also [2]). In other words, $f(s)$ occupies always the same position within s (for $s \in \mathcal{B} \restriction A$). If for every $s \in \mathcal{B} \restriction A$, $f(s)$ has more than one element, then for every $B \in [A]^\infty$, $\{f(\iota_{\mathcal{B}}(Sh^n(B))) : n \in \mathbb{N}\}$ is not pairwise disjoint.

Conversely, if for every $s \in \mathcal{B}$, $f(s)$ always picks one element of s, then given $A \in [\mathbb{N}]^\infty$, the family $\{f(\iota_{\mathcal{B}}(Sh^n(X))) : n \in \mathbb{N}\}$ is pairwise disjoint, because it is a collection of singletons none of them can come from two different shifts of A. □

Notice that if the rank of $\mathcal{T}$ is 1, then there is a set A with $[X]^\infty \subseteq \Phi''[A]^\infty$ for some $X \in [\mathbb{N}]^\infty$.

The case when the barrier $\mathcal{B}$ is of infinite rank is more complex. In some cases, which will be characterized below, it is possible to find a very thin set A with $\Phi''[A]^\infty$ containing a set of the form $[X]^\infty$.

The collection $\{D \in [\mathbb{N}]^\infty : \{f(\iota_{\mathcal{B}}(S^{(n)}(D))) : n \in \mathbb{N}\}$ is pairwise disjoint$\}$ is a Borel subset of $[\mathbb{N}]^\infty$, and therefore, by the Galvin-Prikry Theorem [4], there is $A \in [\mathbb{N}]^\infty$ such that $[A]^\infty$ is contained in this set or in its complement. So, we need only to consider the following two cases:

(i) for every $D \in [\mathbb{N}]^\infty$, $\{f(\iota_{\mathcal{B}}(S^{(n)}(D))) : n \in \mathbb{N}\}$ is a pairwise disjoint family.

(ii) for every $D \in [\mathbb{N}]^\infty$, $\{f(\iota_{\mathcal{B}}(S^{(n)}(D))) : n \in \mathbb{N}\}$ is not pairwise disjoint.

Theorem 3 *Let $\Phi \colon [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ be a shift-invariant continuous function, and let $\mathcal{B}, \phi, \mathcal{T}, f$, and ψ be the corresponding barriers and functions defined as above. If for every $D \in [\mathbb{N}]^\infty$ the family $\{f(\iota_{\mathcal{B}}(S^{(n)}(D))) : n \in \mathbb{N}\}$ is not pairwise disjoint, then for no $A \in [\mathbb{N}]^\infty$ there is $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \Phi''[A]^\infty$.*

Proof. The theorem follows from Lemma 1. □

Theorem 4 *Let $\Phi \colon [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ be a shift-invariant continuous function, and let $\mathcal{B}, \phi, \mathcal{T}, f$, and ψ be the corresponding barriers and functions defined as above. If for every $D \in [\mathbb{N}]^\infty$ the family $\{f(\iota_{\mathcal{B}}(S^{(n)}(D))) : n \in \mathbb{N}\}$ is pairwise disjoint, then for every $A \in [\mathbb{N}]^\infty$ there is $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \Phi''[A]^\infty$.*

Proof. Assume that for every $D \in [\mathbb{N}]^\infty$ the family $\{f(\iota_{\mathcal{B}}(S^{(n)}(D))) : n \in \mathbb{N}\}$ is pairwise disjoint. Consider the collection of finite sets given by

$\mathcal{B}_0 = \{s \cap \min f(s) : s \in \mathcal{B}\}$. We can assume that the empty set is not an element of $\mathcal{B}_0$, since by using the Galvin-Prikry theorem again, it can be assumed that the first element of s is $f(s)$ for every $s \in \mathcal{B}$, or for no s this happens. The first case cannot occur since it contradicts the hypothesis of the theorem.

Let A be a given set in $[\mathbb{N}]^\infty$, listed in increasing order by $A = \{a_n : n \in \mathbb{N}\}$. Our objective is to find a set $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \Phi''[A]^\infty$.

Define by induction a subset $A' \in [A]^\infty$ as follows. Let $a'_0 = a_0$. Suppose we have defined a'_n, then let a'_{n+1} be the first element of A such that for every $s \subseteq A \cap a'_n + 1$, if t is the only initial segment of A/a'_n such that $s \cup t \in \mathcal{B}$, then $a'_{n+1} \geq \min f(s \cup t)$. Let $A' = \{a'_0, a'_1, \dots\}$. We have that for every $n \in \mathbb{N}$ and every $s \subseteq \{a'_0, \dots, a'_n\}$, for every $k > n$, there is a finite subset t of $A \cap (a'_n, a'_k)$ such that $s \cup t \in \mathcal{B}_0$.

We may assume that A' satisfies the conclusions of Theorem 2, in particular, that if $s_0, s_1 \in \mathcal{B}_0$, and t_0, t_1 are the corresponding extensions within $A'/\max(s_0 \cup s_1)$ to elements of $\mathcal{B}$, then $f(t_0) = f(t_1)$.

Let $X' = \Phi(A')$, we will define a subset $X \subseteq X'$ and show that $[X]^\infty \subseteq \Phi''[A]^\infty$. The set X will be obtained inductively thinning out X'.

Let $X' = \{x_i : i \in \mathbb{N}\}$ be listed increasingly. To simplify notation let, for each k, $\iota(k)$ denote $\iota_\mathcal{B}(S^{(k)}(A'))$, and $mf(k) = \min(f(\iota(k)))$. Let also $\iota_0(k) = \iota(k) \cap mf(k)))$, in words, $\iota(k)$ is the only element of $\mathcal{B}$ which is an initial segment of the k-th shift of A', and $\iota_0(k)$ is the portion of $\iota(k)$ below $f(\iota(k))$ (which is a set in $\mathcal{B}_0$); $mf(k)$ is the first element of $f(\iota(k))$.

The set X will have the following properties. If X is listed increasingly by $X = \{x_{i_k} : k \in \mathbb{N}\}$, then for any k:

(i) $\iota(i_0) < \iota(i_1) < \cdots < \iota(i_k)$, and

(ii) i_{k+1} is such that $\iota(i_{k+1})$ is above $\mathcal{R}(i_0, \dots, i_k)$, where $\mathcal{R}$ is defined below.

Given $\{i_0, \dots, i_k\}$, for every set $a \subseteq A \cap \max\, \iota(k) + 1$ there is a unique end extension $E(a)$ of a within $A'/\max\, \iota(k)$ which is in $\mathcal{B}$. Put $\mathcal{R}(i_0, \dots, i_k) = \cup\{E(a) : a \subseteq A \cap \max\, \iota(k) + 1\}$.

To define X, put $i_0 = 0$. Recall that $x_0 = \psi f(\iota(0))$; i_1 is the first i such that $\iota(i)$ is above $\mathcal{R}(i_0)$, in particular, $\iota(i_1)$ is above $\iota(0)$. Suppose we have defined $i_0, \dots, i_k$. Then, i_{k+1} is the first i such that $\iota(i)$ is above $\mathcal{R}(i_0, \dots, i_k)$. This completes the definition of $X = \{x_{i_k} : k \in \mathbb{N}\}$.

Given $Y \in [X]^\infty$, we want to find a set $D \in [A]^\infty$ such that $\Phi(D) = Y$. By our hypothesis, $\{f(\iota_\mathcal{B}(S^{(n)}(A'))) : n \in \mathbb{N}\}$ is a pairwise disjoint family, and the set X is listed in increasing order by $\{\psi(f(\iota(i_k))) : k \in \mathbb{N}\}$. We want $D \in [A]^\infty$ satisfying

$$f(\iota_\mathcal{B}(S^{(n)}(D))) = f(\iota_\mathcal{B}(S^{(j_n)}(A')))$$

for every $n \in \mathbb{N}$, where $Y = \{x_{j_n} : n \in \mathbb{N}\}$ is listed in increasing order. The set D will not necessarily be a subset of A', although it will be a subset of A. Notice that $\{j_n : n \in \mathbb{N}\}$ is a subsequence of $\{i_n : n \in \mathbb{N}\}$, the sequence of indices of elements of X.

To construct D, we will produce an infinite sequence of its initial segments. Put $D_0 = \iota_{\mathcal{B}}(S^{(j_0)}(A'))$ as an initial segment of D.

Suppose we have defined the initial segments $D_0 \subseteq \cdots \subseteq D_k$, so that for every $m \leq k$, D_m without its first m elements is in $\mathcal{B}$, and

$$f(D_m \setminus \{\text{first } m \text{ elements of } D_m\}) = f(\iota(j_m)),$$

and x_{j_m+1}, the next element in X after x_{j_m}, is such that $\iota(j_m + 1)$ is above D_m. Recall that $\psi(f(\iota(j_m))) = x_{j_m}$.

Now consider $\iota(j_{k+1}) = \iota_{\mathcal{B}}(S^{(j_{k+1})}(A'))$, by inductive hypothesis it is above D_k.

Let s' be D_k with its first $k + 1$ elements removed, and finally let $s = s' \cup \iota_0(j_{k+1})$. By construction of A', there is an end extension t of s which is in $\mathcal{B}_0$, obtained adding to s some elements from A lying above $\iota_0(j_{k+1})$ and below $mf(j_{k+1})$. Note that these elements from A are above D_k.

By Theorem 2, the only element $r \in \mathcal{B}$ which extends t within the set $A'/mf(j_{k+1})$, is such that $f(r) = f(\iota_{\mathcal{B}}(S^{(j_{k+1})}(A')))$; we let $D_{k+1} = D_k \cup r$ be the next initial segment of D. This ends the inductive definition of D. For every k, $\psi(f(\iota_{\mathcal{B}}(S^{(k)}(D))) = x_{j_k}$, and thus $\Phi(D) = Y$.□

4 Borel chromatic numbers

Let X be a set. Any binary relation $R \subseteq X^2$ on X which is symmetric and irreflexive determines a graph $\mathcal{G} = (X, R)$: the elements of X are the vertices of $\mathcal{G}$, and for $x, y \in X$, $\{x, y\}$ is an edge of $\mathcal{G}$ if xRy.

A coloring of $\mathcal{G}$ is a map $c : X \to K$ such that $xRy \Rightarrow c(x) \neq c(y)$. If the cardinality of K is k, we say that c is a k-coloring. The chromatic number of $\mathcal{G}$ is the smallest cardinal k for which $\mathcal{G}$ admits a k-coloring.

If X is a standard Borel space (i.e. a completely metrizable separable space, together with its σ-algebra of Borel sets), the Borel chromatic number of $\mathcal{G} = (X, R)$ is the smallest cardinal k for which there is a Borel coloring $c : X \to K$ where K is a set (of colors) of cardinality k.

If $F : X \to X$ is a Borel function, the graph $\mathcal{G}_F = (X, F)$ is defined saying that $\{x, y\}$ is an edge if $y = F(x)$.

The graph $([\mathbb{N}]^\infty, S)$, where S is the shift function given by $F(A) = A \setminus \{\min(A)\}$, provides an example of a graph with chromatic number 2, and Borel chromatic number $\aleph_0$. Since this graph is acyclic, choosing a vertex in each connected component, a 2-coloring can be obtained: for each connected component, give the selected vertex a color and then alternate the two colors as moving away from it. By the Galvin-Prikry Theorem ([4]), the Borel chromatic number of this graph is $\aleph_0$ (see [6, 8]). The same is true for any graph of the form $([X]^\infty, S)$, where $X \in [\mathbb{N}]^\infty$.

If $\mathcal{A} \subseteq [\mathbb{N}]^\infty$ is a Borel set, then by results of [6], the Borel chromatic number of the induced graph $(\mathcal{A}, S)$ can only take the values $1, 2, 3$, or $\aleph_0$ (depending on the Borel set $\mathcal{A}$). Question 8.3 of [6] asks whether the Borel chromatic number of $(\mathcal{A}, S)$ is $\aleph_0$ if and only if there is a set $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \mathcal{A}$. Clearly, if there is a set $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \mathcal{A}$, then the Borel chromatic number of $(\mathcal{A}, S)$ is $\aleph_0$, so the content of the question is whether the converse holds. Namely, is it true that for any Borel subset $\mathcal{A}$ of $[\mathbb{N}]^\infty$, if the Borel chromatic number of $(\mathcal{A}, S)$ is $\aleph_0$ then there is $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \mathcal{A}$? Using Theorem 3 we give a negative answer.

It is enough to find a one-one continuous function $\Phi : [\mathbb{N}]^\infty \to [\mathbb{N}]^\infty$ which commutes with the shift and satisfies the conditions of theorem 3.

For example, let $\mathcal{B} = [\mathbb{N}]^2$ and let $\varphi : \mathcal{B} \to \mathbb{N}$ be $\varphi(\{n, m\}) = 2^n(2m+1)$ (where $n < m$). There is a unique continuous Φ commuting with the shift corresponding to $\mathcal{B}$ and φ in the following way: for every $X \in [\mathbb{N}]^\infty$,

$$\Phi(X)(n) = \varphi(\iota_{\mathcal{B}}(S^{(n)}(X))).$$

It is easy to verify that Φ is one-one, and therefore $\Phi''[\mathbb{N}]^\infty$ is a Borel set (see, for example, [5] 15.1). Notice that this Φ is canonized by $\mathcal{B}$ itself and the identity function $Id : \mathcal{B} \to \mathcal{B}$. The hypothesis of Theorem 3 is satisfied, and thus there is no $X \in [\mathbb{N}]^\infty$ such that $[X]^\infty \subseteq \Phi''[\mathbb{N}]^\infty$. Φ commutes with the shift, so $\Phi''[\mathbb{N}]^\infty$ is closed under the shift, since if $Y = \Phi(A)$, then $S(Y) = \Phi(S(A))$. Therefore, the Borel chromatic number of the graph $(\Phi''[\mathbb{N}]^\infty, S)$ must be $\aleph_0$, otherwise, given a finite Borel coloring of $(\Phi''[\mathbb{N}]^\infty, S)$, taking pre-images by Φ we would get a finite coloring of $([\mathbb{N}]^\infty, S)$, a contradiction.

The problem of characterizing those Borel subsets $\mathcal{A}$ of $[\mathbb{N}]^\infty$ for which the graph $(\mathcal{A}, S)$ has infinite Borel chromatic number remains open.

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