Weakened Markus–Yamabe Conditions for Two–Dimensional Global Asymptotic Stability

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Abstract

For a general two–dimensional autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, it is difficult to find easily verifiable sufficient conditions guarenteeing global asymptotic stability (GAS) of an equilibrium point. This paper considers three conditions which imply GAS for a large class of systems, weakening the so–called Markus–Yamabe condition. The new conditions are: (1) the system admits a unique equilibrium point, (2) it is locally asymptotically stable, and (3) the trace of the Jacobian matrix of $\mathbf{f}$ is negative everywhere. We show that GAS is obtained when the components of $\mathbf{f}$ are polynomials of degree two or represent a Liénard system. However, GAS is not obtained under these conditions for other classes of planar differential systems.

2000 AMS Subject Classification : 34D23.
Keywords and phrases: Global asymptotic stability.

Running title: 2D Global Asymptotic Stability.

1 Introduction and statement of the main results

Since the time of Liapunov, it has become evident that finding conditions which guarentee global asymptotic stability (GAS) of an equilibrium point in a differential system, even in two dimensions, is difficult. Liapunov's approach is probably the most wide–spread general method used, though constructing a Liapunov function usually requires ingenuity, experience, and some luck. For the two–dimensional autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

we seek a set of easily verifiable conditions which may give GAS. A non–intuitive result to this end was proven in 1993, the so-called Markus–Yamabe conjecture in two dimensions (see [5], [7], [8]). This result shows that GAS is obtained if the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x})$ have negative real part for all $\mathbf{x}$ in the plane. The aim of this paper is to weaken the Markus–Yamabe condition and still obtain GAS in two dimensions.

The Markus–Yamabe condition is equivalent to having trace $D\mathbf{f} < 0$ and det $D\mathbf{f} > 0$ at every point. The trace condition itself is equivalent to having each region of finite area shrink under the flow, while the determinant condition has no known geometric interpretation. Several results (see [3], [6], [10], [11]) obtain GAS by dropping the determinant condition, yet asking that an equilibrium point is unique, locally asymptotically stable (LAS), and adding a new condition in a neighborhood of infinity. The new requirements on the equilibrium point are reasonable since they are necessary for GAS and relatively easy to check. Conditions near of infinity, however, may be more difficult to check, non-intuitive, and unnecessary. It is tempting to drop any condition near infinity and make the following open problem.

Open Problem 1. *Suppose* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the following conditions:*

(C_1) $f(\mathbf{x}) = \mathbf{0}$ *if and only* $\mathbf{x} = \mathbf{p}$.

(C_2) The equilibrium point $\mathbf{p}$ *is LAS.*

(C_3) Trace $D\mathbf{f} < 0$ *for all* $\mathbf{x} \in \mathbb{R}^2$.

Then, when $\mathbf{p}$ *is GAS?*

If the components of $\mathbf{f}$ are polynomials of degree at most 2 and at least one of them has degree 2, then system (1) is called a *quadratic polynomial differential system* or simply a *quadratic system.*

Our main results are stated in the following theorem.

Theorem 2. *The following three statements hold.*

(a) Every quadratic system satisfying assumptions (C_1)–(C_3) is GAS.

(b) Every Liénard system of the form

$$\dot{x} = y - g(x), \qquad \dot{y} = -x, \tag{2}$$

with $g(x) = a_1x + a_2x^2 + \cdots + a_dx^d$ and satisfying assumptions (C_1)–(C_3) is GAS.

(c) The differential system

$$\dot{x} = -\frac{x(x+1)}{(1+y^2)^{3/2}}, \qquad \dot{y} = 4x + \frac{(2x-1)y}{\sqrt{1+y^2}}, \tag{3}$$

satisfies assumptions (C_1)–(C_3) and it is not GAS.

In Section 2 we show that the open problem is true for quadratic systems, a non–trivial result since there are 111 different quadratic phase portraits with no limit cycles having a unique finite singular point (see [4]). In Section 3 we show that the open problem also holds for Liénard systems with polynomial components. Finally, in Section 4 we provide a negative answer to the open question with an algebraic function **f** which is non–rational showing statement (c) of Theorem 2.

Recently, the authors have been informed that there are polynomial differential systems of degree 7 satisfying assumptions (C_1)–(C_3) and for which the system is not GAS, see [2].

The question which remains open is to know: *what is the largest family of planar differential systems for which the open problem has a positive answer?* In particular, *what is the maximum degree of polynomial differential systems for which the assumptions (C_1)–(C_3) imply that all the systems with that degree are GAS*?

2 Quadratic polynomial differential systems

If system (1) is linear, then condition (C_1) is met only if the Jacobian matrix at the origin has no zero eigenvalue. It is well–known that the only configurations in this case also satisfying conditions (C_2) and (C_3) are globally attracting nodes or foci. So, the Open Problem 1 holds for linear systems.

The next proposition shows statement (a) of Theorem 2.

Proposition 3. *The only quadratic systems having a unique equilibrium point and satisfying the three conditions of the open problem (up to affine equivalence and time re–scaling) can be written as either*

$$\dot{x} = -x \ , \quad \dot{y} = -by - lx^2 \ , \quad b > 0 \ ;$$

or

$$\dot{x} = -x \ , \quad \dot{y} = -x - y - lx^2 \ .$$

Moreover, their equilibrium point is GAS.

To prove Proposition 3 we will use the next theorem which provides the local phase portraits of semi–hyperbolic equilibrium points; see [1] for a proof.

Theorem 4. *Let* $(0,0)$ *be an isolated equilibrium point of the system*

$$\dot{x} = F(x,y), \quad \dot{y} = y + G(x,y)$$

where F *and* G *are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in* x *and* y*. Let* $y = g(x)$ *be the solution of the equation* $y + G(x,y) = 0$ *in a neighborhood of* $(0,0)$*, and assume that the series expansion of the function* $F(x, g(x))$ *has the form* $a_m x^m + \cdots$ *, where* $m \geq 2$*,* $a_m \neq 0$*. Then*

(a) If m *is odd and* $a_m > 0$*, then* $(0,0)$ *is a topological node.*

(b) If m *is odd and* $a_m < 0$*, then* $(0,0)$ *is a topological saddle.*

(c) If m *is even, then* $(0,0)$ *is a saddle–node.*

Proof of Proposition 3: A key point in our proof is the following classification of quadratic systems having a unique equilibrium point up to affine equivalence and scaling the time variable (see [4] for more details):

(I.e) $\dot{x} = y - x^2 + xy, \quad \dot{y} = ax + by + Q_2(x,y)$ with $a \neq 0$;

(I.s) $\dot{x} = y - x^2 + xy, \quad \dot{y} = by + Q_2(x,y)$ with $b \neq 0$;

(I.h) $\dot{x} = y - x^2 + xy, \quad \dot{y} = Q_2(x,y)$;

(II.e) $\dot{x} = xy, \quad \dot{y} = ax + by + Q_2(x,y)$ with $a \neq 0$;

(II.s) $\dot{x} = xy, \quad \dot{y} = by + Q_2(x,y)$ with $b \neq 0$;

(II.h) $\dot{x} = xy, \quad \dot{y} = Q_2(x,y)$;

(III.e) $\dot{x} = y + x^2$, $\dot{y} = \pm x + y + Q_2(x, y)$ with $n = 0$ and, either $m \neq 0$ and $(l - b)^2 \pm 4m < 0$, or $m = 0$ and $l = b$;

(III.s) $\dot{x} = y + x^2$, $\dot{y} = y + Q_2(x, y)$ with either $n \neq 0$ and $m^2 - 4n(l-1) < 0$, or $n \neq 0$, $m = 0$ and $l = 1$, or $n = 0$, $m \neq 0$ and $l = 1$, or $n = m = 0$ and $l \neq 1$;

(III.h) $\dot{x} = y + x^2$, $\dot{y} = Q_2(x, y)$ with either $n \neq 0$ and $m^2 - 4nl < 0$, or $n \neq 0$, $m = l = 0$, or $n = l = 0$, $m \neq 0$, or $n = m = 0$ and $l \neq 0$;

(IV.e) $\dot{x} = y$, $\dot{y} = \pm x + by + Q_2(x, y)$ with $b \geq 0$ and $l = 0$;

(IV.s) $\dot{x} = y$, $\dot{y} = y + Q_2(x, y)$ with $l \neq 0$;

(IV.h) $\dot{x} = y$, $\dot{y} = Q_2(x, y)$ with $l \neq 0$;

(V.e) $\dot{x} = x^2 - 1$, $\dot{y} = d + by + lx^2 + mxy$ with $m \neq 0$ and $d + l \neq 0$;

(V.s) $\dot{x} = x^2 - 1$, $\dot{y} = d + ax + by + lx^2 + mxy + y^2$ with $(b+m)^2 - 4(d+a+l) = 0$ and $(b - m)^2 - 4(d - a + l) < 0$;

(VII.s) $\dot{x} = x^2$, $\dot{y} = y + Q_2(x, y)$ with $n = 0$;

(VII.h) $\dot{x} = x^2$, $\dot{y} = x + Q_2(x, y)$ with $n = 1$;

(VIII.e1) $\dot{x} = x$, $\dot{y} = by + Q_2(x, y)$ with $b \neq 0$ and $n = 0$;

(VIII.e2) $\dot{x} = x$, $\dot{y} = x + y + Q_2(x, y)$ with $n = 0$;

(VIII.s) $\dot{x} = x$, $\dot{y} = Q_2(x, y)$ with $n \neq 0$;

Homogeneous quadratic : $\dot{x} = P_2(x, y)$, $\dot{y} = Q_2(x, y)$;

where $P_2(x, y) = Lx^2 + Mxy + Ny^2$ and $Q_2(x, y) = lx^2 + mxy + ny^2$.

Proving the Open Problem 1 for quadratic systems involves going through each sub–case in the above classification. The expression "Trace" will refer to the trace of the Jacobian matrix of the system.

(I.e) $\dot{x} = y - x^2 + xy$, $\dot{y} = ax + by + Q_2(x, y)$ with $a \neq 0$. Since Trace $= -2x + y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 1/2$ and $n = -1/2$. This reduces the system to $\dot{x} = y - x^2 + xy$, $\dot{y} = ax + by + lx^2 + 2xy - y^2/2$. If (x_0, y_0) is an equilibrium point, the first equation yields $y_0 = x_0^2/(x_0 + 1)$, which in conjunction with the second equation implies

$$x_0 \left(a + \frac{bx_0}{x_0 + 1} + lx_0 + \frac{2x_0^2}{x_0 + 1} - \frac{x_0^3}{2(x_0 + 1)^2} \right) = 0. \qquad (4)$$

If $x_0 = 0$, then $y_0 = 0$. If $x_0 \neq 0$, then equation (4) may be written as

$$(3 - 2l)x_0^3 + 2(a + b + 2l + 2)x_0^2 + 2(2a + b + l)x_0 + 2a = 0.$$

To satisfy condition (C_1), we require $l = 3/2$, leaving

$$2(a + b + 5)x_0^2 + (4a + 2b + 3)x_0 + 2a = 0. \qquad (5)$$

Condition (C_2) implies $a < 0$ since this case assumes that $a \neq 0$ and the determinant of the Jacobian at the origin equals $-a$. Solving for x_0 in equation (5) yields a discriminant equalling

$$(4a + 2b + 3)^2 - 16a(a + b + 5) = 4(b + 3/2)^2 - 56a > 0,$$

hence condition (C_1) is violated unless $a + b + 5 = 0$. Equation (5) would further require that $4a + 2b + 3 = 0$. Solving for a and b yields $a = 7/2$, contradicting the earlier observation that $a < 0$. Therefore, this case cannot satisfy all the conditions.

(I.s) $\dot{x} = y - x^2 + xy$, $\dot{y} = by + Q_2(x, y)$ with $b \neq 0$. Since Trace $= -2x + y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 2$ and $n = -1/2$, yielding $\dot{x} = y - x^2 + xy$, $\dot{y} = by + lx^2 + 2xy - y^2/2$. Making the change of variables $X = y - bx$ and $Y = y$ transforms the system into

$$\begin{aligned} \dot{X} &= (l + b)(Y - X)^2/b^2 + (2 - b)Y(Y - X)/b - Y^2/2, \\ \dot{Y} &= bY + l(Y - X)^2/b^2 + 2Y(Y - X)/b - Y^2/2. \end{aligned}$$

Solving $bY + l(Y - X)^2/b^2 + 2Y(Y - X)/b - Y^2/2 = 0$ for Y in a neighborhood of $(0, 0)$ yields

$$Y = -lX^2/b^3 + \cdots,$$

so Theorem 4 implies $(0, 0)$ is a saddle–node, contradicting condition (C_2).

(I.h) $\dot{x} = y - x^2 + xy$, $\dot{y} = Q_2(x, y)$. Since Trace $= -2x + y + mx + 2ny$, condition (C_3) is never satisfied.

(II.e) $\dot{x} = xy$, $\dot{y} = ax + by + Q_2(x, y)$ with $a \neq 0$. Since Trace $= y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$, $n = -1/2$ and $b \neq 0$. Since both $(0, 0)$ and $(0, 2b)$ are equilibria, condition (C_1) is satisfied only if $b = 0$, a contradiction.

(II.s) $\dot{x} = xy$, $\dot{y} = by + Q_2(x, y)$ with $b \neq 0$. Since Trace $= y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$ and $n = -1/2$. Since both $(0, 0)$ and $(0, 2b)$ are equilibria, condition (C_1) is satisfied only if $b = 0$, a contradiction.

(II.h) $\dot{x} = xy$, $\dot{y} = Q_2(x, y)$. Since Trace $= y + mx + 2ny$, condition (C_3) is never satisfied.

(III.e.i) $\dot{x} = y + x^2$, $\dot{y} = \pm x + by + lx^2 + mxy$ with $m \neq 0$ and $(l - b)^2 \pm 4m < 0$. Since Trace $= 2x + b + mx$, condition (C_3) is satisfied only if $m = -2$ and

$b < 0$, therefore we must choose the plus sign, leaving the system $\dot{x} = y + x^2$, $\dot{y} = x + by + lx^2 - 2xy$. At the equilibrium point $(0,0)$, the Jacobian's determinant is -1, contradicting condition (C_2).
(III.e.ii) $\dot{x} = y + x^2$, $\dot{y} = \pm x + by + bx^2$. Since Trace $= 2x + b$, condition (C_3) is never satisfied.
(III.s) $\dot{x} = y + x^2$, $\dot{y} = y + Q_2(x,y)$ with either $n \neq 0$ and $m^2 - 4n(l-1) < 0$, or $n \neq 0$, $m = 0$ and $l = 1$, or $n = 0$, $m \neq 0$ and $l = 1$, or $n = m = 0$ and $l \neq 1$. Since Trace $= 2x + 1 + mx + 2ny$, condition (C_3) is satisfied only if $m = -2$ and $n = 0$. Only one of the four possible sub–cases fits, implying $l = 1$. Since Trace $= 1$, we time–reverse the system to yield $\dot{x} = -y - x^2$, $\dot{y} = -y - x^2 + 2xy$. Making the change of variables $X = x - y$ and $Y = y$ transforms the system into $\dot{X} = 2XY - 2Y^2$, $\dot{Y} = Y + X^2 - Y^2$. Solving $Y + X^2 - Y^2 = 0$ for Y in a neighborhood of $(0,0)$ yields

$$Y = \frac{1 - \sqrt{1 + 4X^2}}{2} = -X^2 + \cdots,$$

so Theorem 4 implies $(0,0)$ is a saddle–node, contradicting condition (C_2).
(III.h) $\dot{x} = y + x^2$, $\dot{y} = Q_2(x,y)$. Since Trace $= 2x + mx + 2ny$, condition (C_3) is never satisfied.
(IV.e) $\dot{x} = y$, $\dot{y} = \pm x + by + Q_2(x,y)$ with $b \geq 0$ and $l = 0$. Since Trace $= b + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$ and $n = 0$. This reduces the system to a linear one, with which we have already dealt.
(IV.s) $\dot{x} = y$, $\dot{y} = y + Q_2(x,y)$ with $l \neq 0$. Since Trace $= 1 + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$, $n = 0$, and the system is time–reversed, leaving us with $\dot{x} = -y$, $\dot{y} = -y - lx^2$. Making the change of variables $X = y - x$ and $Y = y$ transforms the system into $\dot{X} = l(Y - X)^2$, $\dot{Y} = Y + l(Y - X)^2$. Solving $Y + l(Y - X)^2 = 0$ for Y in a neighborhood of $(0,0)$ yields

$$Y = \frac{2lX - 1 + \sqrt{(2lX - 1)^2 - 4l^2X^2}}{2l} = -lX^2 + \cdots,$$

so Theorem 4 implies $(0,0)$ is a saddle–node, contradicting condition (C_2).
(IV.h) $\dot{x} = y$, $\dot{y} = Q_2(x,y)$. Since Trace $= mx + 2ny$, condition (C_3) is never satisfied.
(V.e) $\dot{x} = x^2 - 1$, $\dot{y} = d + by + lx^2 + mxy$ with $m \neq 0$ and $d + l \neq 0$. Since Trace $= 2x + b + mx$, condition (C_3) is satisfied only if $m = -2$. To meet condition (C_1), we must take $b = \pm 2$, with the x–coordinate of the equilibrium point equaling $x = \mp 1$. The determinant of the Jacobian at this point is negative in both cases, hence condition (C_2) is violated.
(V.s) $\dot{x} = x^2 - 1$, $\dot{y} = d + ax + by + lx^2 + mxy + y^2$. Since Trace $= 2x + b + mx + 2y$, condition (C_3) is never satisfied.

(VII.s) $\dot{x} = x^2$, $\dot{y} = y + Q_2(x, y)$ with $n = 0$. Since Trace $= 2x + 1 + mx$, condition (C_3) is satisfied only if $m = -2$ and the system is time–reversed, leaving us with $\dot{x} = -x^2$, $\dot{y} = -y - lx^2 + 2xy$. Solving $-y - lx^2 + 2xy = 0$ for y in a neighborhood of $(0, 0)$ yields

$$y = \frac{-lx^2}{1 - 2x} = lx^2 + \cdots$$

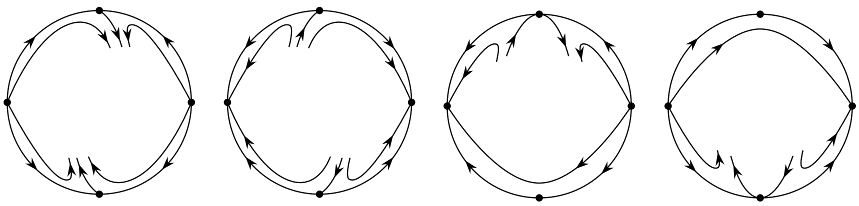
so Theorem 4 implies that $(0, 0)$ is a saddle–node, contradicting condition (C_2).
(VII.h) $\dot{x} = x^2$, $\dot{y} = x + Q_2(x, y)$ with $n = 1$. Since Trace $= 2x + mx + 2y$, condition (C_3) is never satisfied.
(VIII.e1) $\dot{x} = x$, $\dot{y} = by + Q_2(x, y)$ with $b \neq 0$ and $n = 0$. Since Trace $= 1 + b + mx$, condition (C_3) is satisfied only if $m = 0$. The unique equilibrium point $(0, 0)$ satisfies condition (C_2) only if $b > 0$ and the system is time–reversed, giving $\dot{x} = -x$, $\dot{y} = -by - lx^2$. One may show directly that this new system admits GAS at $(0, 0)$. First, $x = Ce^{-t}$ for some constant C. Using this to solve for y yields

$$y = \begin{cases} De^{-bt} - (lC^2 e^{-2t})/(b - 2), & b \neq 2 \\ De^{-2t} - lC^2 te^{-2t}, & b = 2 \end{cases}$$

for some constant D. Conditions (C_1)–(C_3) are all satisfied if and only if $b > 0$.
(VIII.e2) $\dot{x} = x$, $\dot{y} = x + y + Q_2(x, y)$ with $n = 0$. Since Trace $= 2 + mx$, condition (C_3) is satisfied only if $m = 0$ and the system is time–reversed, yielding $\dot{x} = -x$, $\dot{y} = -x - y - lx^2$. One may show directly that this new system admits GAS at $(0, 0)$. First, $x = Ce^{-t}$ for some constant C. Using this to solve for y yields $y = De^{-t} - Cte^{-t} + lC^2 e^{-2t}$ for some constant D. The conditions (C_1)–(C_3) are all satisfied.
(VIII.s) $\dot{x} = x$, $\dot{y} = Q_2(x, y)$ with $n \neq 0$. Since Trace $= 1 + mx + 2ny$, condition (C_3) is satisfied only if $n = 0$ which contradicts the assumption of this case.
Homogeneous quadratic: $\dot{x} = P_2(x, y)$, $\dot{y} = Q_2(x, y)$. Since Trace $= 2Lx + My + mx + 2ny$, condition (C_3) is never satisfied. ∎

3 Liénard systems

In this section we show that the open problem 1 holds for the Liénard system (2); i.e. we prove statement (b) of Theorem 2. A study (see [9]) of such systems in a neighborhood of infinity on the Poincaré sphere forms the backbone of the proof. There are four possibilities; see Figure 1.



d odd, $a_d > 0$ | d odd, $a_d < 0$ | d even, $a_d > 0$ | d even, $a_d < 0$

Figure 1: Poincaré spheres for Liénard systems.

The proof follows straightforwardly. Condition (C_3) implies d is odd, and also, by the Bendixson Theorem (see for instance [13]) that no periodic orbits exist. Conditions (C_1) and (C_2), with the Poincaré–Bendixson Theorem (see for instanced [12]), imply that the first picture of Figure 1 is the only possibility, and that $(0, 0)$ is globally asymptotically stable.

4 A negative answer to the open problem

In this section we prove statement (c) of Theorem 2. It is easy to check that the unique equilibrium point of system (3) is $(0, 0)$, so (C_1) holds. Since the eigenvalues of the linear part of this system at $(0, 0)$ are both equal to -1, $(0, 0)$ is a stable node. Therefore, (C_2) is satisfied. The trace at any point (x, y) is $-2/(1 + y^2)^{3/2}$. Hence, the system also satisfies (C_3). Finally, the equilibrium point $(0, 0)$ is not globally asymptotically stable since the line $x = -1$ is invariant.

Acknowledgments: The first author visited his coauthors in Barcelona during May, 2002, with the support of a Harris Fellowship from Grinnell College. The second author is partially supported by MCYT grant number BFM2002–04236–C02–01 and by a CIRIT grant number 2001SGR 00173. Finally, the third author has been supported by a Polish KBN Grant 2 PO3A 010 22, by a Marie Curie Fellowship of the European Community contract number HPMF–CT–2000–00818.

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