UNIFORM APPROXIMATION ON CLOSED SUBSETS OF $\mathbb{C}$ BY POLYANALYTIC FUNCTIONS

A. BOIVIN, P. M. GAUTHIER, AND P. V. PARAMONOV

Abstract. We give necessary and/or sufficient conditions on a closed set F in $\mathbb{C}$ in order that any function f, continuous on F and n-analytic in F^0, be the uniform limit on F of a sequence of n-analytic entire or n-analytic meromorphic functions.

1. Introduction

Let $n \in \mathbb{N}$ be a fixed natural number, and let $\overline{\partial} = \partial/\partial\overline{z}$ denote the Cauchy-Riemann operator. A function f is said to be *n-analytic* (or *polyanalytic of order n*) in an open set $G \subset \mathbb{C}$ if f is n-times continuously differentiable and satisfies $\overline{\partial}^n f = 0$ in G.

It follows easily that f can always be written as

$$f(z) = f_0(z) + \overline{z} f_1(z) + \ldots + \overline{z}^{n-1} f_{n-1}(z), \tag{1}$$

where $f_0, \ldots, f_{n-1}$ are holomorphic functions in G, and that this representation is unique.

When the functions f_j in (1) are either all (analytic) polynomials, or all rational functions, or all meromorphic functions, we obtain the class of n-analytic polynomials, n-analytic rational functions and n-analytic meromorphic functions, respectively. Note that n-analytic rational functions are not defined as the quotient of n-analytic polynomials when $n \geq 2$, and that n-analytic meromorphic functions may be continuous at a singularity, for

---

1991 *Mathematics Subject Classification.* 30E10, 30G30.

Key words and phrases. polyanalytic functions, uniform approximation, Carathéodory domain, Nevanlinna domain.

The research of the first author was partially supported by a grant from NSERC of Canada.

The research of the second author was partially supported by grants from NSERC of Canada and FCAR Québec.

The research of the third author was partially supported by the grant SS – 2040.2003.1 of Russia.

example the 3-analytic function

$$f(z)=\begin{cases}\overline{z}^2/z & z\neq 0;\\ 0 & z=0;\end{cases}$$

at $z=0$.

Let F be a closed subset of $\mathbb{C}$ and denote by F^0 its interior. Let $C(F)$ be the space of all continuous functions on F and for $f\in C(F)$, let $\|f\|_F=\sup_{z\in F}|f(z)|$. Set

$$\mathcal{A}_n(F)=\{f\in C(F)\mid \overline{\partial}^n f=0 \text{ on } F^\circ\},$$

$$\mathcal{E}_n(F)=\{f\in C(F)\mid \forall\varepsilon>0, \exists g\in\mathcal{A}_n(\mathbb{C}) \text{ with } \|f-g\|_F<\varepsilon\},$$

and

$$\mathcal{M}_n(F)=\{f\in C(F)\mid \forall\varepsilon>0, \text{there exists a } n\text{-analytic meromorphic function } h \text{ in } \mathbb{C} \text{ having no singularities in } F \text{ with } \|f-h\|_F<\varepsilon\}.$$

That is $\mathcal{E}_n(F)$ (respectively $\mathcal{M}_n(F)$) is the set of functions in $C(F)$ that can be approximated uniformly on F by *n-analytic entire* functions (respectively by *n-analytic meromorphic* functions on $\mathbb{C}$ having no singularities on F).

For a compact subset X of $\mathbb{C}$, set

$$\mathcal{P}_n(X)=\{f\in C(X)\mid \forall\varepsilon>0, \text{there exists a } n\text{-analytic polynomial } p \text{ with } \|f-p\|_X<\varepsilon\}.$$

For X and Y compact sets in $\mathbb{C}$ with $X\subseteq Y$, let $\mathcal{R}_n^Y$ denote the set of n-analytic rational functions without singularities on Y and set

$$\mathcal{R}_n^Y(X)=\{f\in C(X)\mid \forall\varepsilon>0,\ \exists r\in\mathcal{R}_n^Y \text{ with } \|f-r\|_X<\varepsilon\}.$$

Note that $\mathcal{P}_n(X)\subseteq\mathcal{R}_n^X(X)\subseteq\mathcal{A}_n(X)$, that $\mathcal{E}_n(F)\subseteq\mathcal{M}_n(F)\subseteq\mathcal{A}_n(F)$ and that $\mathcal{E}_n(X)=\mathcal{P}_n(X)$.

In this paper, we give, for $n\geq 2$, several necessary or sufficient conditions on X or F, (apparently independent of n) in order that equality occurs in some of the inclusions above. In particular, for compact sets X, our results considerably generalize the main results of [4] (see Section 2).

We continue this section with a few more definitions and concepts that will be required to state our results. For any proper subset $S\subset\mathbb{C}$, we denote by S^0 its interior, ∂S its boundary and $\overline{S}$ its closure. The Riemann sphere $\mathbb{C}\cup\{\infty\}$ and the open unit disk $\{|z|<1\}$ will be denoted by $\overline{\mathbb{C}}$ and Δ respectively. For $X\subset\mathbb{C}$, X compact, $\widehat{X}$ will stand for the topological hull of X, that is, the union of X with all the bounded components of $\mathbb{C}\setminus X$ (holes of X).

Let f be a bounded holomorphic function on Δ. By Fatou's theorem the *nontangential limit* $f(e^{it})$ exists dt-almost everywhere (a.e.) on $\partial\Delta$. The following concept was introduced in [4].

Definition 1. A bounded simply connected domain $\Omega \subset \mathbb{C}$ is called a *Nevanlinna domain* if there exists a conformal mapping $k : \Delta \to \Omega$ of Δ onto Ω and two bounded holomorphic functions u and v in Ω (with $v \not\equiv 0$) such that dt-a.e. on $\partial\Delta$ one has

$$\overline{k(e^{it})} = \frac{(u \circ k)(e^{it})}{(v \circ k)(e^{it})}. \tag{2}$$

As pointed out in [4], this definition is independent of the choice of k, and (for Nevanlinna domains) the quotient u/v is uniquely defined in Ω. Indeed, if k_1, u_1, v_1 and k_2, u_2, v_2 satisfy (2), then k_1, u_2, v_2 also satisfy (2). By the Luzin-Privalov theorem (see [9, Chapter 3, §D3]), $(u_1/v_1)\circ k_1 = (u_2/v_2)\circ k_1$ and hence $u_1/v_1 = u_2/v_2$.

The class of all Nevanlinna domains will be denoted by $\mathcal{N}$. For examples of domains in this class, see [4, §3] and the references therein.

Definition 2. A domain $\Omega \subset \mathbb{C}$ is called a *Carathéodory domain* if Ω is bounded and $\partial\Omega = \partial\Omega_\infty$, where Ω_∞ is the unbounded component of $\mathbb{C}\setminus\overline{\Omega}$.

Definition 3. Let A be a subset of $\mathbb{C}$. A non-empty subset $I \subset A$ is a *border* of A (or an A-*border*) if each $a \in I$ has a neighbourhood V in $\mathbb{C}$ such that there is a homeomorphism h of $V \cap A$ onto a relatively open subset of the closed upper half-plane and $V \cap I = \{z \in V \mid \operatorname{Im} h(z) = 0\}$. Thus, a border is locally a Jordan arc. We further require that *a border be locally rectifiable.*

Note that if Ω is an open set in $\mathbb{C}$, I is an $\overline{\Omega}$-border and f is a bounded holomorphic function in Ω, then the nontangential limit $f(\zeta)$ exists for almost every ζ on I with respect to length.

For a bounded domain Ω, a relatively open subset I of $\partial\Omega$ and $\varepsilon > 0$, set

$$\Omega_\varepsilon = \{z \in \Omega \mid \operatorname{dist}(z, \partial\Omega \setminus I) > \varepsilon\}$$

and

$$I_\varepsilon = \{z \in I \mid \operatorname{dist}(z, \partial\Omega \setminus I) > \varepsilon\}.$$

If $I = \partial\Omega$, we set $\Omega_\varepsilon = \Omega$ and $I_\varepsilon = \partial\Omega$.

Definition 4. Let Ω be a bounded domain in $\mathbb{C}$ and let I be a $\overline{\Omega}$-border. We say that Ω is *I-Nevanlinna* (denoted $\Omega \in \mathcal{N}(I)$) if for each sufficiently small $\varepsilon > 0$, there exist two bounded holomorphic functions u and v in Ω_ε (with $v \not\equiv 0$) such that a.e. on I_ε one has

$$\overline{\zeta} = \frac{u(\zeta)}{v(\zeta)}.$$

It follows from the Luzin-Privalov theorem that u/v is uniquely defined in the sense that if u_1, v_1 and u_2, v_2 satisfy the condition of the definition, then u_1/v_1 and u_2/v_2 coincide in Ω_ε, for sufficiently small ε. Letting ε go to zero, we have that u/v is uniquely defined in Ω. It is not obvious whether u/v can be written as the quotient of bounded holomorphic functions in Ω (see [8]).

Note also that if Ω is a Nevanlinna domain such that $\partial\Omega$ is a border of $\overline{\Omega}$, then (Ω is a Jordan domain and) $\Omega \in \mathcal{N}(\partial\Omega)$.

Definition 5. Let Ω be a bounded domain in $\mathbb{C}$ and let γ_1 and γ_2 be two $\overline{\Omega}$-border Jordan arcs on the boundary of Ω. Suppose that for $\ell = 1, 2$, there exist open neighbourhoods U_ℓ of γ_ℓ in $\mathbb{C}$ such that $\Omega_\ell = \Omega \cap U_\ell \in \mathcal{N}(\gamma_\ell)$. Let u_ℓ/v_ℓ be the meromorphic function appearing in the definition of γ_ℓ-Nevanlinna ($\ell = 1, 2$).

γ_1 and γ_2 are said to be *Nevanlinna Ω-independent* if there exists no path in Ω along which the two meromorphic elements $(u_\ell/v_\ell, \Omega_\ell)$, $\ell = 1, 2$, are meromorphic continuations of each other.

Remark 1. It follows from the Luzin-Privalov theorem that if γ_1 and γ_2 are Nevanlinna Ω-independent, then $\Omega \notin \mathcal{N}(I)$ for each $\overline{\Omega}$-border set I containing $\gamma_1 \cup \gamma_2$.

Notice also that if γ_1 and γ_2 are analytically independent $\overline{\Omega}$-border arcs on $\partial\Omega$ (see [4, §3]), then γ_1 and γ_2 are Nevanlinna Ω-independent. In particular, every polygon Ω has two Nevanlinna Ω-independent Jordan arcs on its boundary.

2. Main theorems

Our first two theorems give rather strong necessary conditions for the equalities $\mathcal{A}_n(F) = \mathcal{E}_n(F)$ and $\mathcal{A}_n(F) = \mathcal{M}_n(F)$ to hold, reducing the (necessity part of the) problem of approximating on closed sets by n-analytic entire functions and n-meromorphic functions to an approximation problem on compact sets. Though both Theorem 1 and Theorem 2 below are true for $n = 1$, much better results are known in this case. In fact, when $n = 1$, a complete characterization of the closed sets F for which either $\mathcal{A}_1(F) = \mathcal{E}_1(F)$ or $\mathcal{A}_1(F) = \mathcal{M}_1(F)$ holds has been obtained by N.U. Arakelian and A.H. Nersessian respectively. See the discussion following Problem 1.

Theorem 1. *Let F be a closed subset of $\mathbb{C}$ and let $n \in \mathbb{N}$ be fixed. If $\mathcal{A}_n(F) = \mathcal{E}_n(F)$, then for each Carathéodory domain Ω, one has*

$$\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega}) \tag{3}$$

and

$$\mathcal{A}_n(\overline{F \cap \Omega}) = \mathcal{R}_n^{\overline{\Omega}}(\overline{F \cap \Omega}). \tag{3'}$$

In particular, if in addition $\mathbb{C} \setminus \overline{\Omega}$ is connected, then

$$\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{P}_n(F \cap \overline{\Omega}) \tag{4}$$

and

$$\mathcal{A}_n(\overline{F \cap \Omega}) = \mathcal{P}_n(\overline{F \cap \Omega}). \tag{4'}$$

Theorem 2. *Let F be a closed subset of $\mathbb{C}$ and let $n \in \mathbb{N}$ be fixed. If $\mathcal{A}_n(F) = \mathcal{M}_n(F)$, then for each Carathéodory domain Ω, one has*

$$\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{R}_n^{F \cap \overline{\Omega}}(F \cap \overline{\Omega}) \tag{5}$$

Problem 1. Is it true that, when $n \geq 2$, these necessary conditions for approximation are also sufficient?

When $n = 1$, it follows immediately from Arakelian's theorem (see [5, Chapter IV, §2]) that these conditions are not sufficient to have $\mathcal{A}_1(F) = \mathcal{E}_1(F)$. They are not sufficient either for the analogous result to hold when approximating by entire harmonic functions (see [7, Chapter 3, §10]). But condition (5) is both necessary and sufficient for $\mathcal{A}_1(F) = \mathcal{M}_1(F)$ to hold (see [5, Chapter IV, §1D]; note that the statement there must be corrected to say "for each *Carathéodory* domain g", otherwise it is incorrect).

In Section 4, we shall give examples to illustrate our theorems and to show that, in some circumstances, approximation by entire polyanalytic ($n \geq 2$) functions seems easier than by entire analytic ($n = 1$) or entire harmonic functions. For instance, it is not necessary for the set $\overline{\mathbb{C}} \setminus F$ to be locally connected at ∞ (compare with Arakelian's theorem [5, Chapter IV, §2C] and Gardiner's theorem [7, Corollary 3.21]).

Our next theorem is closely related to the question asked in Problem 1. It generalizes [4, Theorems 2.2].

Theorem 3. *Let $n \in \mathbb{N}$ be fixed. Let X be a compact subset of $\mathbb{C}$, and $\{\Omega_\ell\}_{\ell \in L}$ be the (connected) components of $(\widehat{X})^0$ that are not contained in X. Then*

$$\mathcal{A}_n(X) = \mathcal{P}_n(X). \tag{6}$$

if and only if

$$\mathcal{A}_n(X \cap \overline{\Omega}_\ell) = \mathcal{R}_n^{\overline{\Omega}_\ell}(X \cap \overline{\Omega}_\ell) \tag{7}$$

for each $\ell \in L$.

Remark 2. When $n = 1$, it follows easily from Mergelyan's theorem (see [5, Chapter III, §2]) that (6) $\Longleftrightarrow$ $(X = \widehat{X})$ $\Longleftrightarrow$ (7), and thus that Theorem 3 holds in this case.

Notice also that when X is a *Carathéodory compact set* (i.e. when $\partial X = \partial \widehat{X}$), $\{\Omega_\ell\}_{\ell \in L}$ are precisely the bounded components of $\mathbb{C} \setminus X$, and thus Theorem 3 can be used, with the help of Theorem 1, to deduce Theorem 2.2 in [4].

In order to apply Theorem 3, it is important to be able to answer the following question:

Problem 2. Let Ω be a Carathéodory domain and X be a compact subset of $\overline{\Omega}$ such that $\mathcal{A}_n(X) = \mathcal{R}_n^{\overline{\Omega}}(X)$. When is it true then that $\mathcal{A}_n(X \cup \partial\Omega) = \mathcal{R}_n^{\overline{\Omega}}(X \cup \partial\Omega)$?

In [4], Problem 2 was studied when Ω is a Jordan domain and X is strictly contained in Ω (see [4, Theorem 4.3]). We have obtained the following stronger result.

Theorem 4. *Let $n \in \mathbb{N}$, $n \geq 2$ be fixed, let Ω be a Carathéodory domain and let X be a compact set in $\overline{\Omega}$ such that*

$$\mathcal{A}_n(X) = \mathcal{R}_n^{\overline{\Omega}}(X) \tag{8}$$

and such that the set $E := \partial\Omega \setminus X$ is a $\overline{\Omega}$-border. Let $\{D_\ell\}_{\ell \in L}$ be all the components of $\Omega \setminus X$ for which $E_\ell := \partial D_\ell \setminus X \neq \emptyset$. If $D_\ell \notin \mathcal{N}(E_\ell)$ for each $\ell \in L$, then

$$\mathcal{A}_n(X \cup \partial\Omega) = \mathcal{R}_n^{\overline{\Omega}}(X \cup \partial\Omega). \tag{9}$$

Corollary 1. *Let Ω be a Carathéodory domain with connected complement of it's closure, and such that $\partial\Omega$ contains two Nevanlinna Ω-independent arcs γ_1 and γ_2. Let X be a compact subset of Ω with $\mathcal{A}_n(X) = \mathcal{P}_n(X)$. Then $\mathcal{A}_n(K) = \mathcal{P}_n(K)$, where $K = X \cup \partial\Omega$.*

In [4, Theorem 4.3, Proposition 4.4] the previous result was obtained for Jordan domains Ω and analytically independent γ_1 and γ_2. Corollary 1, and a fortiori Theorem 4, fails when $n = 1$.

3. Proofs

3.1. **Proof of Theorem 1.** We will only establish (3), the proof of (3′) being essentially the same. Moreover (4) and (4′) follow immediately from (3) and (3′) respectively, from [3, Theorem 2] and Runge's Theorem.

Assume that $n \in \mathbb{N}$ is fixed, that $\mathcal{A}_n(F) = \mathcal{E}_n(F)$ and that $f \in \mathcal{A}_n(F \cap \overline{\Omega})$. Using Tietze's extension theorem, we first extend f to a function, still denoted by f, which is continuous on $\mathbb{C}$ and has compact support. We now apply the approximation scheme of A.G. Vitushkin [13], appropriately adapted to our situation. We will only outline the method. The details can be found in [4, Proof of Proposition 2.5].

Put $\mathcal{L} = \overline{\partial}^n$ and let $\Phi(z) = \bar{z}^{n-1}((n-1)!\,\pi z)^{-1}$ be the fundamental solution for $\mathcal{L}$, let $\partial := \partial/\partial z$ and let $B(a, r)$ denote the open disk with centre a and radius $r > 0$.

Now fix an arbitrary $\delta \in (0, 1)$ and construct a δ-partition of unity as follows. For each $j = (j_1, j_2) \in \mathbb{Z}^2$, set $a_j = j_1\delta + ij_2\delta$, $B_j = B(a_j, \delta)$ and choose $\varphi_j \in C_0^\infty(B(a_j, \delta))$ such that $0 \le \varphi_j \le 1$, $\sum_{j\in\mathbb{Z}^2} \varphi_j \equiv 1$ and $\|\overline{\partial}^r \partial^s \varphi_j\| \le A\delta^{-(r+s)}$, $1 \le r + s \le n$, where A depends only on n.

Let $f_j = \Phi * (\varphi_j \mathcal{L} f)$. Set $J = \{j \in \mathbb{Z}^2 \mid B(a_j, \delta) \cap \operatorname{supp} f \neq \emptyset\}$, and note that J is a finite set. We have $\mathcal{L} f_j = \varphi_j \mathcal{L} f$ in the distributional sense, and $f = \sum_{j\in J} f_j$. Let $J_1 = \{j \in J \mid B_j \subset \Omega\}$, $J_2 = \{j \in J \mid B_j \cap \overline{\Omega} = \emptyset\}$ and $J_3 = \{j \in J \mid B_j \cap \partial\Omega \neq \emptyset\}$. Note that $J = J_1 \cup J_2 \cup J_3$.

If $j \in J_1$, then $f_j \in \mathcal{A}_n(F) = \mathcal{E}_n(F) \subset \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$. If $j \in J_2$, then $f_j \in \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$ by Runge's Theorem.

It remains to consider the case $j \in J_3$. For each $j \in J_3$, one can find $a_j^* \in B(a_j, \delta) \setminus \overline{\Omega}$ and a Jordan curve γ_j in $\overline{B(a_j^*, \delta)} \setminus \overline{\Omega}$ with starting point a_j^* and with ending point in $\partial B(a_j^*, \delta)$. It follows (see [4, Proof of Proposition 2.5]) that we can construct functions g_j, n-analytic outside γ_j, and thus in $\mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$, such that

$$\| \sum_{j\in J_3} (f_j - g_j) \|_{F\cap\overline{\Omega}} \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

The desired approximant is $g = \sum_{j\in J_1\cup J_2} f_j + \sum_{j\in J_3} g_j$.

3.2. **Proof of Theorem 2.** Assuming that $\mathcal{A}_n(F) = \mathcal{M}_n(F)$ and that $f \in \mathcal{A}_n(F \cap \overline{\Omega})$, the proof of Theorem 2 is mutatis mutandis the proof of Theorem 1. The details are left to the reader.

3.3. **Proof of Theorem 3.** Notice that all Ω_ℓ, $\ell \in L$, are Carathéodory domains and so, by Theorem 1, (6) implies (7).

To show the sufficiency of (7), let μ be a finite complex-valued Borel measure supported on X such that $\mu \perp \mathcal{P}_n(X)$. We must prove that $\mu \perp \mathcal{A}_n(X)$ holds whenever (7) holds.

We start by noticing that if the index set L is empty, that is if X is simply connected, then (7) is an empty condition and (6) follows from [3, Theorem 2] and Runge's theorem. So we assume that $L \neq \emptyset$, and we fix $\ell \in L$. By [4, Lemma 2.4], one can find a sequence of complex analytic polynomials $\{q_j\}_{j=1}^\infty$ such that $\|q_j\|_{\widehat{X}} \leq C$, where C is an absolute constant, and, as $j \to \infty$, such that $q_j \to 1$ uniformly on compact subsets of Ω_ℓ, $q_j \to 0$ on compact subsets of $(\widehat{X})^0 \setminus \Omega_\ell$, and, in addition, such that the sequence of measures $\{q_j\mu\}_{j=1}^\infty$ converges weak-$*$ to some measure μ_{Ω_ℓ}. It follows that $\operatorname{supp}\mu_{\Omega_\ell} \subset (X \cap \overline{\Omega}_\ell) \cup \partial\widehat{X}$ and that $\mu_{\Omega_\ell} \perp \mathcal{P}_n(X)$.

Recall that

$$\widehat{\nu}(\zeta) = \frac{1}{2\pi i}\int \frac{d\nu(z)}{z-\zeta}$$

defines the Cauchy transform of a (finite) measure ν. We claim that

$$\widehat{\overline{z}^s\mu}(z_0) = 0, \quad \forall z_0 \in \Omega_\ell \setminus X, \quad s = 0, \dots, n-1. \tag{10}$$

Indeed, since $\widehat{\overline{z}^s\mu}(z_0) = \widehat{\overline{z}^s\mu_{\Omega_\ell}}(z_0)$ whenever $z_0 \in \Omega_\ell \setminus X$ (see [4, proof of Theorem 2.2 (2)]), we can assume without loss of generality that

$$\operatorname{supp}\mu \subset (X \cap \overline{\Omega}_\ell) \cup \partial\widehat{X}. \tag{11}$$

But if (11) is satisfied for μ, we then must have $\operatorname{supp}\mu_{\Omega_\ell} \subset X \cap \overline{\Omega}_\ell$. In fact, since $q_j \to 0$ on $(\widehat{X})^0 \setminus \Omega_\ell$,

$$\widehat{\overline{z}^s\mu_{\Omega_\ell}}(w) = 0, \quad \forall w \in \left[(\widehat{X})^0 \setminus \Omega_\ell\right] \cup \left[\mathbb{C} \setminus \widehat{X}\right], \quad s = 0, \dots, n-1. \tag{12}$$

Since μ has no atoms on $\partial\widehat{X}$ (see [1, Lemma 9]), the function

$$f_{\Omega_\ell}(w) = \frac{1}{\pi}\int \frac{\overline{z}-\overline{w}}{z-w}\, d\mu_{\Omega_\ell}(z)$$

is continuous on $\mathbb{C} \setminus \overline{\Omega}_\ell$ and, by (12), it vanishes outside $\overline{\Omega}_\ell$. It thus follows that $\mu_{\Omega_\ell}|_{\mathbb{C}\setminus\overline{\Omega_\ell}} = 0$ since $\overline{\partial}^2 f_{\Omega_\ell} = \mu_{\Omega_\ell}$ in the sense of distributions. Therefore $\mu_{\Omega_\ell} \perp \mathcal{R}_n^{\overline{\Omega}_\ell}(X \cap \overline{\Omega}_\ell) = \mathcal{A}_n(X \cap \overline{\Omega}_\ell)$ and (10) holds.

Now since ℓ was arbitrary in (10), we have that

$$\widehat{\overline{z}^s\mu}(z_0) = 0, \quad \forall z_0 \in \mathbb{C} \setminus X, \quad s = 0, \dots, n-1,$$

and thus $\mu \perp \mathcal{R}_n^X(X)$. We will complete the proof by showing that $\mathcal{R}_n^X(X) = \mathcal{A}_n(X)$. For this, we again apply Vitushkin's technique, using the notation from the proof of Theorem 1 above.

Take any $f \in \mathcal{A}_n(X)$ and extend it to a function (still denoted by f) continuous on all of $\mathbb{C}$ and with compact support. Let $\{f_j\}$ be the localized functions corresponding to the covering $\{B_j\}$. If $B_j \cap \widehat{X} = \emptyset$ or $B_j \cap \partial\widehat{X} \neq \emptyset$, we deal with the corresponding f_j as in the proof of Theorem 1. It remains to consider the case when $B_j \subset (\widehat{X})^0$. In this case, B_j belongs to some component Ω of $(\widehat{X})^0$. If $\Omega \subset X^0$, then $f_j \equiv 0$ and there is nothing to be done. If $\Omega \not\subset X^0$, then $\Omega = \Omega_\ell$ for some $\ell \in L$. Since $f \in \mathcal{A}_n(X \cap \overline{\Omega}_\ell) = \mathcal{R}_n{}^{\overline{\Omega}_\ell}(X \cap \overline{\Omega}_\ell)$, one can find $\{g_m\}_{m=1}^{\infty}$, g_m being n-analytic in some neighbourhood U_m of $X \cap \overline{\Omega}_\ell$ such that, after an appropriate extension of g_m (still denoted g_m) to a continuous function on all of $\overline{\Omega}_\ell$, one has $\|f - g_m\|_{\overline{\Omega}_\ell} \to 0$ as $m \to \infty$. Using well-known properties of the Vitushkin operator, we obtain that

$$\|\Phi * (\phi_j \mathcal{L} g_m) - f_j\|_X = \|\Phi * (\phi_j(\mathcal{L} g_m - \mathcal{L} f))\|_X \leq C\|g_m - f\|_{B_j} \to 0$$

as $m \to \infty$. Clearly $\Phi * (\phi_j \mathcal{L} g_m) \in \mathcal{R}_n^X(X)$. This completes the proof.

3.4. **Proof of Theorem 4.** We will proceed by contradiction. Let $K = X \cup \partial\Omega$ and suppose that

$$\mathcal{A}_n(K) \neq \mathcal{R}_n^{\overline{\Omega}}(K). \tag{13}$$

We will show that then there must exist an $\ell \in L$ such that $D_\ell \in \mathcal{N}(E_\ell)$.

Assuming (13), it follows that there exists a measure σ supported on K with $\sigma \perp \mathcal{R}_n^{\overline{\Omega}}(K)$ but $\sigma \not\perp \mathcal{A}_n(K)$. We notice that $\sigma|_E$ is a nonzero measure, since otherwise, we would have $\operatorname{supp}\sigma \subset X$, contradicting (8). Since $E = \cup_{\ell \in L} E_\ell$ and L is at most countable, we can choose an ℓ such that $\sigma|_{E_\ell}$ is also a nonzero measure. We shall prove that $D_\ell \in \mathcal{N}(E_\ell)$. Let $X_\ell = \overline{\Omega} \setminus (D_\ell \cup E_\ell)$, so that $\partial X_\ell \subset \partial\Omega \cup \partial D_\ell$. Fix $\varepsilon > 0$ so small that $E_\ell^\varepsilon := \{z \in E_\ell \mid \operatorname{dist}(z, X_\ell) \geq \varepsilon\} \neq \emptyset$ and $\sigma|_{E_\ell^\varepsilon} \neq 0$. By definition, E_ℓ is a $\overline{D}_\ell$-border, and so there exists an at most countable family of disjoint (open) Jordan arcs $\{\gamma_j\}_{j \in J_\ell}$ such that $E_\ell = \cup_{j \in J_\ell} \gamma_j$. Moreover, since E_ℓ^ε is compact in ∂D_ℓ and each γ_j is open in ∂D_ℓ, there exists a finite subset $J \subset J_\ell$ such that $E_\ell^\varepsilon \subset \cup_{j \in J} \gamma_j$.

For each $j \in J$, we can choose points a_j and b_j on γ_j such that γ_j has tangents at a_j and b_j, the path $\widetilde{\gamma}_j$ which is the part of γ_j strictly between a_j and b_j has positive orientation with respect to Ω when "starting" at a_j and "ending" at b_j, and moreover such that $E_\ell^\varepsilon \subset \cup_{j \in J} \widetilde{\gamma}_j =: \Gamma_1$ and $\sigma|_{\Gamma_1} \neq 0$. Notice that

$$\operatorname{dist}(z, X_\ell) < \varepsilon \quad \text{if} \quad z \in E_\ell \setminus \Gamma_1.$$

Let Ω_∞ be the unbounded (connected) component of $\overline{\mathbb{C}} \setminus \overline{\Omega}$. Then Ω_∞ is simply connected since $\partial\Omega_\infty = \partial\Omega$ is connected, and each $\widetilde{\gamma}_j$ is a $\overline{\Omega}_\infty$-border (here we extend Definition 3 to include domains in $\overline{\mathbb{C}}$ which contain ∞). Choose an enumeration of J such that the $\widetilde{\gamma}_j$'s are in consecutive order on $\partial\Omega_\infty$. We may as well assume that this enumeration is $\{1, 2, \dots, N\}$, where $N = |J|$. Notice that the $\widetilde{\gamma}_j$'s have negative orientation with respect to Ω_∞. Therefore, for each $j \in J$, we can find a Jordan rectifiable arc β_j in Ω_∞ which starts at b_j, ends at a_{j+1}, is normal to γ_j at b_j and is normal to γ_{j+1} at a_{j+1} (where $a_{N+1} := a_1$ and $\gamma_{N+1} = \gamma_1$). In addition we can (and will) require that the curve $\Gamma = \cup_{j\in J}(\widetilde{\gamma}_j \cup \beta_j)$ is a Jordan rectifiable curve which surrounds and meets nontangentially the compact set

$$Y = X_\ell \cup (E_\ell \setminus (\cup_{j\in J}\, \widetilde{\gamma}_j))$$

and has positive orientation with respect to the domain G surrounded by Γ. We notice that $Y \subset G \cup (\cup_{j\in J}\{a_j, b_j\})$, $\operatorname{supp}\sigma \subset Y \cup \Gamma_1$, $\Gamma_1 \subset \Gamma$, and recall that $\sigma|_{\Gamma_1} \neq 0$. Moreover, since Γ meets Y nontangentially, for each $z \in Y$,

$$\operatorname{dist}(z, \Gamma) \asymp \operatorname{dist}(z, \cup_{j\in J}\{a_j, b_j\}). \tag{14}$$

Let $\sigma_0 = (\prod_{j\in J}(z - a_j)(z - b_j))\sigma$. Then $\sigma_0 \perp \mathcal{P}_n(Y \cup \Gamma)$, since this is also true for σ. Recall that $\widehat{\mu}(\zeta) = (2\pi i)^{-1} \int (z - \zeta)^{-1} d\mu(z)$ defines the Cauchy transform of a measure μ. We need the following Lemma.

Lemma 1. *Let μ be a measure such that $\left(\prod_{j\in J}|z - a_j||z - b_j|\right)^{-1}\mu$ is a finite measure on Y. Then the measure $\widehat{\mu}(\zeta)d\zeta|_\Gamma$ is finite and*

$$(\mu + \widehat{\mu}(\zeta)d\zeta|_\Gamma) \perp \mathcal{P}_1(Y \cup \Gamma). \tag{15}$$

Proof of Lemma 1. Let us first prove that $\widehat{\mu}(\zeta)d\zeta|_\Gamma$ is a finite measure:

$$\begin{aligned}\int_\Gamma |\widehat{\mu}(\zeta)||d\zeta| &= \frac{1}{2\pi}\int_\Gamma \left|\int_Y \frac{d\mu(z)}{z - \zeta}\right| |d\zeta| \\ &\leq \int_Y \left(\int_\Gamma \frac{|d\zeta|}{|z - \zeta|}\right) |d\mu(z)| \\ &\leq \text{Const} \int_Y \frac{\text{length}(\Gamma)}{\prod_{j\in J}|z - a_j||z - b_j|} |d\mu(z)| \\ &< \infty,\end{aligned}$$

where the penultimate inequality follows from (14).

This last estimate allows us to use Fubini's theorem in order to prove (15).

Indeed, if p is a complex polynomial, then

$$\begin{aligned}
&\int_Y p(z)d\mu(z) + \int_\Gamma p(\zeta)\widehat{\mu}(z)d\zeta \\
&\quad = \int_Y p(z)d\mu(z) + \int_\Gamma p(\zeta)\left(\frac{1}{2\pi i}\int_Y \frac{d\mu(z)}{z-\zeta}\right)d\zeta \\
&\quad = \int_Y p(z)d\mu(z) + \int_Y \left(\frac{1}{2\pi i}\int_\Gamma \frac{p(\zeta)}{z-\zeta}d\zeta\right)d\mu(z) \\
&\quad = 0
\end{aligned}$$

by Cauchy's formula. □

Let us apply Lemma 1 to the measures $\mu_s := \overline{z}^s\sigma_0|_Y$, $s = 0, \dots, n-1$. We then have

$$(\overline{z}^s\sigma_0 - (\mu_s + \widehat{\mu}_s(\zeta)d\zeta|_\Gamma)) \perp \mathcal{P}_1(\Gamma),$$

so that, by an analog of the F. and M. Riesz theorem on rectifiable Jordan domains due to I.I. Privalov [11, Chapter 3, §7], for each $s \in \{0, \dots, n-1\}$ there exists a function h_s in the the Smirnov class $E_1(G)$ such that one has

$$\overline{z}^s\sigma_0 = \mu_s + \widehat{\mu}_s(\zeta)d\zeta|_\Gamma + h_s(\zeta)d\zeta|_\Gamma\,. \tag{16}$$

First setting $s = 0$ in (16), then multiplying the resulting equation by $\overline{z}^s$, $s = 0, \dots, n-1$, and observing that $\mu_s := \overline{z}^s\mu_0$, we thus also have

$$\overline{z}^s\sigma_0 = \overline{z}^s\mu_0 + \overline{\zeta}^s\widehat{\mu}_0(\zeta)d\zeta\Big|_\Gamma + \overline{\zeta}^s h_0(\zeta)d\zeta\Big|_\Gamma\,.$$

Subtracting this last equality from (16), we get, for $s = 0, \dots, n-1$,

$$\overline{\zeta}^s(\widehat{\mu}_0(\zeta) + h_0(\zeta))\,d\zeta|_\Gamma = (\widehat{\mu}_s(\zeta) + h_s(\zeta))\,d\zeta|_\Gamma\,.$$

Since $\sigma_0|_{\Gamma_1} = (\widehat{\mu}_0(\zeta) + h_0(\zeta))\,d\zeta|_{\Gamma_1} \neq 0$, by the Luzin-Privalov theorem, we have that $\widehat{\mu}_0(z)+h_0(z) \not\equiv 0$ in D_ℓ, and so the nontangential limit $\widehat{\mu}_0(\zeta)+h_0(\zeta)$ is nonzero at almost all points $\zeta \in \Gamma_1$, which gives that

$$\overline{\zeta}^s = \frac{\widehat{\mu}_s(\zeta) + h_s(\zeta)}{\widehat{\mu}_0(\zeta) + h_0(\zeta)}$$

(nontangentially) a.e. on Γ_1. Since each $\widehat{\mu}_s$ is bounded outside an ε-neighbourhood of Y and each h_s is a quotient of bounded holomorphic functions in G, a simple calculation gives (taking just $s = 1$) that $\overline{\zeta} = u(\zeta)/v(\zeta)$ nontangentially $d\zeta$ - a.e. on Γ_1 for some holomorphic functions u and v bounded in $\{z \in D_\ell \mid \operatorname{dist}(z, K) > \varepsilon\}$, which precisely means that $D_\ell \in \mathcal{N}(E_\ell)$. Theorem 4 is proved.

3.5. **Proof of Corollary 1.** It follows from the hypotheses that $\gamma_1 \cap \gamma_2 = \emptyset$. Let $X_1 = X \cup (\partial\Omega \setminus (\gamma_1 \cup \gamma_2))$. Corollary 1 will follow immediately from Theorem 4 applied to this X_1. It thus only suffices to verify that the hypotheses of Theorem 4 are satisfied in this case.

Since $\mathcal{A}_n(X) = \mathcal{P}_n(X)$, $\widehat{X}$ is contained in Ω and $\partial\Omega \setminus (\gamma_1 \cup \gamma_2)$ has connected complement, it follows from Runge's theorem that

$$\mathcal{A}_n(X_1) = \mathcal{P}_n(X_1).$$

Moreover, in this case, $E := \partial\Omega \setminus X_1 = \gamma_1 \cup \gamma_2$, and the collection $\{D_\ell\}_{\ell \in L}$ consists of a single domain $D_1 = \Omega \setminus \widehat{X}$. But by Remark 1, $D_1 \notin \mathcal{N}(\gamma_1 \cup \gamma_2)$. Thus Theorem 4 applies and Corollary 1 is proved.

4. Examples

Example 1. We first give an example that shows that Theorem 1 would fail if we would replace either $\mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$ or $\mathcal{R}_n^{\overline{\Omega}}(\overline{F \cap \Omega})$ by $\mathcal{P}_n(F \cap \overline{\Omega})$ or $\mathcal{P}_n(\overline{F \cap \Omega})$ in (3) or (3′) respectively.

Let U_1 be an open bounded ribbon which winds around the outside of the circle $\{|z| = 1\}$ and accumulates on that circle. One says that U_1 is a "cornucopia" (see [6, Chapter VI, §5, Figure 4] or [4, §3]). Note in passing that U_1 is a Carathéodory domain with the property that $\mathbb{C} \setminus \overline{U}_1$ is <u>not</u> connected.

Let $F = \{|z| \leq 1\} \cup \overline{U_1}$ and $\Omega = U_1$. We remark that F is a Carathéodory compact set, i.e. that $\partial F = \partial \widehat{F}$, and that $F \cap \overline{\Omega} = \overline{F \cap \Omega} = \overline{\Omega}$. For all $n \in \mathbb{N}$, we have $\mathcal{A}_n(F) = \mathcal{P}_n(F)$ by [3, Theorem 2], and $\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$ by [4, Proposition 2.5], but $\mathcal{A}_n(F \cap \overline{\Omega}) \neq \mathcal{P}_n(F \cap \overline{\Omega})$ by [4, Theorem 2.2(2)].

Example 2. The following is an application of Theorem 3 which cannot be obtained by using [4, Theorem 2.2(2)] instead.

We again start with a cornucopia. This time the open ribbon U_2 winds around the outside of the ellipse $\{x^2 + 2y^2 = 1\}$, and accumulates on that ellipse. Let X be the union of $\overline{U}_2$ with a segment $[-a, a]$, where $a > 0$ and the segment does not contain the foci of the ellipse. Let $\Omega = \{x^2 + 2y^2 < 1\}$ be the interior of the ellipse. Notice that $\Omega \notin \mathcal{N}$ (see [4, §3]), that Ω is the only component of $(\widehat{X})^0$ which is not contained in X, that X is not a Carathéodory compact set (so [4, Theorem 2.2(2)] does not apply in this situation), and that $X \cap \overline{\Omega} = \partial\Omega \cup [-a, a]$. Fix $n \in \mathbb{N}$, $n \geq 2$. From [4, Theorem 4.3(2)], it follows that

$$\mathcal{A}_n(X \cap \overline{\Omega}) = \mathcal{P}_n(X \cap \overline{\Omega}) = \mathcal{R}_n^{\overline{\Omega}}(X \cap \overline{\Omega})$$

and consequently, from Theorem 3, that $\mathcal{A}_n(X) = \mathcal{P}_n(X)$.

Example 3. We now give an example of a nowhere dense closed set F in $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus F$ is connected, but *not* locally connected at ∞, and still F is a set of Carleman approximation (see [5, Chapter IV, §3]) by n-analytic functions when $n \geq 2$. In particular, $\mathcal{A}_n(F) = C(F) = \mathcal{E}_n(F)$. This is in sharp contrast with (the case $n = 1$) Arakelian's theorem [5, Chapter IV, §2] and with harmonic Carleman approximation [7, Theorem 4.6].

For $k = 2, 3, 4, \dots$, let

$$A_k = \left\{x = \tfrac{1}{k},\ 0 \leq y \leq k\right\}$$
$$B_k = \left\{x = \tfrac{1}{2}\left(\tfrac{1}{k} + \tfrac{1}{k-1}\right),\ 0 \leq y \leq k\right\}$$
$$C_k = \left\{\tfrac{1}{k} \leq x \leq \tfrac{1}{2}\left(\tfrac{1}{k} + \tfrac{1}{k-1}\right),\ y = k\right\}$$

and set

$$F = \left(\bigcup_{k=2}^{\infty}(A_k \cup B_k \cup C_k)\right) \cup \left\{x = 0,\ 0 \leq y < \infty\right\}.$$

The set F is often refer to as an Arakelian glove. Clearly, $\overline{\mathbb{C}} \setminus F$ is connected, but not locally connected at ∞. From [2, Proposition 5], to prove that F is a n-analytic Carleman set, it suffices to show the existence of a "compatible exhaustion" $\{K_j\}_{j=1}^{\infty}$. For this purpose, set $K_1 = \emptyset$,

$$K_j = \{|x| \leq j,\ |y| \leq j\}, \quad \text{when} \quad j \geq 2,$$

and let $X_j = K_j \cup (K_{j+2} \cap F)$. It remains to check (see [2, equation (10)]) that

$$\mathcal{A}_n(X_j) = \mathcal{P}_n(X_j). \tag{17}$$

When $j = 1$, this follows from [3, Theorem 2] and Runge's theorem. Now fix $j \in \mathbb{N}$, $j \geq 2$, and let Ω be the Jordan domain whose closure is

$$\begin{aligned}\overline{\Omega} = X_j \setminus \Bigg[\Bigg(\bigcup_{k=j+3}^{\infty}\Big(&\left\{x = \tfrac{1}{k},\ k < y < \infty\right\} \\ &\cup \left\{x = \tfrac{1}{2}(\tfrac{1}{k} + \tfrac{1}{k-1}),\ k < y < \infty\right\}\Big)\Bigg) \\ &\cup \left\{x = 0,\ y > k\right\}\Bigg].\end{aligned}$$

We now apply Theorem 4 with this Ω and $X = K_j$. Since, in this case, $\Omega \setminus X$ consists of two rectangles, which we denote $D_{j,1}$ and $D_{j,2}$, and since $D_{j,i} \notin \mathcal{N}(\partial D_{j,i} \setminus X)$ for $i = 1, 2$, one has $\mathcal{A}_n(X \cup \partial\Omega) = \mathcal{R}_n^{\overline{\Omega}}(X \cup \partial\Omega) = \mathcal{P}_n(X \cup \partial\Omega)$. Now we apply Theorem 3 to obtain (17), and this completes the example.

References

[1] E. Bishop, *Boundary measures of analytic differentials*, Duke Math. J. **27** (1960), 331–340.

[2] A. Boivin, P.M. Gauthier and P. Paramonov, *Approximation on closed sets by analytic or meromorphic solutions of elliptic equations and applications*, Canad. J. Math. **54**(5) (2002), 945–969.

[3] J.J. Carmona, *Mergelyan approximation theorem for rational modules*, J. Approx. Theory **44** (1985), 113–126.

[4] J.J. Carmona, P.V. Paramonov and K.Yu. Fedorovski, *On uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions* (Russian), Mat. Sb. **193**:10 (2002), 75–98; English translation in Sb. Math. **193**:10 (2002), 1469–1492.

[5] D. Gaier, "Vorlesungen über Approximation in Komplexen", Birhäuser Verlag, 1980; English translation: "Lectures on Complex Approximation", Birkäuser, 1987.

[6] T.W. Gamelin, "Uniform Algebras", 2nd Ed., Chelsea, 1984.

[7] S.J. Gardiner, "Harmonic Approximation", LMS Lecture Notes Series vol. 221, Cambridge University Press, 1995.

[8] P.M. Gauthier and W. Hengartner, *Local harmonic majorants of functions subharmonic in the unit disc*, J. Analyse Math. **26** (1973), 405–412.

[9] P. Koosis, "Introduction to H_p Spaces", 2nd ed., Cambridge Tracts in Mathematics vol. 115, Cambridge University Press, 1998.

[10] P.V. Paramonov and K.Yu. Fedorovski, *Uniform and C^1-approximability of functions on compact subsets of $\mathbb{R}^2$ by solutions of second order elliptic equations*, Mat. Sb. **190**:2 (1999), 123–144 (Russian); English translation in: Sb. Math. **190**:2 (1999), 285–307.

[11] I.I. Privalov, "Boundary Properties of Analytic Functions" (Russian), 2nd ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; (German translation) "Randeigenschaften Analytischer Funktionen", Deutscher V. der Wiss., Berlin, 1956.

[12] H.S. Shapiro, "The Schwarz Function and its Generalization to Higher Dimensions", Wiley, 1992.

[13] A.G. Vitushkin, *Analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk. **22**:6 (1967), 141–199 (Russian); English translation in Russian Math. Surveys **22**:6 (1967), 139–200.

Department of Mathematics, University of Western Ontario
E-mail address: boivin@uwo.ca

Département de mathématiques et de statistique, Université de Montréal
E-mail address: gauthier@dms.umontreal.ca

Department of Function Theory and Functional Analysis, Mechanics and Mathematics Faculty, Moscow State University
E-mail address: petr@paramonov.msk.ru