Forcing with finite conditions

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Abstract

We present a generalisation to ω_2 of Baumgartner's forcing for adding a CUB subset of ω_1 with finite conditions.

The following well-known result appears in Baumgartner, Harrington, Kleinberg [2]:

Theorem 1. Suppose that $X \subseteq \omega_1$. Then the following are equivalent:

a. X contains a CUB subset in an outer model which preserves ω_1.
b. X is stationary.

Proof. (a) implies (b) because any two CUB sets must intersect. Conversely, consider the forcing P whose conditions are closed, countable subsets of X, ordered by end-extension. Clearly P adds a CUB subset to X; we must show that ω_1 is preserved.

First a general comment about ω_1-preservation. We say that D is *predense below* p iff every condition below p is compatible with an element of D. Then ω_1-preservation is a consequence of the following:

$(*)$ For any p and D_i, $i < \omega$ with each D_i predense below p, there are $q \leq p$ and *countable* d_i, $i < \omega$ with $d_i \subseteq D_i$ and d_i predense below q for each $i < \omega$.

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For if $(*)$ holds and p forces σ to be a function from ω to ω_1, then we can consider $D_i = \{q \mid \text{for some } \alpha < \omega_1,\ q \text{ forces } \sigma(i) = \alpha\}$; by $(*)$, there is $q \leq p$ and a countable β such that q forces $\sigma(i) < \beta$ for each $i < \omega$, and therefore q forces that σ is bounded.

Now to see that P preserves ω_1, suppose that $\langle D_i \mid i < \omega\rangle$ is a sequence of sets which are predense below p and choose a continuous elementary chain $\langle M_j \mid j < \omega_1\rangle$ of countable elementary submodels of H_θ, θ large, so that $X, p, \langle D_i \mid i < \omega\rangle$ belong to M_0 and $M_j \in M_{j+1}$ for each j. As $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is CUB, we can choose j so that $\alpha = M_j \cap \omega_1 \in X$. Then as each $D_i \cap M_j$ is predense below p on $P \cap M_j$, we can choose $p = p_0 \geq p_1 \geq \ldots$ so that p_{i+1} belongs to M_j and extends an element r_i of D_i for each $i < \omega$, and in addition $\bigcup_i p_i$ has supremum α. Then $q = \bigcup_i p_i \cup \{\alpha\}$ is a condition extending p, and for each i, $\{r_i\} \subseteq D_i$ is predense below q, proving $(*)$. $\square$

Next we ask the following.

Question. Suppose that X is a *constructible*, stationary subset of ω_1. Then is there a *constructible* forcing P which preserves ω_1 and adds a CUB subset to X?

Baumgartner answered this question affirmatively.

Theorem 2. Let X be a stationary subset of ω_1. Then there is a forcing P which adds a CUB subset to X, such that P preserves ω_1 and P belongs to $L[X]$.

Proof. We add a CUB subset to X *with finite conditions*, using a technique of Uri Avraham [1] (a variant of Baumgartner's original technique). A condition is a finite set p of disjoint closed intervals in ω_1 whose left endpoints belong to X. (We allow the one-point intervals $[\alpha, \alpha]$, $\alpha \in X$.) A condition q extends p iff q contains p.

It is clear that for generic G, $C_G =$ the set of all left endpoints of intervals in $\cup G$ is an unbounded subset of X. Each countable ordinal either belongs to some interval in G or fails to be a limit point of X; it follows that C_G is closed. It only remains to show that ω_1 is preserved.

Suppose that p is a condition and D_i, $i < \omega$ are predense below p. Choose a continuous elementary chain $\langle M_j \mid j < \omega_1\rangle$ of countable elementary submodels of H_θ, θ large, so that $X, p, \langle D_i \mid i < \omega\rangle$ belong to M_0 and

$M_j \in M_{j+1}$ for each j. As $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is CUB, we can choose j so that $\alpha = M_j \cap \omega_1 \in X$. Let q be the condition $p \cup \{[\alpha, \alpha]\}$. If r extends q and $r_0 = r \restriction \alpha$ then every extension s_0 of r_0 in $P \cap M_j$ is compatible with r. This is because $[\alpha, \alpha]$ belongs to q. It follows that $d_i = D_i \cap M_j$ is predense below q for each i, as if r extends q then we can choose $s_0 \leq r_0$ which extends a condition in d_i, and therefore since s_0 is compatible with r, r is compatible with an element of d_i. Hence ω_1 is preserved. $\square$

A nice corollary to this result is: If X is a stationary subset of ω_1 then there is a cardinal-preserving forcing that adds a CUB subset to X. This does not follow from the proof of Theorem 1, as the forcing used there will collapse $2^{\aleph_0}$ if CH fails.

Now we look at the situation for ω_2. Unfortunately there is no analogue for Theorem 1.

Theorem 3. (See [3].) Suppose that $0^\#$ exists. Then

$\{X \subseteq \omega_2^L \mid X \in L$ and X has a CUB subset in an inner model where $\omega_2 = \omega_2^L\}$

is not constructible, and indeed has L-degree $0^\#$. In particular, there are X which belong to the above set but have no CUB subset in any set-generic extension of L in which $\omega_2 = \omega_2^L$.

However there is a nice sufficient condition for a subset of ω_2 to contain a CUB in a cardinal-preserving extension. $X \subseteq \omega_2$ is *very stationary* iff for all α in some stationary $X_0 \subseteq X \cap \operatorname{cof} \omega_1$, $X \cap \alpha$ contains a CUB subset of α. Using a variant of $\diamondsuit$ at ω_2 in L one can construct disjoint very stationary subsets of ω_2 in L. And one has:

Theorem 4. If $X \subseteq \omega_2$ is very stationary then there is a set-forcing extension, preserving both ω_1 and ω_2, and in which X contains a CUB subset.

Proof. In analogy with the proof of Theorem 1, force with closed subsets of X of ordertype less than ω_2, ordered by end-extension. The very stationarity of X is used as in the proof of Theorem 1 to show that if p is a condition and D_i, $i < \omega_1$, are predense below p then there is $q \leq p$ extending an element of D_i for each i. It follows that no new ω_1-sequences are added by the forcing and therefore both ω_1 and ω_2 are preserved. $\square$

Now we pose the following question: Is there a version for ω_2 of Baumgartner's forcing (as modified by Avraham) to add a CUB with finite conditions? The next result provides a positive answer. We say that D_0 *witnesses* that $D \subseteq \omega_2$ is very stationary iff $D_0 \subseteq D \cap \mathrm{cof}\ \omega_1$ is stationary and $D \cap \alpha$ contains a CUB subset of α for each $\alpha \in D_0$.

Theorem 5. Assume that ω_2 is the L-successor to ω_1, $D \subseteq \omega_2$ is constructible and there is a constructible witness that D is very stationary. Then there is a constructible forcing P that preserves cofinalities and adds a CUB subset of D.

Proof. Let $D_0 \subseteq D$, $D_0 \in L$ witness that D is very stationary. We may assume that successor elements of D have cofinality ω, by replacing D with $(D \cap \mathrm{Lim}\ D) \cup \{\alpha + \omega \mid \alpha \in D\}$.

A condition is a pair $p = (A, S)$, where:

1. A is a finite set of disjoint closed intervals whose left endpoints belong to D and whose left endpoints of cofinality ω_1 belong to D_0. (We allow the one-point intervals $[\alpha, \alpha]$, $\alpha \in D$.) Let L denote the set of left endpoints of intervals in A.

2. S is a finite set of countable, constructible Σ_1 elementary submodels x of some L_β, β limit, $\beta < \omega_2$, such that:
   2a. $x \cap \alpha$ is unbounded in α whenever $\alpha \in x$ has cofinality ω.
   2b. $\sup(x \cap \alpha)$ belongs to D whenever α belongs to $(x \cap D_0) \cup \{\omega_2\}$.
   2c. If $\alpha < \beta$ are adjacent elements of D, α belongs to x and $\beta < \sup(x \cap \mathrm{Ord})$ then β belongs to x.

3. For each interval $I = [\alpha, \beta]$ in A and each $x \in S$:
   3a. If I intersects x then I belongs to x.
   3b. If $I = [\alpha, \beta]$ does not intersect x and $\alpha < \sup(x \cap \mathrm{Ord})$ then α_x belongs to L, where $\alpha_x =$ the least ordinal $\geq \alpha$ in x.

4. Let F be the set of all elements of L of cofinality ω_1, together with ω_2. For nice x, the *F-height of x* is the least element of F greater than $\sup(x \cap \mathrm{Ord})$.
   4a. If x belongs to S and α belongs to F then $x \cap L_\alpha$ also belongs to S.
   4b. Suppose that $x, y \in S$ have the same F-height. Then $x \in y$, $y \in x$ or $x = y$.

Write $p = (A_p, S_p)$ and let F_p denote the F of 4 above. q extends p iff A_q contains A_p and S_q contains S_p.

Claim 1. Suppose that p belongs to P.
(a) For any CUB $C \subseteq \omega_2$ there exists $\alpha \in C$ of cofinality ω_1 such that p belongs to L_α and p^*, obtained from p by adding the interval $[\alpha, \alpha]$ to A_p (and otherwise leaving p unchanged), is a condition extending p.
(b) Let α and p^* be as in part (a). Then if q^* extends p^* there is q extending p in L_α such that every extension of q in L_α is compatible with q^*.

Proof of Claim 1:

(a) Choose α to be any $\alpha \in C \cap D_0$ such that p belongs to L_α. Property 1 is satisfied by p^* as α is greater than the right endpoint of any interval in A_p. Property 2 is the same for p^* as for p. Property 3a is the same for p^* as for p, as α does not belong to any element of S_p. Property 3b is the same for p^* as for p, as α is greater than $\sup x$ for any $x \in S_p$. And property 4 is the same for p^* as for p, as α does not belong to any element of S_p.

(b) Suppose that q^* extends p^* and define q by:

A_q is $A_{q^*} \cap L_\alpha$,
S_q is $S_{q^*} \cap L_\alpha$.

a. q is a condition in L_α which extends p.

We show that q is a condition. We need only verify properties 3b and 4.
3b. Assume that $I \cap x = \emptyset$ and the left endpoint β of $I = [\beta, \gamma]$ is less than $\sup x$, where I belongs to $A_{q^*} \cap L_\alpha$ and x belongs to $S_{q^*} \cap L_\alpha$. Then since q^* is a condition, β_x is the left endpoint of some interval J in S_{q^*}. But since $[\alpha, \alpha]$ belongs to A_{q^*}, the right endpoint of J is less than α and therefore J belongs to $S_{q^*} \cap L_\alpha = S_q$.
For property 4, first note that $F_q = F_{q^*} \cap \alpha$, together with ω_2.
4a. If x is in S_q and $\beta \in F_q$ then $x \cap L_\beta$ is in S_{q^*} and therefore also in $S_q = S_{q^*} \cap L_\alpha$, since $x \cap L_\beta$ is an element of L_α.
4b. If $x, y \in S_q$ have the same F_q-height then since they both belong to L_α, they have the same F_{q^*}-height. Thus the desired conclusion follows as $x, y \in S_{q^*}$ and q^* is a condition.

Since q^* extends p^* and therefore p, it follows that q also extends p, since p belongs to L_α.

Now suppose that r is an extension of q, and r belongs to L_α. We must find a common extension of r and q^*. We define t by

$A_t = A_r \cup A_{q^*}$,
$S_t = S_r \cup S_{q^*}$.

b. t is a condition extending both r and q^*.

Clearly t, if a condition, extends both r and q^*. We now verify that t is a condition, by verifying properties 1-4.
1. The intervals in A_t are disjoint, as r is a condition extending q, all intervals in A_r have right endpoint less than α and all intervals in A_{q^*} not in A_q have left endpoint at least α.
2. Clear.
3. Suppose that I is an interval in $A_t - A_r$ and x belongs to S_r. Then $\sup(x \cap \mathrm{Ord})$ is less than α and the left endpoint of I is at least α. So property 3 is vacuous in this case. Suppose that I belongs to A_r and x belongs to $S_t - S_r$. Then $x \cap L_\alpha$ belongs to $S_q \subseteq S_r$ and therefore property 3 holds for I and $x \cap L_\alpha$. It follows that 3a holds for I and x, since if I intersects x it must also intersect $x \cap L_\alpha$. And 3b holds for I and x: If I is disjoint from x and the left endpoint β of I is less than $\sup(x \cap \mathrm{Ord})$ then I is also disjoint from $x \cap L_\alpha$ and either (i) β is less than $\sup(x \cap \alpha)$, in which case $\beta_x = \beta_{x \cap \alpha}$ and therefore the result follows since r is a condition, or (ii) $\beta_x = \alpha$, in which case the result follows since $[\alpha, \alpha]$ belongs to A_{q^*}, or (iii) $\beta_x = \alpha_x$, in which case the result follows since q^* is a condition. The remaining cases, where I belongs to A_r and x belongs to S_r, or where I belongs to $A_t - A_r$ and x belongs to $S_t - S_r$, are immediate since r and q^* are conditions.
4a. We must show that if x belongs to S_t and $\beta \in F_t$ then $x \cap L_\beta$ belongs to S_t. If x belongs to S_r then either β is in F_r, in which case $x \cap L_\beta$ belongs to $S_r \subseteq S_t$ since r is a condition, or β is at least α, in which case $x \cap L_\beta = x \in S_r \subseteq S_t$. If x belongs to S_{q^*} then either β is in F_{q^*}, in which case the result follows since q^* is a condition, or β is in F_r, in which case $x \cap L_\beta = (x \cap L_\alpha) \cap L_\beta \in S_r \subseteq S_t$, since $x \cap L_\alpha \in S_q \subseteq S_r$.
4b. We must show that if $x, y \in S_t$ have the same F_t-height, then $x \in y, y \in x$ or $x = y$. If x belongs to S_r then the F_t height of x is at most α and therefore y also belongs to S_r; thus x, y have the same F_r-height and the result follows since r is a condition. If x belongs to $S_{q^*} - S_r$ then the F_t-height of x is greater than α, and therefore y also belongs to S_{q^*}; thus x, y have the same F_{q^*}-height and the desired conclusion follows since q^* is a condition.

This completes the proof of Claim 1.

Claim 2. Suppose that p belongs to P.

(a) For any CUB $C \subseteq P_{\omega_1}(\omega_2)$ there exists a constructible $x \in C$ such that p belongs to x and p^*, defined by adding $x \cap L_\alpha$ to S_p for all $\alpha \in F_p$ (and otherwise leaving p unchanged) is a condition extending p.
(b) Let x and p^* be as in part (a). Then if q^* extends p^* there is q in x extending p such that every extension of q in x is compatible with q^*.

Proof of Claim 2:

(a) Using the fact that ω_2 equals ω_2^L, choose $x \in C$ to be a constructible elementary submodel of (L_{ω_2}, D) which contains p. Also require that if α belongs to $(x \cap D_0) \cup \{\omega_2\}$ then $\sup(x \cap \alpha)$ belongs to D. The latter is possible using the fact that D_0 witnesses that D is very stationary. Define p^* as in the statement of (a). Clearly p^*, if a condition, extends p. To verify that p^* is a condition we need only check properties 3 and 4.
3. As p belongs to x, each interval in A_p belongs to x and therefore the conclusion of 3a holds for x. It follows easily that 3a also holds for $x \cap L_\alpha$ whenever α belongs to F_p. 3a holds for other elements of S_{p^*} since p is a condition. Property 3b is vacuous for $x \cap \alpha$, $\alpha \in F_p$ and holds for other elements of S_{p^*} since p is a condition.
4a. This is true for x by definition of p^*, and for other elements of S_{p^*} since p is a condition.
4b. Suppose that $y, z \in S_{p^*}$ have the same F_{p^*}-height ($= F_p$-height). If both y, z belong to S_p then the desired conclusion follows since p is a condition. Assume that $y = x \cap L_\alpha$ where α belongs to F_p. If z belongs to S_p then z belongs to x and since it has the same F_p-height as y, must also belong to L_α; hence z belongs to y. If z is of the form $x \cap L_\beta$, $\beta \in F_p$, and has the same F_p-height as y then $z = y$, since the F_p-height of $x \cap L_\beta$ equals β for any $\beta \in F_p$.

(b) Let q^* extend p^* and define q as follows:

A_q is $A_{q^*} \cap x$,
S_q is $S_{q^*} \cap x$

Subclaim 1. q is a condition in x extending p.

q, if a condition, surely extends p since q^* extends $p^* \leq p$ and p belongs to x. To check now that q is a condition we need only verify properties 3b and 4.
3b. Suppose that I belongs to A_q, I is disjoint from y and the left endpoint α of I is less than $\sup y$, where y belongs to S_q. Then α_y is the left endpoint of some $J \in A_{q^*}$ since q^* is a condition. Since J intersects y, J must belong

to y and therefore also to x, since y belongs to x. Thus J belongs to A_q.
4a. If y belongs to S_q and α belongs to $F_q \cap \omega_2$ then $y \cap L_\alpha$ belongs to S_{q^*} since q^* is a condition. Since both y and α belong to x, we get $y \cap L_\alpha \in S_q$. If y belongs to S_q then $y \cap L_{\omega_2} = y$ and therefore belongs to S_q.
4b. Suppose that $y \in S_q$ has F_q-height α and F_{q^*}-height β. Suppose that β is less than $\sup x$. Then either β equals α or is the left endpoint of some interval in A_{q^*} disjoint from x. In the latter case, β_x is the left endpoint of some interval in $A_{q^*} \cap x = A_q$ and therefore β_x belongs to F_q, since Property 2a for x implies that it must have uncountable cofinality. Thus $\beta_x = \alpha$. So we conclude that the F_{q^*}-height of y is the least $\beta \in F_{q^*}$ such that either β is less than $\sup x$ and $\beta_x = \alpha$, or β is greater than $\sup x$. Therefore the F_{q^*}-height of $y \in S_q$ is uniquely determined by the F_q-height of y. If $y, z \in S_q$ have the same F_q-height then they therefore also have the same F_{q^*}-height, and since q^* is a condition, either $y \in z$, $z \in y$ or $y = z$, as desired.

Now suppose that r in x extends q. We must find a common extension t of r and q^*. We define t by:

$A_t = A_r \cup A_{q^*}$
$S_t = S_r \cup \{y \cap L_\alpha \mid y \in S_{q^*},\, \alpha \in F_r\}$.

Subclaim 2. t is a condition extending both r and q^*.

Clearly t, if a condition, extends both r and q^*. We show now that t is a condition.
1. Suppose that I is an interval in A_{q^*} but not in A_r. Then I is disjoint from x. If the left endpoint α of I is at least $\sup x$, then I is disjoint from all intervals in A_r, since the latter belong to x. Otherwise α_x is the left endpoint of some interval J in A_q. It follows that the intervals in A_r are disjoint from I, as they belong to x and are either equal to or disjoint from J. Thus A_t consists of pairwise disjoint intervals.
2a. Clear.
2b. We must show that if y belongs to S_t and $\alpha \in (y \cap D_0) \cup \{\omega_2\}$ then $\sup(y \cap \alpha)$ belongs to D. This is clear if y belongs to S_r since r is a condition. It also holds if y belongs to S_{q^*} since q^* is a condition. This implies the result for arbitrary $y \in S_t$ since elements of F_r belong to D_0.
2c. Clear, since elements of F_r belong to $D_0 \subseteq \mathrm{Lim}\ D$.
3a. Suppose that I is an interval in A_t, y belongs to S_t and I intersects y. We must show that I belongs to y. First we consider the case where I belongs to A_r and y belongs to $S_t - S_r$. Write $I = [\alpha, \beta]$ and $y = z \cap L_\gamma$, where z belongs to $S_{q^*} - S_r$ and γ belongs to F_r. Let β^* be the least element of F_{q^*} greater than α. Since we have shown that A_t consists of pairwise

disjoint intervals, it follows that β^* is greater than β. Therefore the the F_{q^*}-heights of $x \cap L_{\beta^*}$ and $z \cap L_{\beta^*}$ are both β^*, the former since β belongs to x and the latter since z intersects $I = [\alpha, \beta]$. Thus either $z \cap L_{\beta^*}$ belongs to $x \cap L_{\beta^*}$, $x \cap L_{\beta^*}$ belongs to $z \cap L_{\beta^*}$ or $x \cap L_{\beta^*} = z \cap L_{\beta^*}$. In the first case, I intersects $y \cap L_{\beta^*} \in S_q \subseteq S_r$, and therefore since r is a condition, I belongs to $y \cap L_{\beta^*} \subseteq y$, as desired. In the second and third cases, y contains $x \cap L^*_\beta$ as a subset and therefore I as an element. Now consider the case where I belongs to $A_t - A_r$ and y belongs to S_r. Then I belongs to A_{q^*} and intersects x, which belongs to S_{q^*}. Thus I belongs to x, contradicting the fact that I does not belong to $A_q \subseteq A_r$. The other two cases, where either I belongs to A_r and y belongs to S_r, or where I belongs to $A_t - A_r$ and y belongs to $S_t - S_r$, follow since both r and q^* are conditions.

3b. Suppose that $I = [\alpha, \beta]$ belongs to A_t, y belongs to S_t, I is disjoint from y and α is less than $\sup(y \cap \mathrm{Ord})$. We must show that α_y is the left endpoint of some interval in A_t. First we consider the case where I belongs to A_r and y belongs to $S_t - S_r$. Write $y = z \cap L_\gamma$ where z belongs to S_{q^*} and γ belongs to F_r. Let β^* be the least element of F_{q^*} greater than β. If $\alpha_y = \beta^*$ then α_y is the left endpoint of some interval in A_{q^*} and we are done. If $\alpha_y > \beta^*$, then let J be the interval of A_{q^*} with left endpoint β^*. Since q^* is a condition and $\alpha_y = \alpha_z$, J is not an element of z and therefore is disjoint from z. Since q^* is a condition, $\alpha_y = (\beta^*)_z$ is the left endpoint of some interval of A_{q^*}, as desired. Finally, if $\beta < \alpha_y < \beta^*$, it follows that $x \cap L_{\beta^*}$ and $z \cap L_{\beta^*}$ both have F_{q^*}-height β^*, and therefore $x \cap L_{\beta^*} \in z \cap L_{\beta^*}$, $z \cap L_{\beta^*} \in x \cap L_{\beta^*}$ or $x \cap L_{\beta^*} = z \cap L_{\beta^*}$. The first and third of these possibilities contradict our hypothesis that $I \in x$ is disjoint from y. In the second possibility, $z \cap L_{\beta^*}$ belongs to S_q and since $\alpha_y = \alpha_z$ is less than β^*, we have that α_y is the left endpoint of some interval in A_r since r is a condition. Now we consider the case where I belongs to $A_t - A_r$ and y belongs to S_r. Thus I belongs to A_{q^*} and must be disjoint from x, else it would belong to x and therefore to A_r. As α is less than $\sup x$, it follows that α_x is the left endpoint of some interval in A_{q^*}, which in fact belongs to A_r. If $\alpha_y = \alpha_x$ then we are done. Otherwise α_y equals $(\alpha_x)_y$, which must be the left endpoint of an interval in A_r, since r is a condition. The remaining two cases, where either I belongs to A_r and y belongs to S_r, or where I belongs to $A_t - A_r$ and y belongs to $S_t - S_r$, follow since both r and q^* are conditions.

4a. We must show that if y belongs to S_t and α belongs to F_t then $y \cap L_\alpha$ belongs to S_t. This is clear if y belongs to S_r and α belongs to F_r, or if y belongs to F_{q^*} and α belongs to F_{q^*}, since r and q^* are conditions. This is also true if y belongs to S_{q^*} and α belongs to F_r, by definition of S_t. And we may assume that y belongs to $S_r \cup S_{q^*}$. So we need only

check the case where y belongs to S_r, α belongs to F_{q^*} and α is less than $\sup(y \cap \mathrm{Ord})$. If α belongs to x then it also belongs to F_r so we are done since r is a condition. Otherwise α_x is defined and belongs to F_r. So since r is a condition, $y \cap L_\alpha = y \cap L_{\alpha_x}$ belongs to S_r.
4b. We must show that if $y, z \in S_t$ have the same F_t-height then either $y \in z$, $z \in y$ or $y = z$. Note that y, z also have the same F_r-height and the same F_{q^*}-height. If y, z both belong to S_r then the desired conclusion follows since r is a condition. Suppose that y, z are of the form $y^* \cap L_\alpha$, $z^* \cap L_\beta$, respectively, where y^*, z^* belong to S_{q^*} and $\alpha, \beta \in F_r$. We may assume that α, β are the F_r-heights of y, z, respectively, and therefore $\alpha = \beta$. Let α^* be the common F_{q^*}-height of y, z. Then $y^* \cap L_{\alpha^*}$, $z^* \cap L_{\alpha^*}$ have F_{q^*}-height α^* and therefore since q^* is a condition, we have $y^* \cap L_{\alpha^*} \in z^* \cap L_{\alpha^*}$, $y^* \cap L_{\alpha^*} = z^* \cap L_{\alpha^*}$ or $z^* \cap L_{\alpha^*} \in y^* \cap L_{\alpha^*}$. In the second case, we get $y = z$. In the first case y belongs to $z^* \cap L_{\alpha^*}$ since it is an initial segment of $y^* \cap L_{\alpha^*}$; as $y \in L_\alpha$ we get $y \in z^* \cap L_\alpha = z$. The third case is identical to the first case, with y and z switched. Finally assume that y belongs to S_r and $z = z^* \cap L_\alpha$ where $z^* \in S_{q^*}$ and $\alpha \in F_r$. We may assume that α is the F_r-height of z, which is also the F_r-height of y. Let α^* be the common F_{q^*}-height of y and z. Then α^* is also the F_{q^*}-height of $x \cap L_{\alpha^*}$, since x contains y, and is the F_{q^*}-height of $z^* \cap L_{\alpha^*}$. Since q^* is a condition, we have $z^* \cap L_{\alpha^*} \in x \cap L_{\alpha^*}$, $x \cap L_{\alpha^*} \in z^* \cap L_{\alpha^*}$ or $z^* \cap L_{\alpha^*} = x \cap L_{\alpha^*}$. In the first case, $(z^* \cap L_{\alpha^*}) \cap L_\alpha = z$ belongs to S_r, so we are done since r is a condition. In the second and third cases we have $y \in z^* \cap L_\alpha = z$.

This completes the proof of Claim 2.

Claim 1 implies that ω_2 is preserved. Claim 2 implies that ω_1 is preserved. As P has cardinality ω_2, all cofinalities are preserved.

Claim 3. Let G be P-generic and $C_G = \{\gamma \mid \gamma$ is a left endpoint of some interval in $\cup\{A_p \mid p \in G\}\}$. Then C_G is a CUB subset of D.

Proof of Claim 3.

It is clear that C_G is unbounded. We show that C_G is closed.

Suppose that p is a condition and for the sake of contradiction, $p \Vdash (\alpha \in \mathrm{Lim}\ C_G$ and $\alpha \notin C_G)$. We may assume that for each $y \in S_p$, α_y is either a left endpoint of some interval in A_p or is forced by p to not belong to C_G. For $q \leq p$, α does not belong to any interval in A_q, else q forces either that α belongs to C_G or is not the limit of elements of C_G.

Suppose that y belongs to S_p, α is not in $\operatorname{Lim} y$ but α is less than $\sup(y \cap \operatorname{Ord})$. Then α_y must be a left endpoint of some interval in A_p, else p forces that α is not a limit point of C_G. Let β be the least element of F_p greater than α and consider $S = \{y \in S_p \mid \alpha \leq \sup(y \cap \operatorname{Ord}) < \beta\}$. Then the elements of S form an $\in$-chain.

Assume first that $y \cap \alpha$ is cofinal in α for some $y \in S$, and let y_0 be the $\in$-least such. Note that if α_{y_0} is a left endpoint of some interval in A_p then α must belong to D, by requirement 2b on conditions. We show that we can extend p to force either that α belongs to C_G or that α is not a limit point of C_G, achieving the desired contradiction. Note that $D \cap y_0 \cap \alpha$ must be cofinal in α, as there are cofinally many $\gamma < \alpha$ which are forced by extensions of p into C_G and for any such $\gamma \notin y_0$, γ_{y_0} belongs to $D_0 \subseteq D$. By property 2c for y_0, $D \cap y_0 \cap \alpha \cap \operatorname{cof} \omega$ is also cofinal in α.

If α_{y_0} is not the left endpoint of some interval in A_p, then let γ be an element of $D \cap y_0 \cap \alpha \cap \operatorname{cof} \omega$ greater than the right endpoint of any interval of A_p with left endpoint less than α, and larger than $\sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$. We claim a condition results when the interval $I = [\gamma, \alpha_{y_0}]$ is added to p: I is disjoint from intervals of A_p with left endpoint less than α by choice of γ. And it is disjoint from intervals of A_p with left endpoint greater than α since by assumption α_{y_0} is not the left endpoint of an interval of A_p, and therefore neither is any ordinal between α and α_{y_0}. I does not intersect any $y \in S_p$ with $\sup(y \cap \beta) < \alpha$ by choice of γ. I does not intersect any $y \in S_p$ with $\sup(y \cap \alpha) < \alpha \leq \sup(y \cap \beta)$, as for such y we have $\alpha_y > \alpha_{y_0}$ since α_y must be a left endpoint of some interval of A_p and α_{y_0} is not. Any other $y \in S_p$ contains y_0 and therefore also the interval I as an element. For those $y \in S_p$ disjoint from I with $\gamma < \sup(y \cap \operatorname{Ord})$, $\gamma_y = \alpha_y$ is a left endpoint of an interval of A_p.

If α_{y_0} is the left endpoint of some interval in A_p, then let $I = [\alpha, \alpha]$. We claim that a condition results when we add I to p: Of course I is disjoint from all intervals of A_p since α does not belong to any such interval. Trivially, if I intersects $y \in S_p$ then it belongs to y. If I is disjoint from $y \in S_p$ and $\alpha < \sup(y \cap \operatorname{Ord})$, then $\alpha_y \geq \alpha_{y_0}$, as otherwise $y \cap L_\beta \in S_\beta$, $y_0 \in y \cap L_\beta$ and therefore $\alpha = \sup(y_0 \cap \alpha_{y_0}) \in y$, against our hypothesis. So α_y must be a left endpoint of some interval of A_p, else α_{y_0} could not be.

Lastly, we treat the case where $y \cap \alpha$ is not cofinal in α for all $y \in S$. In this case we choose $I = [\gamma, \alpha]$, where γ is an element of $D \cap \alpha \cap \operatorname{cof} \omega$ greater than the right endpoint of any interval of A_p with left endpoint less than α, and larger than $\sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$. We claim

that a condition results when we add I to p: I is disjoint from all intervals in A_p by choice of γ. I is disjoint from each $y \in S_p$, as $y \cap \alpha$ is contained in γ by choice of γ and the case hypothesis, and if α were to belong to y, we would have $\alpha = \gamma_y$ and therefore p would force that α is not a limit of elements of C_G, since α is not the left endpoint of any interval of A_q for all $q \leq p$. If y belongs to S_p and $\gamma < \sup(y \cap \mathrm{Ord})$ then α_y must be the left endpoint of an interval of A_p, as we observed earlier.

This completes the proof of Theorem 5.

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