A uniform study on the cyclicity of period annulus of the reversible quadratic Hamiltonian systems*

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Dedicated to Professor Shui-Nee Chow
on the occasion of his 60th birthday

Abstract

The cyclicity of the period annulus of reversible quadratic Hamiltonian systems under quadratic perturbations was studied by several authors for different cases by using different methods. In this paper, we study this problem in a uniform way.

2000 Mathematics Subject Classification. 34C07, 34C08, 37G15.

Key words and phrases. cyclicity of period annulus, reversible quadratic Hamiltonian systems.

1 Introduction

Following [15] and [11] the quadratic centers are classified into four classes: Hamiltonian (Q_3^H), reversible (Q_3^R), Lotka–Volterra (Q_3^{LV}) and codimension four (Q_4). It was pointed out by Ilive in [9] that if $X_H \in Q_3^H \cap Q_3^R$, then except one case (the Bogdanov–Takens system, which has been extensively

*The first author is partially supported by grants NSFC–10231020 and RFDP of China and the second author is partially supported by the grants BFM2002–04236–C02–02 and 2001SGR 00173.

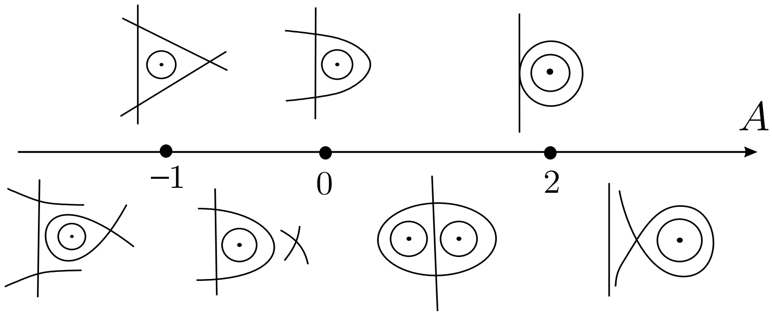


Figure 1. The phase portraits of X_{H_A} for the different values of A.

studied in many papers, see [1, 13, 12, 9]), the cubic Hamiltonian $H(x, y)$ can be transformed into the following one–parameter family

$$H_A(x, y) = x[3(A - 2) - 3(A - 1)x + Ax^2 + y^2], \quad A \in (-\infty, \infty). \quad (1.1)$$

The phase portraits of the corresponding vector fields $X_{H_A} = \frac{\partial H_A}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H_A}{\partial x}\frac{\partial}{\partial y}$ are shown in figure 1. We consider the quadratic perturbations of X_{H_A} as follows.

$$\begin{aligned} \dot{x} &= 2xy + \epsilon f(x, y, \epsilon), \\ \dot{y} &= -(3(A - 2) - 6(A - 1)x + 3Ax^2 + y^2) + \epsilon g(x, y, \epsilon), \end{aligned} \quad (1.2)$$

where ϵ is a small parameter, f and g are quadratic polynomials in x, y with coefficients depending analytically on ϵ. A natural question is: *how many limit cycles can appear for small ϵ from the period annulus (or annuli if $A \in (0, 2)$), surrounding the center (or centers if $A \in (0, 2)$)*? Here the limit cycles that emerge from the center (or centers) are included. The sharp upper bound of the number of such limit cycles is called the *cyclicity* of the period annulus (or annuli).

Ilive [11] investigated the perturbations of quadratic centers, he pointed out, among other, that $X_{H_{-1}} \in Q_3^H \cap Q_3^R \cap Q_3^{LV} \backslash Q_4$ (the Hamiltonian triangle, see figure 1), and the cyclicity of $X_{H_{-1}}$ is determined by a third order Milnikov function (in fact, he already proved in [10] that the cyclicity of the period annulus is equal to three in this case); if $A \in (-\infty, \infty) \backslash \{-1\}$ then $X_{H_A} \in Q_3^H \cap Q_3^R \backslash \{Q_3^{LV} \cup Q_4\}$, and the cyclicity of the period annulus of X_{H_A} is equal to the sharp upper bound of the number of zeros of the second order Milnikov function (see Theorem 3 of [11])

$$J_A(h) = \iint_{\mathrm{Int}(\delta_A(h))} (\alpha + \beta x + \gamma x^{-1})\, dx\, dy, \quad (1.3)$$

for all constants α, β and γ, and $\delta_A(h)$ is an oval contained in the level curves $\{H_A = h$ with $h \in (h_c, h_s)\}$, where h_c and h_s denote the corresponding values of the level curves which shrinks to a center C_A, or expands to the homoclinic or heteroclinic loop connected to one or two saddles respectively if $A \neq 2$, and to a Non–Morsean point if $A = 2$.

Based on this formula and on the Picard–Fuchs equation, and using different techniques, the papers [17], [9] and [16] investigated the cyclicity of the period annulus of X_{H_A} under quadratic perturbations for the cases of $A \in (-1, 0)$, $A = 0$, and $A = 2$, respectively, and [4] studied the same problem for the cases of $A \in (-\infty, -1)$ and $A \in (0, 2)$. On the other hand, using the Picard–Fuchs equation and the Picard–Lefschetz formula (in complex domain), [7] studied the cyclicity of X_{H_A}, $A \in (2, \infty)$, under degree n polynomial perturbations. Combining the above results together, and restricting them to quadratic perturbations, one has the following conclusion.

Theorem 1.1 *If $A \in (-\infty, \infty)\backslash\{-1\}$, then for small ε the maximum number of limit cycles for system (1.2) which emerge from the center and the period annulus altogether is equal to two.*

Remark 1.1 *If $A \in (-\infty, -1) \cup (2, \infty)$, then the period annulus is bounded by a homoclinic loop. By the result of [14] the maximal number of limit cycles in theorem 1.1 also includes the limit cycles bifurcating from the loop.*

In this paper, borrowing some techniques from [3] and using some results of [4], we will give a uniform and easier proof of theorem 1.1. To explain the idea of the proof, we reformulate theorem 1.1 in a geometric way. As mentioned above, the maximal number of limit cycles in theorem 1.1 is equal to the sharp upper bound of the number of zeros of the integral (1.3). We define

$$I_k(h) = \iint_{\mathrm{Int}(\delta_A(h))} x^k \, dx \, dy, \quad k = 0, 1, -1. \tag{1.4}$$

and

$$p(h) = \frac{I_1(h)}{I_0(h)}, \; q(h) = \frac{I_{-1}(h)}{I_0(h)}, \; \omega(h) = \frac{I_1''(h)}{I_0''(h)}, \; \nu(h) = \frac{I_{-1}''(h)}{I_0''(h)}. \tag{1.5}$$

Note that $I_0(h)$ is the area of $\mathrm{Int}(\delta_A(h))$ and $I_0'(h)$ is the period of $\delta_A(h)$, so $I_0(h) > 0$ and $I_0'(h) > 0$ for $h \in (h_c, h_s)$. Since the period function of the quadratic Hamiltonian vector field X_{H_A} is monotone (see [2]), we also have $I_0''(h) \neq 0$. Hence $p, q, \omega, \nu \in C^\infty(h_c, h_s)$, and (1.3) can be written as

$$J_A(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h) = I_0(h)(\alpha + \beta p(h) + \gamma q(h)), \tag{1.6}$$

and we also have

$$J''_A(h) = \alpha I''_0(h) + \beta I''_1(h) + \gamma I''_{-1}(h) = I''_0(h)(\alpha + \beta\omega(h) + \gamma\nu(h)). \quad (1.7)$$

We define two curves in the (p, q)–plane and in the (ω, ν)–plane, respectively, as follows

$$\Sigma_A = \{(p, q)(h) : h \in (h_c, h_s)\}, \ \Omega_A = \{(\omega, \nu)(h) : h \in (h_c, h_s)\}. \quad (1.8)$$

Then the number of zeros of $J_A(h)$ for $h > h_c$ is equal to the number of intersection points (counting their multiplicities) of Σ_A and the straight line $L_{\alpha\beta\gamma} : \{\alpha + \beta p + \gamma q = 0\}$ in the (p, q)–plane.

Definition 1.1 *A plane curve is called sectorial, if it is smooth, and when running it, the tangential vector rotates an angle less than π.*

Therefore, if X_{H_A} has only one period annulus, then Theorem 1.1 follows from

Theorem 1.2 *For any $A \in (-\infty, \infty)\backslash\{-1\}$ the curve Σ_A is sectorial and strictly convex with non–zero curvature.*

It is hard to study Σ_A directly, but it is easier if we link it with Ω_A. From (1.7) we see that in the (ω, ν)–plane the intersection of Ω_A with the straight line $L'_{\alpha\beta\gamma} : \{\alpha + \beta\omega + \gamma\nu = 0\}$ corresponds to a zero of the function $J''_A(h) = 0$. We identify the (p, q)–plane with the (ω, ν)–plane, then the two straight lines $L_{\alpha\beta\gamma}$ and $L'_{\alpha\beta\gamma}$ are identified (we will make this identification throughout the paper). We will prove that both curves Σ_A and Ω_A are smooth, and that the following basic lemma holds.

Lemma 1.1 *Curves Σ_A and Ω_A have no common tangent line for $A \in (-\infty, \infty)\backslash\{-1\}$.*

The idea of the proof of theorem 1.2 We denote by ξ_t the tangent line of Σ_A at the point $(p, q)(t) \in \Sigma_A$, $t \in (h_c, h_s)$. Note that $J_A(h_c) = 0$ for any A, hence if we take ξ_t to be $L_{\alpha\beta\gamma}$, then $J_A(h)$ has a double zero at $h = t$ plus a zero at $h = h_c$. By Rolle's Theorem this implies that ξ_t intersect Ω_A at some point $(\omega, \nu)(t')$ with $t' \in (h_c, t)$. It is easy to prove that if $0 < t - h_c \ll 1$ then ξ_t intersect Ω_A at a unique point transversally. On the other hand, we will show that the two endpoints of Ω_A keep on the different sides of ξ_t for $t \in (h_c, h_s)$. Therefore, lemma 1.1 implies $\xi_t \cap \Omega_A$ consists of a unique point for all $t \in (h_c, h_s)$, counting the multiplicity. This proves the convexity and non–zero curvature property of Σ_A. Since $p'(h) \neq 0$ for all A (see lemma 3.1 of [4]), the curve Σ_A is obviously sectorial. Hence, theorem 1.2 follows.

Remark 1.2 *If $A \in (0,2)$, then X_{H_A} has two period annuli, hence there are two curves Σ_A and Σ'_A. Once we prove both of them are convex with non–zero curvature, then it is easy to prove that any straight line intersect $\Sigma_A \cup \Sigma'_A$ at most in two points, counting their multiplicities. In fact each of these two curves is located in one of the two opposite sectors, sharing one vertex and having two crossing straight lines as boundary, and the tangent line at $h = h_c$ and the asymptotic line for $h \to h_s$ of each curve intersect at a point inside the same sector (see figure 4 of [4]). Therefore, in this paper we will only prove theorem 1.2.*

The proof of lemma 1.1 depends on the Picard–Fuchs equation and some techniques, among them the use of a Riccati equation satisfied by $\omega(h)$ is important. A Riccati equation was used in the study of the cyclicity problem, probably for the first time, by Iliev in [10].

The paper is organized as follows. We make some preliminaries and recall some known results in section 2, study the properties of the curves Σ_A and Ω_A in section 3 (we will use the equivalent forms $\bar{\Sigma}_A$ and $\bar{\Omega}_A$), and prove theorem 1.2 in section 4.

2 Preliminaries

2.1 Phase portraits of X_{H_A}

The next result is easily obtained from system (1.2).

Lemma 2.1 *The following statements hold for X_{H_A} (i.e. $\epsilon = 0$ of (1.2)):*

(1) *For any A, X_{H_A} has an invariant straight line $\{(x,y) : x = 0\}$, and has a center at $(x,y) = (1,0)$, corresponding to the level value of $h = h_1 = A - 3$.*

(2) *For any $A \neq 0$, X_{H_A} has a singularity at $((A-2)/A, 0)$, corresponding to $h = h_2 = (A+1)(A-2)^2/A^2$, which is a saddle if $A < 0$ or $A > 2$, or a center if $0 < A < 2$.*

(3) *For $A < 2$, X_{H_A} has two saddles on the invariant straight line at $(0, \pm\sqrt{3(2-A)})$, and has a heteroclinic loop, corresponding to $h = 0$ if $-1 < A < 2$.*

(4) *As $A \to 0$, the singularity $((A-2)/A, 0)$ goes to infinity. As $A \to 2$, the three singularities $((A-2)/A, 0)$ and $(0, \pm\sqrt{3(2-A)})$ combine together and form a degenerate singularity at the origin.*

The 7 phase portraits of X_{H_A} are shown in figure 1, depending on the values of $A \in (-\infty,-1), -1, (-1,0), 0, (0,2), 2$ and $(2,\infty)$, respectively. Hence, to study the cyclicity of the period annulus surrounding the center $(1,0)$ we need to take

$$(h_c, h_s) = \begin{cases} (h_1, h_2), & \text{if } A < -1 \text{ or } A > 2, \\ (h_1, 0), & \text{if } A \in (-1, 2], \end{cases} \tag{2.1}$$

where $h_1 = A-3$ and $h_2 = (A+1)(A-2)^2/A^2$. In the rest of the paper, we always take the value of (h_c, h_s) as in (2.1), if no other values are specified. Note that $0 \in (h_c, h_s)$ only in the case $A \in (2,3)$.

2.2 The Picard–Fuchs equations

The Picard–Fuchs equation for $I_0(h)$, $I_1(h)$ and $I_{-1}(h)$ was deduced in [4] as follows:

$$G(h)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_{-1} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \\ I_{-1} \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned}
G(h) &= 6h\,(h-h_1)\,(A^2\,h-(A+1)(A-2)^2),\\
a_{00} &= 4A^2h^2-2(A-1)(2A^2-4A-9)h,\\
a_{01} &= -12Ah,\\
a_{02} &= A(A-1)h^2-(A+1)(A-2)(A-3)h,\\
a_{10} &= -A(A-1)h^2+(A-2)(A^2-2A+9)h,\\
a_{11} &= 6A^2h^2-6A(A-1)(A-2)h,\\
a_{12} &= 2h^2,\\
a_{20} &= A(9A^2-18A+1)h-9(A^2-1)(A-2)(A-3),\\
a_{21} &= -6A^2(A-1)h+6A(A+1)(A-2)(A-3),\\
a_{22} &= 2A^2h^2-2(A^2-1)(A-3)h.
\end{aligned}$$

From the definition of $p(h)$ and $q(h)$ (see (1.5)), we have

$$p'(h) = \frac{I_1'(h)}{I_0(h)} - \frac{I_0'(h)}{I_0(h)}p(h),\ \ q'(h) = \frac{I_{-1}'(h)}{I_0(h)} - \frac{I_0'(h)}{I_0(h)}q(h).$$

Combining this with (2.2), we obtain a system of equations for (h,p,q):

$$\begin{aligned}
\dot{h} &= G(h),\\
\dot{p} &= a_{10}+a_{11}p+a_{12}q-p(a_{00}+a_{01}p+a_{02}q),\\
\dot{q} &= a_{20}+a_{21}p+a_{22}q-q(a_{00}+a_{01}p+a_{02}q).
\end{aligned} \tag{2.3}$$

Note that the curve Σ_A, defined in (1.8), is the projection onto (p, q)–plane of an orbit of system (2.3), which is the one–dimensional stable or unstable manifold (depending on the value of A) of the singularity $(h_1, 1, 1)$. From (2.2) we can deduce the following equality (see (3.7) of [4])

$$I''_{-1}(h) = c_1(h, A)I''_0(h) + c_2(h, A)I''_1(h), \tag{2.4}$$

where

$$c_1(h, A) = \frac{6(A-2)}{h} - \frac{4A(A-1)}{(A+1)(A-3)},$$
$$c_2(h, A) = -\frac{9(A-1)}{h} + \frac{8A^2}{(A+1)(A-3)}.$$

From (2.4) and (1.5) we obtain

$$\nu(h) = c_1(h, A) + c_2(h, A)\,\omega(h). \tag{2.5}$$

From (2.2) we also obtain the following 2–dimensional system for (h, ω) (see (3.10) of [4])

$$\dot{h} = T(h), \quad \dot{\omega} = -b_{01}\omega^2 + (b_{11} - b_{00})\omega + b_{10} = \phi(h, \omega), \tag{2.6}$$

where

$$\begin{aligned}
&T(h) = (A+1)(A-3)G(h),\\
&b_{00} = -4A^2(3A^2-6A-5)h^2 + 6(A^2-1)(A-3)(3A^2-6A-5)h-\\
&\qquad 6(A+1)^2(A-2)^2(A-3)^2,\\
&b_{01} = 8A^3(A-1)h^2 - A(A+1)(A-3)(17A^2-34A-3)h+\\
&\qquad 9(A-1)(A-2)(A+1)^2(A-3)^2,\\
&b_{10} = -A(A-1)(A^2-2A+5)h^2 + (A-2)(A+1)^2(A-3)^2h,\\
&b_{11} = -2A^2(3A^2-6A-17)h^2 + 6(A-1)(A+1)^2(A-3)^2h.
\end{aligned}$$

Remark 2.1 *From (2.4) and (2.5) we have that if $A = 3$ then $\omega(h) \equiv 1/3$ for $h \in (h_c, h_s)$. If $A = 2$, then the two equations of system (2.6) have a common factor $2h$. Since we consider $h \in (h_1, 0)$ for $A = 2$, we could, for simplicity, eliminate this factor, and (2.6) becomes*

$$\dot{h} = -36h(h+1), \quad \dot{\omega} = -(32h-9)\omega^2 + (28h-18)\omega - 5h.$$

In the following, we will use this system instead (2.6) in the case of $A = 2$.

2.3 Some related results

By using the existence of the invariant straight line and from (2.3), we have the following results (see Lemmas 3.1 and 4.1 of [4]).

Lemma 2.2 *For* $A \in (-\infty, \infty)\backslash\{-1\}$ *we have*

(1) $p'(h) \neq 0$ *for* $h \in (h_c, h_s)$.

(2) *At its endpoint* $(p,q)(h_1) = (1,1)$ *of* Σ_A, *we have that* $\dfrac{dq}{dp} = -\dfrac{A+3}{A+1}$, $\dfrac{d^2q}{dp^2} = \dfrac{20}{A+1}$.

It is easy to see that if $A \neq 0, 1$ and 2, then system (2.6) has six distinct singularities: three saddles $(h_1, \omega_1), (0, \tilde{\omega})$, (h_2, ω_2') and three improper nodes $(h_1, 1)$, $(0,0)$, (h_2, ω_2), where

$$\begin{aligned} \omega_1 &= \frac{5A^2 - 6A - 3}{5A^2 + 6A + 9}, & \tilde{\omega} &= \frac{2(A-2)}{3(A-1)}, \\ \omega_2 &= \frac{A-2}{A}, & \omega_2' &= \frac{(A-2)(5A^2 - 14A + 5)}{A(5A^2 - 26A + 41)}. \end{aligned} \tag{2.7}$$

We recall that an improper node is a node such that all the orbits arrive to or exit from it in one direction. Let

$$C_\omega = \{(h, \omega(h)) : \omega(h) \text{ is defined in (1.5)},\ h \in (h_c, h_s)\}.$$

Lemma 2.3 *In the* (h, ω)*–plane, the curve* C_ω *is the stable (or unstable) one–dimensional manifold of the saddle* (h_1, ω_1)*, and it connects to another singularity as* $h \to h_s$ *as follows (see figure 2, where the dashed curves show different possible positions of* C_ω*):*

(1) *to the improper node* (h_2, ω_2) *with slope* $-\infty$ *if* $A \in (-\infty, -1) \cup [2, 3)$*, and with slope* $+\infty$ *if* $A \in (3, \infty)$ *(when* $A \in (2,3)$*,* C_ω *passes through the saddle* $(0, \tilde{\omega})$ *and coincides with the unstable manifold of this saddle, with negative slope);*

(2) *to the improper node* $(0,0)$ *with slope* $+\infty$ *if* $A \in (-1, 2)$.

Remark 2.2 *We do not show the cases* $A = 1$ *and* $A = 3$ *in Figure 2. In fact the behavior of* C_ω *for case* $A = 1$ *is the same as for the case* $0 < A < 1$ *(or the case* $1 < A < 2$*), but the value* $\tilde{\omega}$ *tends to* ∞ *(or* $-\infty$*). In the case* $A = 3$*,* C_ω *is just a line segment* $\{\omega = 1/3\}$*, see Remark 2.1.*

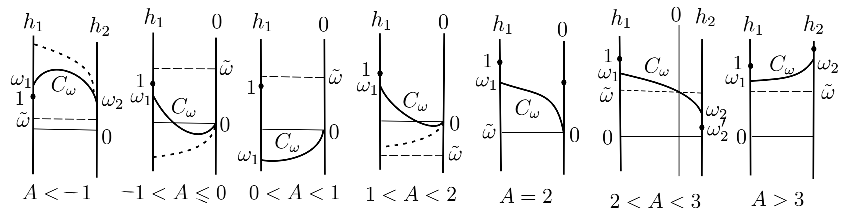


Figure 2. The behavior of C_ω for different values of A.

Proof. From (4.6) of [4] we see that C_ω is the stable or unstable manifold of the saddle (h_1, ω_1), going to the direction $h > h_c = h_1$. We need to study its limit as $h \to h_s$. For this purpose, we can use the same arguments as in section 5 of [4]. So, we list the main points below, and give details for one case (the little complicated case $A \in (2, 3)$).

(1) The function $\phi(h, \omega)$ in (2.6) is a polynomial of degree two in h and in ω. Hence each branch of the 0-clines of (2.6) (i.e. the curves $(h, \omega_0(h))$ in (h, ω)–plane defined by $\phi(h, \omega) = 0$) is monotone, or has at most one extreme point.

(2) From the coefficient of ω^2 in $\phi(h, \omega)$ of (2.6), it is not difficult to find that only for $A \in (0, 2)$ the orbits of system (2.6) can have a vertical asymptotic line in (h_c, h_s), and this case was studied in section 5 of [4]. In the other cases, each branch of the 0-clines connects two singularities on the invariant lines $h = h_c$ and $h = h_s$, respectively, and the two branches of the 0-clines do not intersect for $h \in (h_c, h_s)$.

(3) At the saddles (h_1, ω_1) and $(0, \tilde{\omega})$ the 0-clines of (2.6) have slopes

$$\omega'_{01} = 2\omega'(h_1), \quad \omega'_{00} = 2\tilde{\omega}'(0), \tag{2.8}$$

respectively, where

$$\omega'(h_1) = \frac{5(A+1)(A-3)g_1(A)}{24(5A^2+6A+9)^2}, \qquad \tilde{\omega}'(0) = \frac{5(A+1)(A-3)}{108(A-2)(A-1)^2}, \tag{2.9}$$

with $g_1(A) = 7A^3 + 21A^2 - 27A - 9$ (having zeros approximately at -3.90, -0.278 and 1.18), $\omega'(h_1)$ is the slope of C_ω at (h_1, ω_1), and $\tilde{\omega}'(0)$ is the slope of the stable (or unstable) manifold of system (2.6) at the saddle $(0, \tilde{\omega})$.

As an example, we study the case $A \in (2, 3)$ in detail. In this case, $h_c = h_1 < 0 < h_2 = h_s$, $1 > \omega_1 > \tilde{\omega} > 0$ and $\tilde{\omega} > \omega_2 > \omega'_2 > 0$. One branch of the 0-clines, C_0^1, passes through three singularities $(h_1, 1)$, $(0, \tilde{\omega})$ and (h_2, ω_2), and has a unique maximum point between the points $(h_1, 1)$ and $(0, \tilde{\omega})$. Another branch of the 0-clines, C_0^2, passes through other three singularities (h_1, ω_1), $(0, 0)$ and (h_2, ω_2'), and has a unique minimum between the points $(0, 0)$ and (h_2, ω_2'). By (2.8), the stable manifold of the saddle (h_1, ω_1), i.e. C_ω, and the unstable manifold of the saddle $(0, \tilde{\omega})$, denoted by C_ω', must stay in the region between C_0^1 and C_0^2 if $0 < h - h_1 \ll 1$ or $0 < -h \ll 1$. Since $\delta(0)$ is an ordinary oval of X_{H_A} for $2 < A < 3$, we have that $|I_0''(0)|$ and $|I_1''(0)|$ are bounded while $I_0''(0) \neq 0$ (see [2]). On the other hand, if we take $h = 0$ in (3.6) of [4], then we find $I_1''(0) \neq 0$; otherwise, we would get $I_0'(0) = 0$, which contradicts the fact that $I_0'(0)$ is the period of $\delta(0)$. Hence, by (1.5), $\omega(h)$ must tend to a finite non–zero value as $h \to 0$. This implies that, when h increases from h_1 to 0, C_ω cannot intersect the 0–cline C_0^1, then tends to ∞, and cannot go to the improper node $(0, 0)$. Therefore, the only possibility is that C_ω coincides with C_ω'. When h increases from 0 to h_2, C_ω is above the 0–cline C_0^1 (using (2.8) again), hence C_ω decreases and goes to the improper node (h_2, ω_2) with slope $-\infty$. □

We denote by $S_{\tilde{\omega}}$ the segment $\{(h, \omega) : \omega = \tilde{\omega}, h \in (h_c, h_s)\}$.

Lemma 2.4 *$C_\omega \cap S_{\tilde{\omega}} = \emptyset$ if $A \in (-\infty, \infty) \backslash (\{-1\} \cup (2, 3))$; and $C_\omega \cap S_{\tilde{\omega}} = \{(0, \tilde{\omega})\}$ if $A \in (2, 3)$.*

Proof. If $A = 1$, then $S_{\tilde{\omega}}$ tends to ∞ (or $-\infty$, see Remark 2.2), hence the statement is obviously true. The case of $A \in (2, 3)$ was studied in the proof of lemma 2.3. In other cases, the two endpoints of C_ω are located on the same side of $S_{\tilde{\omega}}$, see figure 2. Hence, if $C_\omega \cap S_{\tilde{\omega}} \neq \emptyset$, then there is at least one point on $S_{\tilde{\omega}}$, at which the vector field (2.6) would be tangent to $S_{\tilde{\omega}}$. We prove that this is impossible. If $A = 0$, then from (2.7) and (2.6) we have $\tilde{\omega} = 4/3$ and $\phi(h, 4/3) = 30h \neq 0$ for $h \in (h_c, h_s)$. If $A \in (-\infty, \infty) \backslash (\{-1\} \cup \{0\} \cup \{1\} \cup (2, 3))$, then

$$\phi(h, \tilde{\omega}) = -\frac{5A(A-1)(A+1)^2(A-3)^2 h(h-h^*)}{9(A-1)^2},$$

where $h^* = (A+1)(A-2)(A-3)/(A(A-1))$, and

$$(h_1 - h^*)(h_2 - h^*) = \frac{4(A+1)(A-2)(A-3)}{A^3(A-1)^2} > 0, \text{ if } A < -1 \text{ or } A > 3;$$

and

$$(h_1 - h^*)(0 - h^*) = \frac{2(A+1)(2-A)(A-3)^2}{A^2(A-1)^2} > 0, \text{ if } A \in (-1, 2) \backslash (\{0\} \cup \{1\}).$$

Hence, h^* does not belong to (h_c, h_s), and the lemma is proved. $\square$

3 Properties of the curves $\bar{\Sigma}_A$ and $\bar{\Omega}_A$

Instead of Σ_A and Ω_A (see (1.8)), for simplicity, we will defined the curves $\bar{\Sigma}_A$ and $\bar{\Omega}_A$. For $A \neq -1$ we let

$$\bar{q}(h) = q(h) + \frac{A+3}{A+1}p(h) - \frac{2(A+2)}{A+1}. \tag{3.1}$$

Correspondingly, let

$$\bar{\nu}(h) = \nu(h) + \frac{A+3}{A+1}\omega(h) - \frac{2(A+2)}{A+1}. \tag{3.2}$$

We define the two curves

$$\bar{\Sigma}_A = \{(p, \bar{q})(h) : h \in (h_c, h_s)\},\ \bar{\Omega}_A = \{(\omega, \bar{\nu})(h) : h \in (h_c, h_s)\}. \tag{3.3}$$

Lemma 3.1 *For $A \in (-\infty, \infty)\backslash\{-1\}$ the following statements hold.*

(1) *As h increasing from $h_c = h_1$ to h_s, the curve $\bar{\Omega}_A$ goes from its endpoint $(\omega, \bar{\nu})(h_1) = (w_1, 0)$ to the endpoint $(\omega, \bar{\nu})(h_2) = (w_2, \bar{\nu}_2)$ if $A \in (-\infty, -1) \cup (2, \infty)$, where $\bar{\nu}_2 = 12/(A(A+1)(A-2))$.*

(2) *Along $\bar{\Omega}_A$ we have*

$$\lim_{h\to h_1} \frac{d\bar{\nu}}{d\omega} = \frac{72(5A^2+6A+9)}{(A-3)g_1(A)},$$

$$\lim_{h\to h_1} \frac{d^2\bar{\nu}}{d\omega^2} = -\frac{28(5A^2+6A+9)^3 g_2(A)}{5(A+1)(A-3)(g_1(A))^3},$$

where $g_1(A)$ is defined below (2.9), and $g_2(A) = 55A^4 + 204A^3 + 162A^2 - 324A - 81$ (having two real roots approximately at -0.2306 and 0.9803).

Proof. Statement (1) is easily deduced from (3.2), (2.5) and lemma 2.3. Statement (2) can be obtained from system (2.6) by using (3.2) and the method of lemma 2 from [5]. $\square$

Lemma 3.2 *For $A \in (-\infty, \infty)\backslash\{-1\}$ the following statements hold.*

(1) *For $0 < h - h_1 \ll 1$ we have the following expansions*

$$p(h) = 1 - \frac{1}{24}(A+1)(h-h_1) + \cdots, \quad \bar{q}(h) = \frac{5}{288}(A+1)(h-h_1)^2 + \cdots.$$

(2) *Along* $\bar{\Sigma}_A$ *we have* $\lim_{h\to h_1} \dfrac{d\bar{q}}{dp} = 0$ *and* $\lim_{h\to h_1} \dfrac{d^2\bar{q}}{dp^2} = \dfrac{20}{A+1}$.

(3) *Along* $\bar{\Sigma}_A$ *we have* $\lim_{h\to h_2} \dfrac{d\bar{q}}{dp} = \dfrac{\bar{q}(h_2) - \bar{\nu}(h_2)}{p(h_2) - \omega(h_2)}$ *if* $A \in (-\infty, -1) \cup (2, \infty)$.

Proof. Statement (1) can be obtained from system (2.3) by using (3.1) and the method of lemma 2 from [5]. Statement (2) follows from statement (1). To prove statement (3), we use the following formula (see the end of section 1 of [8]). Suppose that $f(x,y)$ is a polynomial. Then near the value h_s, corresponding to a saddle loop with a saddle at (x_s, y_s), any integral $F(h) = \iint_{\mathrm{Int}(\delta_A(h))} f(x,y)\,dx\,dy$, has the expansion

$$F(h) = F(h_s) + f(x_s, y_s)(h - h_s)\ln|h - h_s| + c_3(h - h_s) + \cdots,$$

where c_3 is some coefficient which we do not need in our discussion. If $A \in (-\infty, -1) \cup (2, \infty)$, then by lemma 2.1(2), $(x_s, y_s) = ((A-2)/A, 0)$ is a saddle with a saddle loop, corresponding to the level value $h_s = h_2$. From this formula and definition (1.4) we have that

$$I_k(h) = I_k(h_2) + (x_s)^k (h - h_s)\ln|h - h_s| + c_k(h - h_s) + \cdots, \ k = 0, 1, -1.$$

Hence, by using the above formula and definition (1.5) we have

$$\lim_{h\to h_2} \frac{dq}{dp} = \lim_{h\to h_2} \frac{I'_{-1}(h) - I'_0(h)q(h)}{I'_1(h) - I'_0(h)p(h)} = \frac{(x_s)^{(-1)} - q(h_2)}{x_s - p(h_2)},$$

where $x_s = (A-2)/A$. Combining this with (3.1), we obtain statement (3). □

Consider the following transformation from the (h,ω)–plane to the $(\omega, \bar{\nu})$–plane

$$\omega = \omega, \quad \bar{\nu} = \left(c_1(h, A) - \frac{2(A+2)}{A+1}\right) + \left(c_2(h, A) + \frac{A+3}{A+1}\right)\omega.$$

By (2.4), if $A \neq 3$, the transformation has the form

$$\omega = \omega, \quad \bar{\nu} = \frac{3(h - h_1)}{h(A-3)}\,[3(A-1)\omega - 2(A-2)]. \tag{3.4}$$

It is easy to see that (3.4) maps the straight line $\{(h,\omega) : h = h_0\}$ ($h_0 \in [h_c, h_s]$) of the (h,ω)–plane into a straight line of the $(\omega, \bar{\nu})$–plane. In particular, maps $\{(h,\omega) : h = h_1\}$ into $L_1 = \{(\omega, \bar{\nu}) : \bar{\nu} = 0\}$, and maps $\{(h,\omega) : h = h_s\}$ into L_2, where

$$L_2 = \begin{cases} \left\{(\omega,\bar\nu) : \bar\nu = \dfrac{36(A-1)(\omega-\tilde\omega)}{(A+1)(A-3)(A-2)^2}\right\}, \\ \qquad\qquad \text{if } A \in (-\infty,-1)\cup(2,\infty)\backslash\{3\}, \\ \\ \{(\omega,\bar\nu) : \omega = \tilde\omega\}, \quad \text{if } A \in (-1,2]\backslash\{1\}. \end{cases} \tag{3.5}$$

We note that if $A \neq 1$ then $L_1 \cap L_2 = \{(\tilde\omega, 0)\}$. From (2.5) and (3.2) we see that the transformation (3.4) maps the curve C_ω of the (h,ω)–plane into the curve $\bar\Omega$ of the $(\omega,\bar\nu)$–plane.

Let

$$D_A = \begin{cases} \{(h,\omega) : h \in [h_1,h_2], \omega > \tilde\omega\} \text{ if } A \in (-\infty,-1)\cup(3,\infty), \\ \{(h,\omega) : h \in [h_1,0), \omega < \tilde\omega\} \text{ if } A \in (-1,1), \\ \{(h,\omega) : h \in [h_1,0), -\infty < \omega < \infty\} \text{ if } A = 1, \\ \{(h,\omega) : h \in [h_1,0), \omega > \tilde\omega\} \text{ if } A \in (1,2], \\ \{(h,\omega) : h \in [h_1,0)\cup(0,h_2], \omega > \tilde\omega\} \text{ if } A \in (2,3). \end{cases} \tag{3.6}$$

Correspondingly, let D'_A be the region in the $(\omega,\bar\nu)$–plane, which is the half–plane $\bar\nu \geq 0$ if $A = 1$, and is the sector limited by the two straight lines L_1 and L_2 with vertex at $(\tilde\omega, 0)$ (containing $\bar\Omega$ inside) if $A \neq -1, 1, 3$. Note that L_1 is included in D'_A and the vertex $(\tilde\omega, 0)$ is not; and L_2 is included only when $A < -1$ and $A > 2$. If $A \in (2,3)$, then both D_A and D'_A are divided into two sub–regions.

The Jacobian of transformation (3.4)

$$\frac{D(\omega,\bar\nu)}{D(h,\omega)} = -\frac{3[2(A-1)\omega - 2(A-2)]}{h^2}$$

is well defined and non–zero if $(h,\omega) \in D_A$ (see the definitions of (3.6) and (2.7)). Hence, we immediately have the following result.

Lemma 3.3 *For any $A \in (-\infty,\infty)\backslash(\{-1\}\cup\{3\})$ the transformation (3.4) from D_A to D'_A is a smooth diffeomorphism. Hence, system (2.6) in D_A becomes the smooth system*

$$\dot\omega = \varphi_1(\omega,\bar\nu), \qquad \dot{\bar\nu} = \varphi_2(\omega,\bar\nu), \tag{3.7}$$

in D'_A.

Lemma 3.4 *For $A \in (-\infty,\infty)\backslash(\{-1\}\cup\{3\})$ we have*

(1) *Any orbit of system (2.6), especially C_ω, is transversal to all straight lines $\{h = h_0, h_0 \in [h_1,h_s)\}$ in D_A.*

(2) *Any orbit of system (3.7), especially* $\bar{\Omega}_A$, *is transversal to all straight lines between* L_1 *and* L_2 *in* D'_A *(*L_1 *is included but* L_2 *is not), the lines are parallel if* $A = 1$, *or are in a sector region with vertex* $(\tilde{\omega}, 0)$ *if* $A \neq 1$.

Proof. Statement (1) follows from the first equation of (2.6), i.e. $\dot{h} \neq 0$, if $h \in (h_1, h_s)$ and $A \in (-\infty, \infty)\backslash(\{-1\} \cup (2, 3])$, and by using the first expression of (2.9). The case $A \in (2, 3)$ was discussed in the proof of lemma 3.3. Statement (2) follows from statement (1) and lemma 3.3. □

We denote by $L^*_{\alpha\beta\gamma}$ the part of the straight line $L'_{\alpha\beta\gamma} = \{\alpha+\beta\omega+\gamma\bar{\nu} = 0\}$ in the $(\omega, \bar{\nu})$–plane, which is contained in D'_A. Let $C_U = \{(h, \omega) : h_1 \leq h \leq h_s,\ \omega = U(h)\}$ where

$$U(h) = U(h; A, \alpha, \beta, \gamma) = \frac{M(h)}{N(h)}, \tag{3.8}$$

with

$$M(h) = [-(A-3)\alpha + 6(A-2)\gamma]h - 6(A-2)(A-3)\gamma,$$
$$N(h) = [(A-3)\beta + 9(A-1)\gamma]h - 9(A-1)(A-3)\gamma.$$

Lemma 3.5 *For* $A \in (-\infty, \infty)\backslash(\{-1\} \cup \{3\})$ *and any constants* α, β *and* γ, $L^*_{\alpha\beta\gamma}$ *is tangent with order* k *to an orbit of system (3.7) (in particular to* Ω_A, *at a point* $(\omega, \bar{\nu})(h_0)$ *for* $h_0 \in (h_1, h_s)$*), if and only if* C_U *is tangent with order* k *to the corresponding orbit of system (2.6) (in particular to* C_ω, *at* $(h_0, \omega(h_0))$*).*

Proof. Under transformation (3.4) the line $L'_{\alpha\beta\gamma}$ becomes

$$\alpha + \beta\omega + \gamma\bar{\nu} = \frac{N(h)\omega - M(h)}{(A-3)h}, \tag{3.9}$$

where $\beta^2 + \gamma^2 \neq 0$, and the linear functions $N(h)$ and $M(h)$ are defined in (3.8). If $N(h_0) \neq 0$, then for h near h_0 we can rewrite the above equality as

$$\alpha + \beta\omega + \gamma\bar{\nu} = \frac{N(h)}{(A-3)h}[\omega - U(h)].$$

This means that transformation (3.4) maps the straight line $L^*_{\alpha\beta\gamma}$ to the curve C_U, and the lemma is proved for h near h_0 by lemma 3.3. Next, we show that we can skip the zero point of $N(h)$ for $h \in [h_1, h_s]$. In fact, if $N(h_0) = 0$ but $M(h_0) \neq 0$, then the equation (3.9) is not satisfied, and we do not need to consider it. If $N(h_0) = M(h_0) = 0$, then the resultant of $N(h)$ and $M(h)$ must be zero, i.e.

$$(A-3)^2[3(A-1)\alpha + 2(A-2)\beta]\,\gamma = 0.$$

If $\gamma = 0$, then $N(h) = (A-3)h\beta$ and $M(h) = (A-3)h\alpha$, the right hand side of (3.9) becomes $\beta\omega + \alpha$, and the above discussion is still valid. Note that in this case $U(h)$ is a constant $-\alpha/\beta$ ($\beta \neq 0$ when $\gamma = 0$). If $3(A-1)\alpha + 2(A-2)\beta = 0$, then $L^*_{\alpha\beta\gamma} \subset D'_A$ is parallel to L_1 when $A = 1$, or passes through the vertex $(\tilde{\omega}, 0)$ of the sector D'_A when $A \neq 1$. By lemma 3.4, there is no orbit of system (3.7) tangent to $L^*_{\alpha\beta\gamma}$, and the assumption of the lemma is not satisfied. □

Lemma 3.6 *For $A \in (-\infty, \infty)\backslash(\{-1\} \cup \{3\})$ and for any constants α, β, and γ, there exist at most three points on $L^*_{\alpha\beta\gamma}$, counting their multiplicities, such that at each of these points the vector field (3.7) is tangent to $L^*_{\alpha\beta\gamma}$.*

Proof. By lemma 3.5 we only need to consider the number of tangent points on C_U (corresponding to $L^*_{\alpha\beta\gamma}$) with respect to the vector field (2.6) in the (h, ω)–plane. By using (2.6) and (3.8) we obtain

$$\dot{\omega} - U'(h)\dot{h}\big|_{\omega=U(h)} = \phi(h, U(h)) - U'(h)T(h) = \frac{h(A-3)^2\,F(h)}{N^2(h)}, \quad (3.10)$$

where $F(h) = F(h; A, \alpha, \beta, \gamma)$ is a polynomial in all its arguments, and of degree 3 in h. Besides, $F(h)$ has the factor $(h - h_1)$ or $(h - h_2)$ if $L^*_{\alpha\beta\gamma}$ has the endpoint $(\omega, \bar{\nu})(h_1)$ or $(\omega, \bar{\nu})(h_2)$, respectively. Note that we may suppose $N(h) \neq 0$ for $h \in [h_1, h_s]$, see the proof of lemma 3.5, and the straight line $\{h = 0\}$ is not included in D_A, see (3.6). □

Lemma 3.7 *In the identified $(p, q)-$ and $(\omega, \bar{\nu})-$plane, if a straight line $L_{\alpha\beta\gamma}$ intersects $\bar{\Sigma}_A$ at least in two points (counting their multiplicities, but the left endpoint $(1, 0)$ of $\bar{\Sigma}_A$ is not included), then $L_{\alpha\beta\gamma}$ must intersect $\bar{\Omega}_A$ at some point $(\omega, \bar{\nu})(h)$ with $h \in (h_c, h_s)$.*

Proof. For this (α, β, γ), the Abelian integral $I_A(h) = I_0(h)(\alpha + \beta p(h) + \gamma\bar{q}(h))$ has at least 3 zeros for $h \in [h_c, h_s]$, since $I(h_c) = 0$. Hence $I''_A(h) = I''_0(h)(\alpha + \beta\omega(h) + \gamma\bar{\nu}(h))$ has at least one zero for $h \in (h_c, h_s)$. This gives the desired conclusion. □

Lemma 3.8 *The following statements hold.*

(1) *For any $A \neq -1$, $\bar{\Sigma}_A\backslash\{(1, 0)\}$ stays on one side of the tangent line at its endpoint $(p, \bar{q})(h_1) = (1, 0)$ (i.e. the line L_1). Hence, for $h > h_1$ we have that $\bar{q}(h) < 0$ if $A < -1$, and $\bar{q}(h) > 0$ if $A > -1$.*

(2) *For any* $A \neq -1$, $\bar{\Omega}_A \backslash \{(\omega_1, 0)\}$ *stays on one side of the tangent line, denoted by* L_3*, at its endpoint* $(\omega, \bar{\nu})(h_1) = (w_1, 0)$.

(3) *For any* $A \in (-\infty, -1) \cup (2, \infty) \backslash \{3\}$, $\bar{\Omega}_A \backslash \{(\omega_1, 0), (\omega_2, \bar{\nu}_2)\}$ *stays on one side of the straight line, denoted by* L_4*, passing through its two endpoints* $(\omega, \bar{\nu})(h_1) = (w_1, 0)$ *and* $(\omega, \bar{\nu})(h_2) = (w_2, \bar{\nu}_2)$*, and stay on one side of* L_2 *and is tangent to* L_2 *at* $(w_2, \bar{\nu}_2)$.

(4) *For any* $A \neq -1$*, if a straight line* L *intersect* $\bar{\Omega}_A$ *at lease in four points (counting their multiplicities), then it is impossible that the intersection point of* $L \cap L_1$ *is located on* L_1 *in the same side of* $(\omega_1, 0)$ *with respect to the point* $(1, 0)$.

Proof. By lemma 3.2(1) and (2), we have $(p, \bar{q})(h_1) = (1, 0)$, $\lim_{h \to h_1} d\bar{q}/dp = 0$, and $\lim_{h \to h_1} d^2\bar{q}/dp^2 = 20/(A+1)$. Hence $\bar{\Sigma}_A$ is tangent to $L_1 = \{\bar{q} = 0\}$, and $q(h)(A+1) > 0$ for $0 < h - h_1 \ll 1$. If $\bar{q}(h) = 0$ for some $h > h_1$, then $\bar{\Sigma}_A$ intersects the straight line L_1 at list at three points (counting their multiplicities), hence by the same argument as in the proof of lemma 3.7, L_1 must intersect $\bar{\Omega}_A$ at a point $(\omega, \bar{\nu})(h)$ with $h \in (h_1, h_s)$, and this contradicts lemma 3.4(2). Hence, statement (1) is proved.

We note that for h near h_1 and near h_s, $\bar{\Omega}_A$ is located on the same side of L_3. Hence, if $\bar{\Omega}_A \cap L_3 \neq \emptyset$ for $h > h_1$, then there are at least two points on L_2 (counting their multiplicities), such that the vector field (3.7) is tangent to L_3 in D'_A. By lemma 3.5, this is equivalent to a similar tangency property of vector field (2.6) with C_ω. Along L_3, we have that $\alpha = \omega_1 \gamma (d\bar{\nu}/d\omega)(h_1)$, $\beta = -\gamma (d\bar{\nu}/d\omega)(h_1)$. By using (2.6), lemma 3.1(2) and lemma 3.6, we obtain that the function $F(h)$ of (3.10) has the form

$$\gamma^2 (A+1)^2 (A-3)^2 (h-h_1)^2 \psi_1(h, A)/(g_1(h))^2,$$

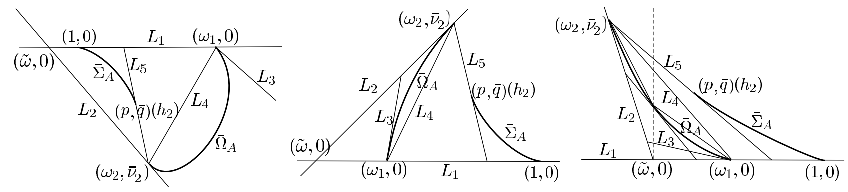
where $\psi_1(h, A)$ is a linear function in h. This proves statement (2).

To prove statement (3), we use the same argument as above. We take the straight line L_4 to be $L_{\alpha\beta\gamma}$, i.e. $\alpha = \gamma \omega_1 \bar{\nu}_2/(\omega_2 - \omega_1)$, $\beta = -\gamma \bar{\nu}_2/(\omega_2 - \omega_1)$. Hence the function $F(h)$ of (3.10) has the form

$$\gamma^2 (A-3)^2 (h-h_1)(h-h_2) \psi_2(h, A)/(A+3)^2 (A-2)^2,$$

where $\psi_2(h, A)$ is also a linear function in h. On the other hand, it is easy to verify that for h near h_1 and near h_2, $\bar{\Omega}_A$ is located on the same side of L_4. Since L_2 is contained in the boundary of D'_A, $\bar{\Omega}_A \backslash \{(\omega_2, \bar{\nu}_2)\}$ certainly stay on one side of L_2, and by lemmas 2.3 and 3.3 $\bar{\Omega}_A$ is tangent to L_2 at the point $(\omega_2, \bar{\nu}_2)$.

The proof of statement (4) is similar. If $L \cup \bar{\Omega}_A$ consists of at least 4 points, then there are at least 3 points on L at which the vector field (3.7)



(a) $A < -1$ (b) $A > 3$ (c) $2 < A < 3$

Figure 3. The relative positions of $\bar{\Sigma}_A$ and $\bar{\Omega}_A$ for $A \in (-\infty,-1) \cup (2,\infty)$.

is tangent to L. Hence, by lemma 3.5, the vector field (2.6) is tangent to C_U, corresponding to L, at least at 3 points. If in the $(\omega,\bar{\nu})$– plane the point $L \cap L_1$ is located on L_1 in the same side of $(\omega_1,0)$ with respect to the point $(1,0)$, then in the (h,ω)–plane, C_U intersects the invariant line $\{h = h_1\}$ at a point M on this line in the same side of the saddle (h_1,ω_1) with respect to the other singularity $(h_1,1)$. Note that C_ω is the stable or unstable manifold of the saddle (h_1,ω_1), hence the directions of the vector field (2.6) with respect to C_U at the point M and at the nearest intersection point of $C_U \cap C_\omega$ are opposite, hence between these two points there is at least one more tangent point on C_U with respect to the vector field (2.6), and this contradicts lemma 3.6. Note that by the same reason M cannot coincide with the saddle point (h_1,ω_1). Therefore, the proof of the lemma is finished. □

As we defined above, in the identified $(p,\bar{q})$– and $(\omega,\bar{\nu})$–plane, the sector D'_A is formed by the two straight lines L_1 and L_2 (if $A \neq -1, 1$), L_3 is the tangent line of $\bar{\Omega}_A$ at its endpoint $(\omega_1,0)$, and L_4 is the line joining the two endpoints of $\bar{\Omega}_A$. Now we let L_5 be the tangent line of $\bar{\Sigma}_A$ at its endpoint $(p,\bar{q})(h_2)$ if $A < -1$ or $A > 2$. Besides, we denote by D_Σ the region in D'_A, which is formed by L_1, L_2 and L_5, denote by D_Ω the region in D'_A, which is formed by L_3, L_4 and L_2. See figure 3.

Lemma 3.9 *If $A \in (-\infty,-1) \cup (2,\infty) \backslash \{3\}$, then we have : (1) $\bar{\Omega}_A \subset D_\Omega$. (2) $D_\Sigma \cap D_\Omega$ consists of a unique point $(\omega,\bar{\nu})(h_2) = (\omega_2,\bar{\nu}_2)$. (3) $\bar{\Sigma}_A \subset D_\Sigma$. (4) If $A > 2$, then the boundary of D_Ω is a triangle. In particular, if $A \in (2,3)$, then each of the two pieces of $\bar{\Omega}_A$ (contained in each of the two parts of D'_A, respectively) is also located in the triangle region formed by the two tangent lines at its two endpoints and the straight line passing through them, see figure 3(c).*

Proof. Statement (1) follows from lemma 3.8(2),(3) and the fact that L_2 is contained in the boundary of D'_A.

By lemma 3.2(3), L_5 and L_4 pass through the same point $(\omega, \bar{\nu})(h_2) = (\omega_2, \bar{\nu}_2)$. Hence, to prove statement (2), it is enough to prove that in D'_A the line L_5 is on the left of L_4 if $A < -1$, and L_5 is on the right of L_4 if $A > 2$, and this fact follows from the following claims (see figure 3):

(i) $p(h_2) < \omega_2$ if $A < -1$ and $p(h_2) > \omega_2$ if $A > 2$.

(ii) $\bar{q}(h_2) > \bar{\nu}_2$ if $A < -1$ and $\bar{q}(h_2) < \bar{\nu}_2$ if $A > 2$.

(iii) $p(h_2) < \omega_1$ if $A < -1$, and $p(h_2) > \omega_1$ if $A > 2$.

Now we prove these claims. From lemma 2.1(2) we see that if $A \in (-\infty, -1) \cup (2, \infty)$ then $\mathrm{Int}(\delta_A(h_2))$ is the region bounded by the homoclinic loop of the saddle $((A-2)/A, 0)$, and $\mathrm{Int}(\delta_A(h_2))$ is on the left of the saddle if $A < -1$, and on the right of the saddle if $A > 2$ (see figure 1). From (2.7) we have $\omega(h_2) = \omega_2 = (A-2)/A$. Hence, by the definition of $p(h)$ in (1.5), we obtain claim (i). If claim (ii) is not true, then by lemma 3.2(3) we would find a straight line L, such that $L \cap D_\Omega = \emptyset$ and L intersects $\bar{\Sigma}_A$ at least in two points, and this contradicts lemma 3.7. By using the definition of $p(h)$ in (1.5) and (1.4), claim (iii) is equivalent to $\xi(A) = \iint_{\mathrm{Int}(\delta_A(h_2))} (x - \omega_1) dx\, dy < 0$ if $A < -1$, and $\xi(A) > 0$ if $A > 2$. In polar coordinates (r, θ) given by $x = \omega_2 + r\cos\theta$, $y = r\sin\theta$, the oval $\delta_A(h_2)$ is given by

$$r = r(\theta) = \frac{3A\cos^2\theta - (A-2)\sin^2\theta}{A\cos\theta\,(\sin^2\theta + A\cos^2\theta)},$$

where $\theta \in \tan^{-1}(-t_A, t_A)$, $t_A = \sqrt{3A/(A-2)\cos\theta}$. Let $t = \tan\theta$, then

$$\xi(A) = \int_{-t_A}^{t_A} \left(\frac{1}{3} f^3(t, A) + \frac{\omega_2 - \omega_1}{2} f^2(t, A) \right) dt,$$

where $f(t, A) = ((2-A)t^2 + 3A)/(A(t^2 + A))$. A calculation shows that if $A < -1$, then

$$\xi(A) = \frac{(A+1)[(3-A)\sqrt{3(2-A)}\,\eta(A) - \zeta(A)\,\mathrm{arctanh}(\sqrt{3/(2-A)})]}{4(-A)^{5/2}(5A^2 + 6A + 9)} < 0,$$

where $\eta(A) = 25A^2 - 12A + 27$, and $\zeta(A) = 25A^4 - 112A^3 + 130A^2 + 120A + 45$. And if $A > 2$ then

$$\xi(A) = \frac{(A+1)[(3-A)\sqrt{3(A-2)}\,\eta(A) + \zeta(A)\arctan(\sqrt{3/(A-2)})]}{4A^{5/2}(5A^2 + 6A + 9)} > 0.$$

In both cases, the numerator of $\xi(A)$ has fixed sign, it can be shown by a method, for example, as in the proof of lemma 3.4 of [6]. Thus, claim (iii), and consequently statement (2), are proved.

To prove statement (3), we first note that $\bar{\Sigma}_A$ stays in one side of L_1 by lemma 3.8(1). We claim that $\bar{\Sigma}_A$ also stays in one side of L_5. Otherwise, L_5 intersects $\bar{\Sigma}_A$ at least in two points, this contradicts lemma 3.7, because by statements (1) and (2) $L_5 \cap \bar{\Omega}_A = \{(\omega, \bar{\nu})(h_2)\}$. The same argument implies that $\bar{\Sigma}_A$ also stays in one side of L_2. Otherwise, we would find a straight line, which intersects $\bar{\Sigma}_A$ at least in two points, and has no intersection with D_Ω.

To prove statement (4), we first note that

$$\omega_1 - \tilde{\omega} = \frac{5(A+1)(A-3)^2}{3(5A^2+6A+9)(A-1)}, \quad 1 - \omega_1 = \frac{12(A+1)}{5A^2+6A+9}.$$

Hence, $\tilde{\omega} < \omega_1 < 1$ if $A > 2$, and $\bar{\Omega}_A$ must be located inside the triangle region bounded by L_1, L_2 and L_4, see figures 3(b) and 3(c). On the other hand, by lemma 3.8(2), $\bar{\Omega}_A$ is located on one side of L_3, so L_3 must intersect L_2 at a point between the points $(\tilde{\omega}, 0)$ and $(\omega_2, \bar{\nu}_2)$, and $\bar{\Omega}_A$ is located inside the triangle region bounded by L_2, L_3 and L_4. If $A \in (2, 3)$, then D_A (resp. D'_A) is divided by the straight line $\{h = 0\}$ (resp. $\{\omega = \tilde{\omega}\}$) into two parts. From (3.4) we see that the images of $\{(h, \omega) : \omega = \tilde{\omega}\}$ and $\{(h, \omega) : h = 0\}$ are contained in $\{(\omega, \bar{\nu}) : \omega = \tilde{\omega}\}$. By taking $\omega = \omega(h)$ in (3.4), and using (2.6), (2.8) and the L'Hopital Rule we find that for $2 < A < 3$

$$\lim_{h \to 0} \bar{\nu}(h) = \frac{5(A+1)(3-A)}{12(A-1)(A-2)} > 0, \quad \lim_{h \to 0} \frac{\bar{\nu}'(h)}{\omega'(h)} = -\frac{41A^2 - 82A + 21}{8(A-2)^2} < 0.$$

Hence, we can determine the position of $(\omega, \bar{\nu})(0)$ and the curve $\bar{\Omega}_A$ is smooth at this point. The proof of the property for each arc of $\bar{\Omega}_A$, on the left and on the right of this point, is the same as in the proof of lemma 3.8(2) and (3). Here we also use the fact that the singularity $(\tilde{\omega}, 0)$ is a saddle of system (2.6). $\square$

If $A \in (-1, 2]$, then we need to consider $h \in (h_1, 0)$. In the identified plane, we let $L_6 = \{(p, \bar{q}) : p = p(0)\}$, and let L_0 be the identified $\bar{q}-$ and $\bar{\nu}-$ axes. From (2.7) it is easy to find the two zeros of $\omega_1 = 0$ at $A_1 \approx -0.3798$ and $A_2 \approx 1.5798$.

Lemma 3.10 *If $A \in (-1, 2]$, then the following statements hold.*

(1) $\bar{\Sigma}_A \backslash \{(1, 0)\}$ *is located above L_1 (the tangent line at its endpoint $(1, 0)$), and on the right of its asymptotic line L_6, which is on the right of L_0.*

(2) $\bar{\Omega}_A \backslash \{(\omega_1, 0)\}$ *is located above L_1 and on one side of L_3 (the tangent line at its endpoint $(\omega_1, 0)$), and has the asymptotic line L_0. Besides, $\bar{\Omega}_A$ has the following properties:*

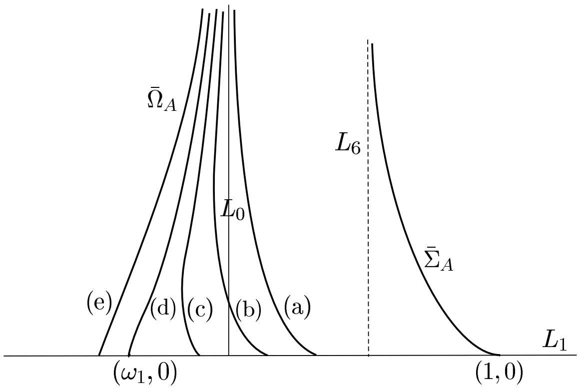


Figure 4. The relative positions and behavior of $\bar{\Sigma}_A$ and $\bar{\Omega}_A$ for $A \in (-1, 2]$.

(i) *If $A = 2$, then $\bar{\Omega}_2$ is located on the right of L_0. If $A \in (-1, A_1] \cup [A_2, 2)$, then $\bar{\Omega}_A \cap L_0$ consists of a unique point. In these two cases along $\bar{\Omega}_A$ near its endpoint $(\omega_1, 0)$ we have $d\bar{\nu}/d\omega < 0$ and $d^2\bar{\nu}/d\omega^2 > 0$, and $\bar{\Omega}_A \backslash \{(\omega_1, 0)\}$ stays on the left of the vertical straight line $L_7 = \{\omega = \omega_1\}$. The behavior of $\bar{\Omega}_A$ is shown as curves (a) and (b) in figure 4, respectively.*

(ii) *If $A \in (A_1, A_2)$, then $\bar{\Omega}_A$ is located on the left of L_0, and along $\bar{\Omega}_A$ near its endpoint $(\omega_1, 0)$, if $d\bar{\nu}/d\omega < 0$ then $d^2\bar{\nu}/d\omega^2 > 0$. See curves (c), (d) and (e) in figure 4.*

(3) *Any tangent line of $\bar{\Omega}_A$ does not pass through the point $(\tilde{\omega}, 0)$ if $A \neq 1$, where $\tilde{\omega} \in (-\infty, 0) \cup (1, \infty)$ if $A \neq 2$, and $\tilde{\omega} = 0$ if $A = 2$. If $A = 1$, then any tangent line of $\bar{\Omega}_1$ is not parallel to L_1.*

(4) *$\omega_1 < p(0)$, hence $\bar{\Sigma}_A \cap \bar{\Omega}_A = \emptyset$.*

Proof. Statement (1) follows from lemmas 3.8(1), 2.2(1) and 3.2(1). Since the heteroclinic loop $\delta(0)$ is located on the right of the invariant line $\{x = 0\}$ of system X_{H_A}, from the definition of $p(h)$ and $q(h)$ in (1.5) we find $p(0) > 0$ and $\lim_{h \to 0} q(h) = \infty$. By (3.1), $\bar{\Sigma}_A$ has the asymptotic line L_6.

Statement (2) follows from lemmas 3.8(2) and the following two facts: (I) L_1 corresponds to a straight line $\{\alpha + \beta\omega + \gamma\bar{\nu} = 0\}$ with $\alpha = \beta = 0$. By (3.8), if $\alpha \to 0$ and $\beta \to 0$, then C_U tends to $\{\omega = \tilde{\omega}\} \cup \{h = h_1\}$ if $A \neq 1$ and tends to $\{h = h_1\}$ if $A = 1$. In any case, by lemmas 2.4 and 3.4, $\bar{\Omega}_A \backslash \{(\omega_1, 0)\}$ is located above L_1. (II) By lemma 2.3, $\lim_{h \to 0} \omega(h) = 0$. Hence, by lemma 3.7 and statement (1) ($\bar{\Sigma}_A$ has the vertical asymptotic line L_6), $\bar{\Omega}_A$ must have the asymptotic line L_0.

Now we prove the sub–statements (2)(i) and (2)(ii). The behavior of $\bar{\Omega}_A$ near its endpoint $(\omega_1, 0)$ is easy to obtain from lemma 3.1(2) and by comparing the values of the zero points of ω_1 (as a function of A): $g_1(A)$ and $g_2(A)$. Note that for h near h_1 and near 0, $\bar{\Omega}_A$ stays on the left of L_0 if $A \in [A_1, A_2]$ $(\omega_1(A_1) = \omega_1(A_2) = 0)$, stays between the two lines L_0 and L_7 if $A = 2$, and is located, respectively, on the left of L_0 and between L_0 and L_7 if $A \in (-1, A_1) \cup (A_2, 2)$. Hence, if we prove that $\bar{\Omega}_A$ can intersect each line L_0 and L_7 at most in one point (counting the multiplicity), then the sub–statements (i) and (ii) follow. We note that if $\bar{\Omega}_A$ intersects the line L_0 (resp. L_7) at least in two points, then along L_0 (resp. L_7) we may find at least two tangent points with respect to the vector fields (3.7), because $\bar{\Omega}_A$ is the stable or unstable manifold of the saddle $(\omega_1, 0)$, hence the directions of the vector fields (3.7) with respect to L_0 (resp. L_7) at the point $(0, 0)$ (resp. (ω_1, σ), $0 < \sigma \ll 1$) and at the nearest intersection point of $\bar{\Omega}_A \cap L_0$ (resp. $\bar{\Omega}_A \cap L_7$) are different. Thus, it is enough to show that along L_0 (resp. L_7) the vector field (3.7) has at most one tangent point for $h \in (h_1, 0)$. In fact, if we take L_0 (resp. L_7) as the line $L_{\alpha\beta\gamma}$, i.e. $\alpha = 0, \gamma = 0$ (resp. $\alpha = -\omega_1\beta, \gamma = 0$), we obtain the function $F(h)$ in (3.10) with a linear function in h as its factor, besides the factors h and $h - h_1$.

Statement (3) follows from lemma 3.4(2) and the fact that from (2.7) we have $\tilde{\omega} > 1$ if $-1 < A < 1$, $\tilde{\omega} > 0$ if $1 < A < 2$, and $\tilde{\omega} = 0$ if $A = 2$.

If $A \in [A_1, A_2]$, then $\omega_1 < 0$, and statements (1) and (2)(ii) obviously imply statement (4). If $A \in (-1, A_1) \cup (A_2, 2]$, we calculate $p(0) - \omega_1$ directly. In this case the period annulus is bounded by $H_A(x, y) = 0$. Using (1.1), (1.5) and (1.4) we have that $p(0) - \omega_1 > 0$ if and only if

$$K_A = \int_0^{x_A} (x - \omega_1)\sqrt{-Ax^2 + 3(A-1)x - 3(A-2)}\, dx > 0,$$

where $x_A = [3(A-1) + \sqrt{3(3 + 2A - A^2)}]/(2A)$. A calculation shows that if $A \in (-1, 0) \supset (-1, A_1)$, then

$$K_A = \frac{3(A+1)(3-A)}{48(5A^2 + 6A + 9)} \left\{ 2\eta(A)\sqrt{3A(A-2)} + 3\varsigma(A) \ln \mu(A) \right\} > 0,$$

where

$$\eta(A) = 25A^2 - 12A + 27, \quad \varsigma(A) = 5A^3 + 15A^2 + 15A - 27,$$
$$\mu(A) = \frac{\sqrt{3A(A+1)(3-A)}}{2(3(1-A) - 2\sqrt{3A(A-2)})}.$$

Similarly, if $A \in (0, 2] \supset (A_2, 2]$, then

$$K_A = \frac{3(A+1)(3-A)}{48(5A^2 + 6A + 9)} \left\{ 2\eta(A)\sqrt{3A(2-A)} + 3\varsigma(A)\tau(A) \right\} > 0,$$

where $\eta(A)$ and $\varsigma(A)$ are the same as above, and

$$\tau(A) = \left(\frac{\pi}{2} + \arctan\left(\frac{3(A-1)}{2\sqrt{3A(2-A)}}\right)\right).$$

□

4 Proof of the main result

Lemma 3.2(2) shows that for any $A \neq -1$ the curve $\bar{\Sigma}_A$ is convex near its endpoint $(1,0)$, corresponding to $h = h_1$. We need to prove that $\bar{\Sigma}_A$ is convex for all $h \in [h_1, h_s)$. We suppose the contrary: there is a $h^* \in (h_1, h_s)$ such that the point $M^* = (p, \bar{q})(h^*)$ is the first zero–curvature point on $\bar{\Sigma}_A$ from the point $(1,0)$. We denote by $\bar{\Sigma}_A^*$ the arc of $\bar{\Sigma}_A$ from $(1,0)$ to M^*. We first prove the following basic lemma.

Lemma 4.1 *In the identified $(p,\bar{q})$– and $(\omega,\bar{\nu})$–plane, for any $A \neq -1$ the curves $\bar{\Sigma}_A^*$ and $\bar{\Omega}_A$ have no common tangent line.*

Proof. First we make a proof in the case $A < -1$, then we use the same idea for the other cases. We denote by T_Σ^h (resp. T_Ω^h) the tangent line of $\bar{\Sigma}_A^*$ (resp. $\bar{\Omega}_A$) at the point $(p,\bar{q})(h)$ (resp. $(\omega,\bar{\nu})(h)$). We suppose that there are $t \in (h_1, h^*]$ and $s \in (h_1, h_s)$, such that $T_\Sigma^t = T_\Omega^s = T$.

We claim that the two endpoints of $\bar{\Omega}_A$, $(\omega_1, 0)$ and $(\omega_2, \bar{\nu}_2)$, must keep on different sides of the tangent line T_Σ^h for all $h \in (h_1, h^*]$. This is obviously true for the endpoint $(\omega_1, 0)$ by the convexity of $\bar{\Sigma}_A^*$. By lemma 3.8(1)–(3) and lemma 3.9 (see figure 3), if T_Σ^h passes through the point $(\omega_2, \bar{\nu}_2)$ or leaves this point at the same side as $(\omega_1, 0)$ does, then $T_\Sigma^h \cap \bar{\Omega}_A = \{(\omega_2, \bar{\nu}_2)\}$ or $T_\Sigma^h \cap \bar{\Omega}_A = \emptyset$, and this contradicts lemma 3.7. We now apply this property to the common tangent line T, then by lemma 3.8(4), the intersection of $T \cap \bar{\Omega}_A$ has only two possibilities: either a tangent point with tangency 2 plus a simple point, or a unique point with tangency 3. Besides, the behavior of $T \cap \bar{\Omega}_A$ has two types, shown in the cases (a) and (b) of figure 5. If the type (a) happens (it may have a tangency 3, or for h increasing $\bar{\Omega}_A$ crosses T at a point, then it meets T again at a tangent point on the left), then we can find $s' \in (h_1, s)$, such that $T_\Omega^{s'}$ passes through the point $(\tilde{\omega}, 0)$, contradicting lemma 3.4(2). If the type (b) happens (it also has two more relative positions of $T \cap \bar{\Omega}_A$), then by lemmas 3.8(2),(3) and 3.9(1) we must find a straight line L' (see figure 5(b)), such that $L' \cap \bar{\Omega}_A$ consists of at least 4 point (counting their multiplicities), and $L' \cap L_1$ at a point on L_1 in the same side of $(\omega_1, 0)$ with respect to the point $(1,0)$, and this contradicts lemma 3.8(4).

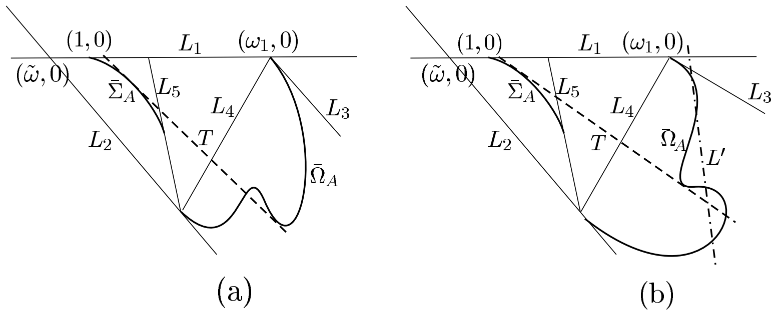


Figure 5. The behavior of $T \cap \bar{\Omega}_A$ for $A \in (-\infty, -1)$.

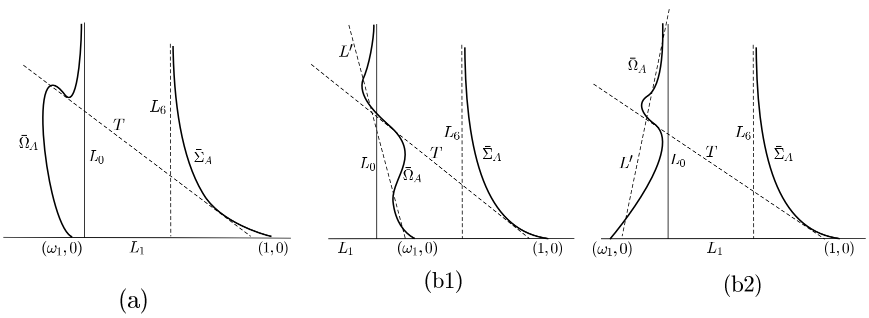


Figure 6. The behavior of $T \cap \bar{\Omega}_A$ for $A \in (-1, 2)$.

The proof in other cases is basically the same, and even more simple. If $A > 2$, then by lemma 3.9(4) D_Ω is bounded by a triangle (or each piece of $\bar{\Omega}_A$ is bounded in the triangle region if $2 < A < 3$). Hence, if $\bar{\Omega}_A$ and $\bar{\Sigma}_A$ have a common tangent line, then we must find another straight line in D'_A, such that it intersects $\bar{\Omega}_A$ at least in 4 points, and it intersects L_1 at the same side as $(\omega_1, 0)$ with respect to $(1, 0)$. We remark that the proof of the case $A = 2$ is exactly the same. The case $A = 3$ is special: from remark 2.1 we see that as $A \to 3$ then $\bar{\Omega}_A$ tends to a vertical segment $\{(\omega, \bar{\nu}) : \omega = 1/3, 0 \le \bar{\nu} \le 1\}$, and the lemma is obviously true.

If $A \in (-1, 2)$, then $\bar{\Omega}_A$ has a vertical asymptotic line L_0. Hence, it is obvious that the endpoint $(\omega_1, 0)$ of $\bar{\Omega}_A$ and the point $(\omega, \bar{\nu})(h)$ for $h \sim 0$ are located on different sides of the tangent line T^h_Σ for all $h \in (h_1, h^*]$. If $T^t_\Sigma = T^s_\Omega = T$ is a common tangent line ($t \in (h_1, h^*]$ and $s \in (h_1, 0)$), then the behavior of $T \cap \bar{\Omega}_A$ has two types shown in figure 6. If type (a) of figure

6 happens, then as h increases from s to 0, the tangent line T^h_Ω moves on $\bar{\Omega}_A$ in such a way that the intersection of $T^h_\Omega \cap L_1$ passes through the interval $\{(\omega, \bar{\nu}) : \bar{\nu} = 0, \omega \in (1, \infty) \cup (-\infty, 0)\}$, and also it passes a position which is parallel to L_1. This contradicts lemma 3.10(3). If the type (b1) or (b2) of figure 6 happens, then, by using lemmas 3.10 and 3.8, we may find a straight line, intersecting $\bar{\Omega}_A$ at least in 4 point, and intersecting L_1 in a point $(\omega, 0) \in L_1$ with $\omega < \omega_1 < 1$. This contradicts lemma 3.8(4). □

Proof of Theorem 1.2 We suppose $A \neq -1$. By lemma 2.2(1), the curve Σ_A is sectorial. From the definitions of Σ_A and $\bar{\Sigma}_A$, it follows that Σ_A is strictly convex with non–zero curvature if and only if $\bar{\Sigma}_A$ does. If $\bar{\Sigma}_A$ is not strictly convex with non–zero curvature, then, as we discussed in the beginning of this section, we may find the first zero–curvature point M^* on it from its endpoint $(1, 0)$, and denote by $\bar{\Sigma}^*_A$ the arc of $\bar{\Sigma}_A$ from $(1, 0)$ to M^*. We let T^h_Σ be the tangent line of $\bar{\Sigma}^*_A$ at the point $(p, \bar{q})(h)$. We take T^h_Σ to be the straight line $L_{\alpha\beta\gamma}$, move it on $\bar{\Sigma}_A$ as h increases from h_1 to h^*, and consider the number of intersection points of $T^h_{\Sigma^*} \cap \bar{\Omega}_A$. If $0 < h - h_1 \ll 1$, then by lemmas 3.4 and 3.1(2) $T^h_\Sigma \cap \bar{\Omega}_A$ consists of a unique point (counting its multiplicity). Since $\bar{\Sigma}_A$ has zero curvature at the point $(p, \bar{q})(h^*)$, the Abelian integral $I_A(h) = I_0(h)(\alpha + \beta p(h) + \gamma \bar{q}(h))$ has at least a triple zero at h^* plus a zero at h_1. Hence $I''_A(h) = I''_0(h)(\alpha + \beta\omega(h) + \gamma\bar{\nu}(h))$ has at least two zeros for $h \in (h_1, h^*)$. This means that $T^h_\Sigma \cap \bar{\Omega}_A$ consists of at least two points. On the other hand, as we have obtained in the proof of lemma 4.1, the endpoint $(\omega_1, 0)$ and the point $(\omega, \bar{\nu})(h)$ for $h \sim 0$ are located on the different sides of the tangent line T^h_Σ for all $h \in (h_1, h^*]$. Therefore, the increasing of the numbers of intersection points of $T^h_\Sigma \cap \bar{\Omega}_A$, as h increases from h_1 to h^*, can be realized only through a tangent position of T^h_Σ to $\bar{\Omega}_A$ for a $h \in (h_1, h^*]$, and this contradicts lemma 4.1. Thus, the proof of the theorem is finished. □

Acknowledgements This work was elaborated when the first author is visiting the Centre de Recerca Matemática and Departament de Matemàtiques, of Universitat Autònoma de Barcelona. He thanks them for the hospitality and financial support.

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