

# Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds

STEFANO FRANCAVIGLIA<sup>12</sup>

## Abstract

Let  $W$  be a compact manifold and let  $\rho$  be a representation of its fundamental group into  $\mathrm{PSL}(2, \mathbb{C})$ . Then the volume of  $\rho$  is defined by taking any  $\rho$ -equivariant map from the universal cover  $\widetilde{W}$  to  $\mathbb{H}^3$  and then by integrating the pull-back of the hyperbolic volume form on a fundamental domain. It turns out that such a volume does not depend on the choice of the equivariant map. Dunfield extended this construction to the case of a non-compact (cusped) manifold  $M$ , but he did not prove the volume is well-defined in all cases.

We prove here that the volume of a representation is always well-defined and depends only on the representation. Moreover, we show that this volume can be easily computed by straightening any ideal triangulation of  $M$ .

We show that the volume of a representation is bounded from above by the relative simplicial volume of  $M$ . Finally, we prove a rigidity theorem for representations of the fundamental group of a hyperbolic manifold. Namely, we prove that if  $M$  is hyperbolic and  $\mathrm{vol}(\rho) = \mathrm{vol}(M)$  then  $\rho$  is discrete and faithful.

## 1 Introduction

Let  $W$  be a compact manifold and let  $\rho$  be a representation of its fundamental group into  $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$ . The volume of  $\rho$  is defined by taking any  $\rho$ -equivariant map from the universal cover  $\widetilde{W}$  to  $\mathbb{H}^3$  and then by integrating the pull-back of the hyperbolic volume form on a fundamental domain. This volume does not depend on the choice of the equivariant map because two equivariant maps are always equivariantly homotopic and the cohomology-class of the pull-back of the volume form is invariant under homotopy.

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<sup>1</sup>Author address: Stefano Francaviglia, Dipartimento di Matematica Applicata “U. Dini”, via Bonanno 25b, I-56126 Pisa, e-mail: s.francaviglia@sns.it.

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In [D] this definition is extended to the case of a non compact cusped 3-manifold  $M$  (see Definitions 4.1 and 2.5). When  $M$  is not compact, some problems of integrability arise if one tries to use the above definition of the volume of a representation. The idea of Dunfield for overcoming these difficulties is to use a particular (and natural) class of equivariant maps, called *pseudo-developing maps* (see Definition 2.5), that have a nice behavior on the cusps of  $M$  allowing to control their volume. Concerning the well-definition of the volume, working with non-compact manifolds, two pseudo-developing maps in general are not equivariantly homotopic and in [D] it is not proved that the volume of a representation does not depend on the chosen pseudo-developing map.

In this paper we show that the volume of a representation is well-defined even in the non-compact case, and we generalize to non-compact manifolds some results known in the compact case. We restrict to the orientable case. The paper is structured as follows.

In Sections 2 and 3 we introduce the notion of pseudo-developing map for a given representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  and the notion of straightening of such a map.

In Section 4 we prove that for each orientable cusped 3-manifold  $M$  and for each representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$ , the volume of  $\rho$  is well-defined and depends only on  $\rho$ . The main theorems are:

**Theorem 4.9** *Let  $D_\rho$  and  $F_\rho$  be two pseudo-developing maps for  $\rho$ . Then  $\text{vol}(D_\rho) = \text{vol}(F_\rho)$ .*

**Theorem 4.10** *For any pseudo-developing map  $D_\rho$  for  $\rho$  we have  $\text{vol}(D_\rho) = \text{Strvol}(D_\rho)$ .*

Roughly speaking, Theorem 4.10 says that the volume of  $\rho$  can be computed by straightening any ideal triangulation of  $M$  and then summing the volume of the straight version of the tetrahedra.

In Section 5, generalizing the techniques used for the proof of Theorem 4.10, we show that the volume of a representation  $\rho$  is bounded from above by the relative simplicial volume:

**Theorem 5.1** *For all representations  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  we have  $|\text{vol}(\rho)| < v_3 \cdot \|(\bar{M}, \partial\bar{M})\|$ , where  $v_3$  is the volume of a regular ideal tetrahedron in  $\mathbb{H}^3$ .*

In Section 6 we prove the following rigidity theorem for representations

of the fundamental group of a hyperbolic manifold:

**Theorem 6.1** *Let  $M$  be a non-compact, complete, orientable hyperbolic 3-manifold of finite volume. Let  $\Gamma \cong \pi_1(M)$  be the sub-group of  $\mathrm{PSL}(2, \mathbb{C})$  such that  $M = \mathbb{H}^3/\Gamma$ . Let  $\rho : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$  be a representation. If  $|\mathrm{vol}(\rho)| = \mathrm{vol}(M)$  then  $\rho$  is discrete and faithful. More precisely there exists  $\varphi \in \mathrm{PSL}(2, \mathbb{C})$  such that for any  $\gamma \in \Gamma$*

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}.$$

In Section 7 we give some corollaries. In particular we show how from Theorem 6.1 one can get a proof of Mostow's rigidity for non-compact manifolds (see [P] and [BCS] for a more general statement and a different proof):

**Theorem 7.1 (Mostow's rigidity for non-compact manifold)** *Let  $f : M \rightarrow N$  be a proper map between two orientable non-compact, complete hyperbolic 3-manifolds of finite volume. Suppose that  $\mathrm{vol}(M) = \deg(f)\mathrm{vol}(N)$ . Then  $f$  is properly homotopic to a locally isometric covering with the same degree as  $f$ .*

Other corollaries that can be useful for checking the hyperbolicity of a 3-manifold are also shown.

The author was informed by B. Klaff that results similar to those proved in this paper have also been established in [Kl].

I would like to thank Carlo Petronio and Joan Porti for the very many very interesting conversations about this theme.

## 2 General definitions

We fix here the class of manifolds we consider, namely the class of ideally triangulated cusped manifolds. Since we work with cusped manifolds, we want to fix a structure on the cusps.

**Definition 2.1 (Cusped manifold)** An orientable manifold  $M$  is called *cusped manifold* if it is diffeomorphic to the interior of a compact manifold with boundary  $\overline{M}$ . A *cusped* of  $M$  is a closed regular neighborhood of a component of  $\partial\overline{M}$ . In the following we require  $M$  to have dimension 3 and  $\partial\overline{M}$  to be a union of tori, so each cusp is homeomorphic to  $T^2 \times [0, \infty)$ .

We define  $\widehat{M}$  as the compactification of  $M$  obtained by adding one point for each cusp of  $M$ . Called  $\widetilde{M}$  the universal cover of  $M$ , we call  $\widetilde{\widehat{M}}$  the space obtained by adding to  $\widetilde{M}$  one point for each lift of each cusp of  $M$ .

We call the points added to  $M$  (or  $\widetilde{M}$ ) *ideal points* of  $M$  (or  $\widetilde{M}$ ). For each ideal point  $p$  of  $M$ , we fix a smooth product structure  $T_p \times [0, \infty)$  on the cusp relative to  $p$ . Such a structure induces a cone structure, obtained from  $T_p \times [0, \infty]$  by collapsing  $T_p \times \{\infty\}$  to  $p$ , on a neighborhood  $C_p$  of  $p$  in  $\widehat{M}$ .

We lift such structures to the universal cover. Let  $\tilde{p}$  be an ideal point of  $\widetilde{M}$  that projects to the ideal point  $p$  of  $M$ . We denote by  $N_{\tilde{p}}$  the cone at  $\tilde{p}$ . The cone  $N_{\tilde{p}}$  is homeomorphic to  $P_{\tilde{p}} \times [0, \infty]$  where  $P_{\tilde{p}}$  covers the torus  $T_p$  and  $P_{\tilde{p}} \times \{\infty\}$  is collapsed to  $\tilde{p}$ .

**Remark 2.2** In the definition of cusped manifold we have included a fixed product structure on the cusps. This is for technical reasons, however we will show that the results about the volume of representations do not depend on the chosen structure.

**Remark 2.3** Let  $\widetilde{M}$  be the universal cover of  $M$ . In the following, when we speak about  $\pi_1(M)$ , we tacitly assume that a base-point and one of its lifts have been fixed. If  $p$  is an ideal point of  $M$ , then  $\pi_1(T_p)$  is well-defined only up to conjugation. Called  $\{\tilde{p}_i\}$  the set of the lifts of  $p$ , there is a one-to-one correspondence between the stabilizers  $\text{Stab}(\tilde{p}_i)$  of  $\tilde{p}_i$  in the group of deck transformations of  $\widetilde{M} \rightarrow M$  and the conjugates of  $\pi_1(T_p)$  in  $\pi_1(M)$ . Such a correspondence is uniquely determined once the base-points have been fixed.

To avoid pathologies, since we are working with cusped manifolds, we need that the maps we use have a nice behavior “at infinity.” Namely, we will often require that a map from a cusp to  $\mathbb{H}^3$  is a cone-map in the following sense.

**Definition 2.4 (Cone-map)** Let  $A$  be a set,  $c \in \mathbb{R}$  and  $C$  be the cone obtained from  $A \times [c, \infty]$  by collapsing  $A \times \{\infty\}$  to a point, which we call  $\infty$ . A map  $f : C \rightarrow \overline{\mathbb{H}^n}$  is a *cone-map* if:

- $f(C) \cap \partial\mathbb{H}^n = \{f(\infty)\}$ ;
- $\forall a \in A$  the map  $f|_{a \times [c, \infty]}$  is either the constant to  $f(\infty)$  or the geodesic ray from  $f(a, c)$  to  $f(\infty)$ , parametrized in such a way that the parameter  $(t - c)$ ,  $t \in [c, \infty]$ , is the arc-length.

We recall here the definition of pseudo-developing map for a representation (see [D]).

**Definition 2.5 (Pseudo-developing map)** Let  $M$  be a cusped manifold and let  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  be a representation. A *pseudo-developing map* for  $\rho$  is a piecewise smooth map  $D_\rho : \widetilde{M} \rightarrow \mathbb{H}^3$  which is equivariant w.r.t. the actions of  $\pi_1(M)$  on  $\widetilde{M}$  via deck transformations and on  $\mathbb{H}^3$  via  $\rho$ . Moreover we require  $D_\rho$  to extend to a continuous map, which we still call  $D_\rho$ , from  $\widehat{\widetilde{M}}$  to  $\overline{\mathbb{H}^3}$  that maps the ideal points to  $\partial\mathbb{H}^3$  (see Remark 2.6 for comments on this property). Finally we require that there exists  $t_{D_\rho} \in \mathbb{R}^+$  such that for each cusp  $N_p = P_p \times [0, \infty]$  of  $\widetilde{M}$ , the restriction of  $D_\rho$  to  $P_p \times [t_{D_\rho}, \infty]$  is a cone-map.

Let  $\gamma \neq \text{id}$  be an isometry of  $\mathbb{H}^3$  and let  $\text{Fix}(\gamma)$  be the set of fixed points of  $\gamma$ . Then  $\text{Fix}(\gamma) \cap \partial\mathbb{H}^3$  consists of either one or two points. Moreover, if  $\gamma_1$  and  $\gamma_2$  commute, then  $\text{Fix}(\gamma_1)$  is  $\gamma_2$ -invariant. It follows that if  $\Gamma$  is an Abelian subgroup of orientation-preserving isometries and  $\gamma \in \Gamma$ , then  $\text{Fix}(\gamma)$  is  $\Gamma$ -invariant. Actually, for almost all Abelian  $\Gamma$  and for any  $\gamma_1, \gamma_2 \in \Gamma \setminus \{\text{id}\}$  we have

$$\text{Fix}(\gamma_1) = \text{Fix}(\gamma_2).$$

The only cases in which this is not true are when  $\Gamma$  is a dihedral group generated by two rotations of angle  $\pi$  around orthogonal axes. Such a group is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and its unique fixed point is the intersection of the axes. It follows that any Abelian group  $\Gamma$  of orientation-preserving isometries has a fixed point in  $\overline{\mathbb{H}^3}$  and, if  $\Gamma$  is not dihedral, then it has a fixed point in  $\partial\mathbb{H}^3$ .

Let now  $p$  be an ideal point of  $\widehat{\widetilde{M}}$ . Since  $\text{Stab}(p)$  is Abelian, then either it is dihedral or it has a fixed point in  $\partial\mathbb{H}^3$ . If  $\rho$  is a representation of  $\pi_1(M)$  and  $D_\rho$  is a pseudo-developing map for  $\rho$ , then  $D_\rho(p)$  is a fixed point of  $\rho(\text{Stab}(p))$ . It follows that, using Definition 2.5, in order for a pseudo-developing map to exist,  $\rho(\text{Stab}(p))$  must have a fixed point in  $\partial\mathbb{H}^3$ .

**Remark 2.6** We included in Definition 2.5 the requirement that  $D_\rho$  maps ideal points to  $\partial\mathbb{H}^3$  only for simplicity. No pathologies do occur if some ideal point is mapped to the interior of  $\mathbb{H}^3$ . Coherently with this fact, from now on we suppose that:

For each boundary torus  $T$ , the group  $\rho(\pi_1(T))$  is not dihedral.

As above we notice that this is only for simplicity and one can easily check that all the results of this paper remain true, *mutatis mutandis*, without this assumption.

**Lemma 2.7** *Let  $M$  be a cusped manifold and let  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  be a representation. Then a pseudo-developing map  $D_\rho$  exists.*

*Proof.* The proof is the same as in [D], we recall it by completeness. We construct a pseudo-developing map inductively on the  $n$ -skeleta. Let  $p$  be an ideal point of  $\widehat{M}$ . Since  $\text{Stab}(p)$  is Abelian and not dihedral, then its  $\rho$ -image has at least one fixed point  $q \in \partial\mathbb{H}^3$ . We define  $D_\rho(p) = q$  and, for all  $\alpha \in \pi_1(M)$  we set  $D_\rho(\alpha(p)) = \rho(\alpha)(q)$ . We do the same for the other ideal points. Now for each ideal point  $p$  we define  $D_\rho$  on  $P_p \times \{0\}$  in any  $\text{Stab}(p)$ -equivariant way and then we make the cone over  $D_\rho(p)$  in such a way that  $D_\rho$  has the cone property. Then we extend  $D_\rho$  in any equivariant way. The extension is possible because  $\mathbb{H}^3$  is contractible.  $\square$

**Remark 2.8** Let  $p$  be an ideal point of  $\widehat{M}$ . If  $\rho(\text{Stab}(p))$  is a parabolic non-trivial group, then it has a unique fixed point. It follows that  $D_\rho(p)$  is uniquely determined. Thus, if all the  $\rho$ -images of the stabilizers of the ideal points are parabolic, then the  $D_\rho$ -images of all the ideal points are uniquely determined.

**Definition 2.9 (Ideally triangulated manifold)** Let  $M$  be a cusped manifold. An ideal triangulation of  $M$  is a triangulation of  $\widehat{M}$  having the set of ideal points as 0-skeleton. An *ideally triangulated manifold* is a cusped manifold equipped with a finite smooth ideal triangulation  $\tau$ . We require the triangulation to be compatible with the product structure. That is, for each cusp  $N_p$  we require  $\tau \cap (T_p \times \{0\})$  to be a triangulation of  $T_p$  and the restriction to  $N_p$  of  $\tau$  to be the product triangulation.

We will often consider the simplices of an ideal triangulation of a manifold  $M$  as subsets of  $\widehat{M}$ .

**Remark 2.10** It is well-known that any cusped manifold can be ideally triangulated (see for example [BP]).

### 3 The straightening

A straightening of a pseudo-developing map  $D$  is a map that agrees with  $D$  on the ideal points and that maps each tetrahedron to a straight one. The straightening is useful to calculate the hyperbolic volume associated to a pseudo-developing map (see Section 4). A particular case is when the manifold  $M$  is complete hyperbolic, because in this case the straightening descends to a map from  $M$  to itself. Here we prove that such a map is onto.

Let  $\Delta \subset \mathbb{H}^3$  be an oriented geodesic ideal tetrahedron. Since  $\Delta$  is the convex hull of its vertices, then the  $\text{Isom}^+(\mathbb{H}^3)$ -class of  $\Delta$  is completely

determined by the  $\text{Isom}^+(\mathbb{H}^3)$ -class of the oriented set of its vertices (the orientation of the vertices is defined up to the action of  $\mathcal{A}_4$ ). Such a class is completely determined by a non-real complex number called modulus, up to a three-to-one ambiguity. Such an ambiguity can be avoided by choosing a preferred pair of opposite edges of  $\Delta$  (see [BP], [F], [PP], [PW] [Th1]). We extend the notion of modulus to the set of flat tetrahedra, that is to those whose vertices are distinct and lie on a hyperbolic plane of  $\mathbb{H}^3$ , by accepting real moduli different from 0 and 1. We want to extend this definition also to the degenerate tetrahedra, *i.e.* to those having two or more coincident vertices. Unfortunately, for such a tetrahedron it is not possible to encode its isometry class in a complex number. Let us agree that when we use a modulus in  $\{0, 1, \infty\}$  for  $\Delta$ , we mean that  $\Delta$  is a degenerate tetrahedron and that the modulus encodes the complete information on the isometry class of  $\Delta$ , *i.e.* who are the coincident vertices of  $\Delta$ .

**Definition 3.1** Let  $\Delta^k$  be the standard  $k$ -simplex. Let  $\varphi : \Delta^k \rightarrow \overline{\mathbb{H}}^n$  be a continuous map that maps the 0-skeleton of  $\Delta^k$  to  $\partial\mathbb{H}^n$ . Let  $Q$  be the Euclidean convex hull of the  $\varphi$ -image of the vertices of  $\Delta^k$ , made in a disc model of  $\mathbb{H}^n$ . Let  $\psi : \Delta^k \rightarrow Q$  be the only simplicial map that agrees with  $\varphi$  on the 0-skeleton.

We say that the map  $\varphi$  is *standard* if there exist two homeomorphisms  $\eta : \text{Im}(\varphi) \rightarrow Q$  and  $\beta : \Delta^k \rightarrow \Delta^k$  such that

$$\eta \circ \varphi \circ \beta = \psi.$$

We say that a foliation  $\mathcal{F}$  of  $\Delta^k$  is *standard* if there exists a standard map  $\varphi : \Delta^k \rightarrow \overline{\mathbb{H}}^n$  such that  $\mathcal{F} = \{\varphi^{-1}(x)\}$ .

**Remark 3.2** For any standard map  $\varphi$  the dimension of  $\mathcal{F} = \{\varphi^{-1}(x)\}$  depends only on the  $\varphi$ -image of the 0-skeleton.

**Remark 3.3** It is not hard to show that for a map  $\varphi$  to be standard does not depend on the disc model we use. In other words  $\varphi$  is standard if and only if  $\gamma\varphi$  is standard for any isometry  $\gamma$ .

Let  $M$  be an ideally triangulated manifold,  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  be a representation, and  $D_\rho$  be a pseudo-developing map for  $\rho$ . Let  $\Delta$  be a tetrahedron of  $\tau$  and  $\tilde{\Delta}$  be one of its lifts. The vertices of  $\tilde{\Delta}$  are ideal points, so their images under  $D_\rho$  lie in  $\partial\mathbb{H}^3$ . The map  $D_\rho$  determines a modulus for  $\tilde{\Delta}$  simply by considering the convex hull of the image of its vertices (the orientation is the one induced by  $M$ ). Note that since  $D_\rho$  is equivariant, then it defines a modulus for  $\Delta$ . For each face  $\sigma$  of  $\Delta$  we call  $\text{Str}_{D_\rho}(\tilde{\sigma})$ , or simply  $\text{Str}(\tilde{\sigma})$ , the straight simplex obtained as the convex hull of the  $D_\rho$ -image of the vertices of  $\tilde{\sigma}$ .

**Definition 3.4 (Straightening)** A straightening of  $D_\rho$  is a continuous, piecewise smooth,  $\rho$ -equivariant map  $\text{Str}(D_\rho) : \widehat{M} \rightarrow \overline{\mathbb{H}^3}$  such that:

1. For each simplex  $\sigma$  of the triangulation,  $\text{Str}(D_\rho)$  maps  $\tilde{\sigma}$  to  $\text{Str}(\tilde{\sigma})$ .
2. The restriction of  $\text{Str}(D_\rho)$  to any simplex  $\sigma$  is standard.
3. For each cusp  $N_{\tilde{p}} = P_{\tilde{p}} \times [0, \infty]$  there exists  $c \in \mathbb{R}$  such that  $\text{Str}(D_\rho)$  restricted to  $P_{\tilde{p}} \times [c, \infty]$  is a cone-map.

**Lemma 3.5** *Let  $M$  be an ideally triangulated manifold. Let  $\rho$  be a representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  and  $D_\rho$  be a pseudo-developing map. Then a straightening  $\text{Str}(D_\rho)$  of  $D_\rho$  exists. Moreover  $\text{Str}(D_\rho)$  is always equivariantly homotopic to  $D_\rho$  via a homotopy that fixes the ideal points.*

*Proof.* A straightening of  $D_\rho$  can be constructed with the same techniques of Lemma 2.7. Regarding the homotopy, since  $D_\rho$  maps non-ideal points to the interior of  $\mathbb{H}^3$ , then one can use a geodesic flow with the time-parameter in  $[0, \infty]$  (for example the convex combination of Definition 4.11) to construct a homotopy with the required properties. □

**Remark 3.6** A straightening in general is not a pseudo-developing map in our setting, because it can map some point of  $\widehat{M}$  to  $\partial\mathbb{H}^3$ . However, if there are no degenerate tetrahedra, then a straightening is also a pseudo-developing map, and the homotopy between  $D_\rho$  and  $\text{Str}(D_\rho)$  can be made coherently with the cone structure of the cusps, *i.e.* in such a way that the intermediate maps along the homotopy between  $D_\rho$  and  $\text{Str}(D_\rho)$  have the cone property on the cusps.

When  $M$  has a complete hyperbolic structure of finite volume, there is a natural notion of straightening of the ideal triangulation. Namely, choose the arc-length as the cone parameter on the cusps of  $M$  and consider  $\mathbb{H}^3$  as the universal cover of  $M$ . Then choose  $\rho$  as the holonomy of the hyperbolic structure of  $M$ ; the identity map of  $\mathbb{H}^3$  clearly is a pseudo-developing map for  $\rho$ . A natural straightening map is a straightening of the identity.

**Proposition 3.7** *Let  $M$  be an ideally triangulated manifold equipped with a complete, finite-volume hyperbolic structure. Then any natural straightening map projects to a map  $\text{Str} : \widehat{M} \rightarrow \widehat{M}$  which is onto. Moreover  $\text{Str}(M) \supset M$ .*

*Proof.* It is easy to see that  $\widehat{M} = \widehat{\mathbb{H}^3}$  naturally embeds into  $\overline{\mathbb{H}^3}$  and that the ideal points lie on  $\partial\mathbb{H}^3$ . Since the straightening is equivariant, then it

projects to a map  $\text{Str} : \widehat{M} \rightarrow \widehat{M}$ . Moreover,  $\text{Str}$  fixes the ideal points. We prove that  $\text{Str}$  is onto. One can easily prove that

$$H_3(\widehat{M}; \mathbb{Z}) \cong H_3(\overline{M}, \partial\overline{M}; \mathbb{Z}) \cong \mathbb{Z}.$$

So we can define the degree of a map  $f : \widehat{M} \rightarrow \widehat{M}$  by

$$f_*([\widehat{M}]) = \deg(f) \cdot [\widehat{M}]$$

where  $[\widehat{M}]$  is the generator of  $H_3(\widehat{M}; \mathbb{Z})$  induced by the orientation of  $M$ . Now note that by Lemma 3.5 the natural straightening is homotopic to the identity via an equivariant homotopy. Because of equivariance, the homotopy projects to a homotopy between  $\text{Str}$  and the identity. It follows that  $\text{Str}_*$  and  $\text{id}_*$  coincide on  $H_*(\widehat{M}; \mathbb{Z})$ , so  $\deg(\text{Str}) = \deg(\text{id}) = 1$ . Now suppose that  $\text{Str}$  is not onto and let  $x$  be a point in  $\widehat{M}$  outside its image. If we consider  $\text{Str}$  as a map from  $\widehat{M}$  to  $\widehat{M} \setminus \{x\}$ , we get  $\text{Str}_*([\widehat{M}]) = 0 \in H_3(\widehat{M} \setminus \{x\}; \mathbb{Z})$  simply because  $H_3(\widehat{M} \setminus \{x\}; \mathbb{Z}) = 0$ . Then  $\text{Str}_*([\widehat{M}])$  is a boundary in  $\widehat{M} \setminus \{x\}$ , consequently it is a boundary also in  $\widehat{M}$ . It follows that  $\text{Str}_*([\widehat{M}]) = 0$ . This implies  $\deg(\text{Str}) = 0$ , that is a contradiction.

The last assertion follows because  $\text{Str}$  is onto and fixes the ideal points.  $\square$

## 4 Volume of representations

For this section we fix an ideally triangulated manifold  $M$  and a representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$ .

In this section we recall the notion of volume of an equivariant map from  $\widetilde{M}$  to  $\mathbb{H}^3$ . We prove that if we restrict to the class of pseudo-developing maps, then the volume of  $\rho$  is well-defined. Namely the volume does not depend neither on the pseudo-developing map nor on the product structure of the cusps. Such a volume can be calculated using a straightening of any pseudo-developing map and it is exactly the algebraic sum of the volumes of the straightened tetrahedra.

**Definition 4.1 (Volume of pseudo-developing map)** Let  $D_\rho$  be a pseudo-developing map for  $\rho$ . Let  $\omega$  be the volume form of  $\mathbb{H}^3$  and let  $D_\rho^*\omega$  be the pull-back of  $\omega$ . Since  $D_\rho$  is equivariant, then  $D_\rho^*\omega$  projects to a 3-form, that we still call  $D_\rho^*\omega$ , on  $M$ . The volume  $\text{vol}(D_\rho)$  of  $D_\rho$  is defined by:

$$\text{vol}(D_\rho) = \int_M D_\rho^*\omega.$$

**Remark 4.2** We will see below that for pseudo-developing maps the volume is always finite. The same definition of volume does not work for any equivariant map from  $\widetilde{M}$  to  $\mathbb{H}^3$  because if the pull-back of the volume form is not in  $L^1$ , then the volume is not well-defined.

**Definition 4.3 (Straight volume)** Let  $D_\rho$  be a pseudo developing map for  $\rho$ . Let  $\{\Delta_i\}$  be the set of the tetrahedra of the ideal triangulation of  $M$  and  $\{\widetilde{\Delta}_i\}$  be a set of lifts of the  $\Delta_i$ 's. Let  $v_i = 0$  if  $\text{Str}(\widetilde{\Delta}_i)$  is a degenerate tetrahedron, and let  $v_i$  be the algebraic volume of  $\text{Str}(\widetilde{\Delta}_i)$  otherwise.

We define the straight volume of  $(D_\rho)$  as  $\text{Strvol}(D_\rho) = \sum_i v_i$ .

**Remark 4.4** Let  $\Delta_1, \dots, \Delta_n$  be the tetrahedra of  $\tau$ . Let  $\mathbf{z} = (z_1, \dots, z_n) \in \{\mathbb{C} \setminus \{0, 1\}\}^n$  be a solution of Thurston's hyperbolicity equations (see [BP], [F], [NZ], [PP], [PW], [Th1]). Then there exists a developing map  $D_{\mathbf{z}} : \widetilde{M} \rightarrow \mathbb{H}^3$  for  $\mathbf{z}$  that is a pseudo-developing map for some holonomy  $\rho(\mathbf{z})$ . Such a map is already straight and we have  $\text{Strvol}(D_{\mathbf{z}}) = \text{vol}(D_{\mathbf{z}}) = \sum v_i = \text{vol}(\mathbf{z})$ , where  $v_i$  is the volume of the geodesic ideal tetrahedron of modulus  $z_i$ .

Let  $C_p = T_p \times [0, \infty] / \sim$  be a cusp of  $M$  and let  $N_{\widetilde{p}} = P_{\widetilde{p}} \times [0, \infty] / \sim$  be one of its lifts in  $\widetilde{M}$ . Then we can identify  $\pi_1(C_p)$  with  $\text{Stab}(\widetilde{p})$ . Let  $f : P_{\widetilde{p}} \times \{0\} \rightarrow \mathbb{H}^3$  be a  $\text{Stab}(\widetilde{p})$ -equivariant map, let  $\xi \in \partial\mathbb{H}^3$  be a fixed point of  $\rho(\text{Stab}(\widetilde{p}))$  and let  $F : N_{\widetilde{p}} \rightarrow \overline{\mathbb{H}^3}$  be the cone-map obtained by coning  $f$  to  $\xi$ . As above, let  $F^*\omega$  be the pull-back of the volume-form on  $C_p$ . Similarly we can pull-back the metric. We call  $A_t^p$  the area of the torus  $T_p \times \{t\}$ .

**Lemma 4.5** *In the previous setting, for  $t > r$  we have:*

$$A_t^p \leq A_r^p e^{-(t-r)} \quad \text{and} \quad \int_{T_p \times [t, \infty)} |F^*\omega| \leq A_t^p.$$

*Proof.* Let  $(x, y)$  be local coordinates on  $P_{\widetilde{p}}$ . Choose the half-space model  $\mathbb{C} \times \mathbb{R}^+$  of  $\mathbb{H}^3$  and assume that  $\xi = \infty$ . In such a model the hyperbolic metric at the point  $(z, s)$  is the Euclidean one rescaled by the factor  $1/s$ . It follows that, called  $\alpha + i\beta$  and  $h$  the complex and real components of  $F$ , we have

$$\alpha(x, y, t) + i\beta(x, y, t) = \alpha(x, y, r) + i\beta(x, y, r) \quad h(x, y, t) = h(x, y, r)e^{(t-r)}.$$

The element of area at level  $t$  is  $d\sigma_t(x, y) = \sqrt{\det({}^T JF_t \cdot H \cdot JF_t)}$ , where  $F_t$  is the restriction of  $F$  to  $P_{\widetilde{p}} \times \{t\}$  and  $H(x, y, t) = \frac{1}{h^2} \text{Id}$  is the matrix of the hyperbolic metric. From direct calculations it follows that  $d\sigma_t(x, y) \leq d\sigma_r(x, y)e^{-t+r}$  and the first inequality follows.

Now note that the volume element  $|F^*\omega|$  at the point  $(x, y, t) \in C_p$  is bounded by the area element of the torus  $T_p \times \{t\}$  multiplied by the length element of the ray  $\{(x, y)\} \times [0, \infty]$ . Since the parameter  $t$  is exactly the arc-length, then the length element is exactly  $dt$ . It follows that

$$\int_{T_p \times [t, \infty)} |F^*\omega| \leq \int_t^\infty A_s^p ds \leq \int_t^\infty A_t^p e^{-(s-t)} ds = A_t^p.$$

This completes the proof.  $\square$

**Remark 4.6** From Lemma 4.5 it follows in particular that

$$\int_{T_p \times [t, \infty)} |F^*\omega| \leq A_0^p e^{-t}.$$

This means that we have an estimate of  $\int_{T_p \times [0, \infty)} |F^*\omega|$  not depending on the point  $\xi = F(p)$  but only on the area of  $T_p \times \{0\}$ .

**Remark 4.7** From Lemma 4.5 it follows that  $\text{vol}(D_\rho)$  is finite for any pseudo-developing map  $D_\rho$ .

The following lemma is proved in [D].

**Lemma 4.8** *If  $D_\rho$  and  $F_\rho$  are two pseudo-developing maps for  $\rho$  that agree on the ideal points, then  $\text{vol}(D_\rho) = \text{vol}(F_\rho)$ .*

This is because any two pseudo-developing maps are equivariantly homotopic. The fact that they coincide on the ideal points allows one to construct a homotopy  $h$  that respects the cone structures of the cusps. Namely, for each ideal point  $\tilde{p}$  of  $M$  we choose any equivariant homotopy between the restrictions of  $D_\rho$  and  $F_\rho$  to  $P_{\tilde{p}} \times \{\bar{t}\}$ , where  $\bar{t} = \max\{t_{D_\rho}, t_{F_\rho}\}$ , we cone such a homotopy to  $D_\rho(\tilde{p})$  along geodesic rays, and we extend the homotopy outside the cusps in any equivariant way. For such a homotopy  $h$  we can use the Stokes theorem on  $M \times [0, 1]$  for  $h^*\omega$  to obtain the thesis. More precisely, let  $K_t$  be  $M \setminus \cup_p (T_p \times (t, \infty))$ , where  $p$  varies on the set of the ideal points; then we have

$$0 = \int_{K_t \times [0, 1]} d(h^*\omega) = \int_{\partial(K_t \times [0, 1])} h^*\omega = \int_{K_t} (D_\rho^*\omega - F_\rho^*\omega) + \int_{\partial K_t \times [0, 1]} h^*\omega$$

and, as in Lemma 4.5, we can prove that the last integral goes to zero as  $t \rightarrow \infty$ .

We now prove that the claim of Lemma 4.8 is true in general.

**Theorem 4.9** *Let  $D_\rho$  and  $F_\rho$  be two pseudo-developing maps for  $\rho$ . Then  $\text{vol}(D_\rho) = \text{vol}(F_\rho)$ .*

*Proof.* For  $t \in [0, \infty)$ , let  $D_\rho^t$  be the map constructed as follows:  $D_\rho^t$  coincides with  $D_\rho$  until the level  $t$  of each cusp. Then for each cusp  $N_p$  we complete  $D_\rho^t$  by coning  $D|_{F_p \times \{t\}}$  to  $F_\rho(p)$  along geodesic rays in such a way that the arc-length is the parameter  $s - t$ , where  $s \in [t, \infty)$ . Now,  $D_\rho^t$  is a pseudo-developing map that agrees with  $F_\rho$  on the ideal points. Thus by Lemma 4.8  $\text{vol}(D_\rho^t) = \text{vol}(F_\rho)$ . Since  $D_\rho^t$  and  $D_\rho$  agree outside the cusps and where they differ they are cones on the same basis (and different vertices), from Lemma 4.5 we get

$$|\text{vol}(D_\rho) - \text{vol}(D_\rho^t)| \leq 2 \sum_p A_t^p \leq 2 \left( \sum_p A_0^p \right) e^{-t}$$

where  $p$  varies on the set of ideal points and  $A_t^p$  is the area of the torus  $T_p \times \{t\}$ . As  $t \rightarrow \infty$  we get the thesis. □

Similar techniques actually allow to prove the following theorem.

**Theorem 4.10** *For any pseudo-developing map  $D_\rho$  for  $\rho$  we have  $\text{vol}(D_\rho) = \text{Strvol}(D_\rho)$ .*

Before proving Theorem 4.10, we give the following definition.

**Definition 4.11** Let  $f, g$  be two maps from a set  $X$  respectively to  $\mathbb{H}^n$  and  $\overline{\mathbb{H}}^n$ . For  $t \in [0, \infty]$  the convex combination  $\Phi_t$  from  $f$  to  $g$  is defined by:

$$\Phi_t(x) = \begin{cases} \gamma_x(t) & t \leq \text{dist}(f(x), g(x)) \\ g(x) & t \geq \text{dist}(f(x), g(x)) \end{cases}$$

where  $\gamma_x$  is the geodesic from  $f(x)$  and  $g(x)$ , parametrized by arc-length.

**Remark 4.12** In Definition 4.11, if  $X$  is a topological space and  $f$  and  $g$  are continuous, then the convex combination from  $f$  to  $g$  is continuous on  $X \times [0, \infty]$  because the function  $\text{dist}(f(x), g(x))$  is well-defined and continuous from  $X$  to  $[0, \infty]$ .

*Proof of 4.10.* For the proof assume that  $t_{D_\rho} = 0$ . We start by fixing a suitable homotopy  $h$  between  $D_\rho$  and  $\text{Str}(D_\rho)$ . Define  $h : \widetilde{M} \times [0, \infty] \rightarrow \mathbb{H}^3$  outside the cusps to be the convex combination from  $D_\rho$  to  $\text{Str}(D_\rho)$  and then for each cusp  $N_{\widehat{p}}$  extend  $h$  by coning  $h((x, 0), s)$  to  $D_\rho(\widehat{p})$  along geodesic rays

in such a way that the parameter  $t \in [0, \infty)$  of the cusp is the arc-length. Let  $D_s(x) = h(x, s)$ . By Lemma 4.8 we have that

$$\int_M D_\rho^* \omega = \int_M D_s^* \omega \quad \text{for } s \in (0, \infty).$$

So we only have to prove that  $\int_M D_s^* \omega \rightarrow \text{Strvol}(D_\rho)$  as  $s \rightarrow \infty$ . Clearly, it suffices to prove that for any tetrahedron  $\Delta$  we have  $\int_\Delta D_s^* \omega \rightarrow v$  where  $v$  is the volume of  $\text{Str}(\Delta)$ . If  $\Delta$  does not collapse in the straightening, then the distance from  $D_\rho$  and  $\text{Str}(D_\rho)$  is bounded outside the cusps and so  $D_s = \text{Str}(D_\rho)$  for  $s \gg 0$ ; since  $\text{Str}(D_\rho)$  is a homeomorphism on  $\Delta$ , then  $\int_\Delta D_s^* \omega$  is exactly the volume of the straight version of  $\Delta$ .

If  $\Delta$  collapses in the straightening, then we have to show that  $\int_\Delta D_s^* \omega \rightarrow 0$ . This follows from direct calculations. We give only the lead-line of them because they are involved but use elementary techniques. Moreover, in the next section, we will give an alternative proof of this theorem (see Theorem 5.1 and Remark 5.9).

Given the convex combination  $\Phi_t$  from a map  $f$  to a map  $g$ , it is possible to calculate the Jacobian of  $\Phi_t$  as a function of the derivatives of  $f$  and  $g$ , the time  $t$  and the distance between  $f$  and  $g$ . This is not completely trivial, for example think of a tetrahedron as a convex combination of two segments: the segments have zero area but in the middle we have quadrilaterals with non-zero area. Using these calculations, we can estimate  $|D_s^* \omega|$  outside the cusps, showing that its integral goes to zero as  $s$  goes to infinity. Looking inside the cusps, by Lemma 4.5 we reduce the estimate to the same estimate as above, made with 2-dimensional objects (the bases of the cusps).  $\square$

**Remark 4.13** Since  $\text{vol}(D_\rho) = \text{Strvol}(D_\rho)$  it follows that such a volume does not depend on the chosen cone structure of the cusps. Moreover, by Theorem 4.9,  $\text{vol}(D_\rho)$  does not depend on the developing map, but only on  $\rho$ . This allows us to give the following definition.

**Definition 4.14** The volume  $\text{vol}(\rho)$  of  $\rho$  is the volume of any pseudo-developing map for  $\rho$ .

As the following corollary shows, for hyperbolic manifolds the volume of the holonomy is exactly the hyperbolic volume.

**Corollary 4.15** *Let  $M$  be a complete hyperbolic manifold of finite volume. If  $\rho$  is the holonomy of the hyperbolic structure then  $\text{vol}(\rho) = \text{vol}(M)$ .*

*Proof.* Consider  $\mathbb{H}^3$  as the universal cover of  $M$  and choose the arc length as the cone parameter of the cusps. Clearly the identity of  $\mathbb{H}^3$  is a pseudo-developing map for  $\rho$ . Obviously we have  $\int_M \text{Id}^*(\omega) = \text{vol}(M)$ .  $\square$

**Corollary 4.16** *Let  $z_i$  be the modulus induced by a pseudo-developing map  $D_\rho$  on the  $i^{\text{th}}$  tetrahedron and let  $v_i$  be the volume of a hyperbolic ideal geodesic tetrahedron of modulus  $z_i$ . Then we have  $\text{vol}(\rho) = \sum v_i$ .*

**Remark 4.17** Even if  $\sum v_i$  depends only on  $\rho$ , the moduli  $z_i$  induced by a pseudo-developing map  $D_\rho$  actually can depend on  $D_\rho$ . Namely, any ideal point  $p$  is mapped to a fixed point of  $\rho(\text{Stab}(p))$  and, if this is not a parabolic group, we have more than one possibility for  $D_\rho(p)$ . Conversely, if each  $\rho(\text{Stab}(p))$  is a non-trivial parabolic group, then by Remark 2.8 it follows that the moduli  $z_i$  are uniquely determined by  $\rho$ .

**Proposition 4.18** *Let  $g$  be a reflection of  $\mathbb{H}^3$  and let  $\bar{\rho}$  be the representation  $g \circ \rho \circ g^{-1}$ . Then  $\text{vol}(\bar{\rho}) = -\text{vol}(\rho)$ .*

*Proof.* If  $D_\rho$  is a pseudo-developing map for  $\rho$ , then  $g \circ D_\rho$  is a pseudo-developing map for  $\bar{\rho}$  and it is easily checked that  $\text{vol}(g \circ D_\rho) = -\text{vol}(D_\rho)$ .  $\square$

We recall here two facts proved in [D].

**Proposition 4.19** *Suppose that  $\rho_t : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$ ,  $t \in [0, 1]$  is a smooth one parameter family of representations. Then  $\text{vol}(\rho_0) = \text{vol}(\rho_1)$ .*

This can be proved by using the one-parameter family  $\rho_t$  to construct a one-parameter family of pseudo-developing maps. Some estimates on the derivative along the  $t$ -direction of the volume of such maps are needed.

**Proposition 4.20** *Suppose that  $\rho$  factors through the fundamental group of a Dehn filling  $N$  of  $M$ . Then the volume of  $\rho$  w.r.t.  $N$  coincides with the volume of  $\rho$  w.r.t.  $M$ .*

Theorem 4.10 extends from ideal to “classical” triangulations, namely to genuine triangulations  $\mathcal{T}$  of  $\bar{M}$ . Consider such a  $\mathcal{T}$  as a triangulation of  $M$  with some simplices at infinity (those in  $\partial\bar{M}$ ). Given a pseudo-developing map  $D_\rho$  for  $\rho$ , define a straightening of  $D_\rho$  relative to  $\mathcal{T}$ , exactly as in Section 3, by considering the convex hulls of the images of the vertices of  $\mathcal{T}$ . Then, one can give the definition of the straight volume relative to  $\mathcal{T}$  of a

developing map  $D_\rho$  exactly as in Definition 4.3, with the unique difference that one has to use the tetrahedra of  $\mathcal{T}$  instead of the ideal tetrahedra of an ideal triangulation of  $M$ . Call such a volume  $\text{Strvol}^{\mathcal{T}}(D_\rho)$ .

Finally, exactly as in Theorem 4.10, one can prove the following fact:

**Proposition 4.21** *Let  $\mathcal{T}$  be a triangulation of  $\overline{M}$  and  $D_\rho$  be a pseudo-developing map for  $\rho$ . Then  $\text{vol}(\rho) = \text{Strvol}^{\mathcal{T}}(D_\rho)$ .*

## 5 Comparison with simplicial volume

Here we generalize the argument used to prove Theorem 4.10 to compare  $\text{vol}(\rho)$  with the simplicial volume of  $M$ , obtaining exactly the expected inequality.

Let  $\|(\overline{M}, \partial\overline{M})\|$  be the simplicial volume of  $\overline{M}$  relative to the boundary (see [BP], [G], [Ku], [Th1] for more details), and let  $v_3$  be the volume of a regular straight ideal tetrahedron of  $\mathbb{H}^3$ .

**Theorem 5.1** *For any representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  we have*

$$|\text{vol}(\rho)| \leq v_3 \cdot \|(\overline{M}, \partial\overline{M})\|.$$

*Proof.* For the proof we fix a representation  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  and a pseudo-developing map  $D_\rho$  for  $\rho$ .

Let  $c = \sum_i \lambda_i \sigma_i$  be a smooth singular chain in  $\overline{M}$ ; here each simplex  $\sigma_i$  is a piecewise smooth map from the standard tetrahedron  $\Delta^3$  to  $\overline{M}$ . The simplicial volume of  $c$  is defined as  $\|c\| = \sum |\lambda_i|$ . The relative simplicial volume of  $(\overline{M}, \partial\overline{M})$  is defined as

$$\|(\overline{M}, \partial\overline{M})\| = \inf\{\|c\| : [c] = [\overline{M}] \in H_3(\overline{M}, \partial\overline{M})\}.$$

The proof has two main steps:

1. Given a smooth cycle  $c = \sum_i \lambda_i \sigma_i$  representing  $[\overline{M}]$ , show that  $\text{vol}(\rho) = \sum_i \int_{\Delta^3} \lambda_i \sigma_i^*(D_\rho^* \omega)$ , where  $\omega$  is the volume form of  $\mathbb{H}^3$ .
2. By replacing  $c$  with its straightening, show that  $\text{vol}(\rho) = \sum_i \lambda_i v_i$ , where  $v_i$  is the volume of a straight version of  $\sigma_i$ .

From Step 2 it follows that

$$|\text{vol}(\rho)| \leq \sum_i |\lambda_i| \cdot |v_i| \leq v_3 \cdot \|c\|.$$

Theorem 5.1 follows taking to the infimum over all  $c$ 's representing  $[\overline{M}]$ .

*Step 1.* Since a pseudo-developing map has the cone property on the cusps, the 3-form  $D_\rho^*\omega$  defined on  $M$  extends to a 3-form on  $\overline{M}$  that vanishes at the boundary. So we can consider the class  $[D_\rho^*\omega] \in H^3(\overline{M}, \partial\overline{M})$ . Since  $[c] = [\overline{M}]$ , then

$$\text{vol}(\rho) = \int_M D_\rho^*\omega = \langle [D_\rho^*\omega], [\overline{M}] \rangle = \langle [D_\rho^*\omega], [c] \rangle = \sum_i \int_{\Delta^3} \lambda_i \sigma_i^*(D_\rho^*\omega).$$

*Step 2.* The idea is the following. Consider a lift  $\tilde{c}$  of  $c$  to  $\widehat{M}$ . Let  $\bar{c} = (D_\rho)_*\tilde{c}$  be the push-forward of  $\tilde{c}$  to  $\mathbb{H}^3$  via  $D_\rho$  and let  $\text{Str}(\bar{c})$  be a straightening of  $\bar{c}$ . Since the straightening is homotopic to the identity, then there exists a chain-homotopy of degree one *i.e.* a map  $H$  from  $k$ -chains to  $(k+1)$ -chains such that  $\text{Str} - \text{Id} = H \circ \partial - \partial \circ H$ . Then we have

$$\begin{aligned} \text{vol}(\rho) &= \langle D_\rho^*\omega, \tilde{c} \rangle = \langle \omega, (D_\rho)_*\tilde{c} \rangle = \langle \omega, \bar{c} \rangle \\ &= \langle \omega, \text{Str}(\bar{c}) \rangle + \langle \omega, \partial H\bar{c} \rangle - \langle \omega, H\partial\bar{c} \rangle \\ &= \sum_i \lambda_i v_i + \langle d\omega, H\bar{c} \rangle - \langle \omega, H\partial\bar{c} \rangle. \end{aligned}$$

The last two summands are zero because  $d\omega = 0$  and, even if  $\partial\bar{c} \neq 0$ , everything can be made  $\rho$ -equivariantly so that the action of  $\rho$  cancels out in pairs the contributions of  $\langle \omega, H\partial\bar{c} \rangle$ .

We formalize now this argument. Let  $C_k(X)$  denote the real vector space of finite singular, piecewise smooth  $k$ -chains in a space  $X$ . Consider the projection  $\overline{M} \rightarrow \widehat{M}$  obtained by collapsing each boundary torus to a point. Given a relative cycle  $c = \sum_i \lambda_i \sigma_i$  in  $C_k(\overline{M})$  *i.e.* a chain  $c$  such that  $\partial c \in C_{k-1}(\partial\overline{M})$ , we also call  $c$  the chain induced on  $C_k(\widehat{M})$  with  $\partial c \in C_k(\text{ideal points})$ . We call  $\tilde{c}$  a lift of  $c$  to  $\widehat{M}$ , that is  $\tilde{c} = \sum_i \lambda_i \tilde{\sigma}_i \in C_k(\widehat{M})$  where each  $\tilde{\sigma}_i$  is a lift of  $\sigma_i$ .

**Remark 5.2** The chain  $\tilde{c}$  in general is not a relative cycle. Nevertheless, since  $c$  is a relative cycle, assuming  $\partial\tilde{c} = \sum_j l_j \eta_j$ , there exists a family  $\{\alpha_j\}$  of elements of  $\pi_1(M)$  such that

$$\sum_j l_j \cdot \alpha_{j*}(\eta_j) \in C_k(\text{ideal points})$$

where  $\pi_1(M)$  acts on  $\widehat{M}$  via deck transformations and  $\alpha_{j*}(\eta_j)$  is the composition of  $\alpha_j$  with  $\eta_j$ .

We set  $\bar{\sigma}_i = (D_\rho)_*(\tilde{\sigma}_i)$  and  $\bar{c} = \sum_i \lambda_i \bar{\sigma}_i = (D_\rho)_*(\tilde{c}) \in C_k(\mathbb{H}^3)$ . We restrict now the class of simplices we want to use.

**Definition 5.3** We call a  $k$ -simplex  $\sigma : \Delta^k \rightarrow \overline{\mathbb{H}}^3$  *admissible* if for any sub-simplex  $\eta$  of  $\sigma$ , if the interior of  $\eta$  touches  $\partial\overline{\mathbb{H}}^3$  then  $\eta$  is constant. A chain is admissible if its simplices are admissible.

**Lemma 5.4** Let  $c' = \sum_i \lambda_i \sigma'_i$  be a relative cycle in  $C_k(\overline{M})$ . Then there exists a cycle  $c = \sum_i \lambda_i \sigma_i$  (with the same  $\lambda_i$ 's) such that  $[c'] = [c] \in H_k(\overline{M}, \partial\overline{M})$  and such that  $\bar{c}$  is admissible.

*Proof.* For any chain  $\beta \in C_k(\overline{M})$ , define  $\text{span}(\beta)$  as the set of all the sub-simplices of  $\beta$  (of any dimension). Given the chain  $c'$ , construct  $c$  as follows: near  $\partial\overline{M}$  push  $c'$  a little inside  $M$ , keeping fixed only the simplices of  $\text{span}(\partial c')$ . This operation can be made via an homotopy, so  $[c] = [c']$ . Moreover, the only simplices of  $c$  that touch  $\partial\overline{M}$  are the ones of  $\text{span}(\partial c)$ . Finally,  $c$  is admissible because, if  $\bar{\sigma}_i(x) \in \partial\overline{\mathbb{H}}^3$ , then from the definition of pseudo-developing map it follows that  $\sigma_i(x)$  is an ideal point. Thus  $x$  lies on a face  $F$  of  $\sigma_i$  such that the simplex  $\eta = (\sigma_i)|_F$  belongs to  $\text{span}(\partial c)$ . It follows that  $\tilde{\eta}$  is a constant map and then also  $\bar{\eta}$  is constant.  $\square$

We call  $\overline{C}_k(\overline{\mathbb{H}}^3)$  the vector space of admissible chains. Note that the boundary operator is well-defined on  $\oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$  (The boundary of an admissible cycle is admissible).

**Definition 5.5** For any admissible simplex  $\sigma : \Delta^k \rightarrow \overline{\mathbb{H}}^3$ , a straightening  $\text{Str}(\sigma) : \Delta^k \rightarrow \overline{\mathbb{H}}^3$  is a simplex agrees with  $\sigma$  on the 0-skeleton, moreover we require  $\text{Str}(\sigma)$  to be a standard map whose image is the convex hull of its vertices. For any chain  $c = \sum_i \lambda_i \sigma_i$  a straightening of  $c$  is a chain  $\text{Str}(c) = \sum_i \lambda_i \text{Str}(\sigma_i)$ .

A straightening of a simplex is admissible because any straight simplex is admissible. The straightening of a simplex is not unique in general. Nevertheless, as the following lemma shows, it is possible to choose a straightening for any simplex compatibly with the boundary operator of  $\oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$ .

**Lemma 5.6** There exists a chain-map  $\text{Str} : \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3) \rightarrow \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$  that maps each simplex to one of its straightenings and such that for any isometry  $\gamma$  of  $\mathbb{H}^3$ ,  $\gamma_* \circ \text{Str} = \text{Str} \circ \gamma_*$ .

*Proof.* Let  $K$  be the set of pairs  $\{(B, f)\}$  where  $B$  is a sub-space of  $\oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$  and  $f : B \rightarrow \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$  is a linear map, such that:

- $\partial(B) \subset B$ .

- $\forall \gamma \in \text{Isom}(\mathbb{H}^3)$ ,  $\gamma_*(B) \subset B$ .
- $\forall \sigma \in B$ ,  $f(\sigma)$  is a straightening of  $\sigma$ .
- $\forall \gamma \in \text{Isom}(\mathbb{H}^3)$ ,  $f \circ \gamma_* = \gamma_* \circ f$ .
- $f \circ \partial = \partial \circ f$ .

Note that  $K$  is not empty because each 0-simplex is admissible and it is itself its unique straightening, so that  $(\overline{C}_0(\overline{\mathbb{H}}^3), Id) \in K$ . We order  $K$  by inclusion (*i.e.*  $(B, f) \prec (C, g)$  iff  $B \subset C$  and  $g|_B = f$ ) and use Zorn's lemma. Let  $\{(B_\xi, f_\xi)\}$  be an ordered sequence in  $K$ . Clearly

$$(B_\infty = \cup_\xi B_\xi, f_\infty = \cup_\xi f_\xi)$$

is an upper bound for  $\{(B_\xi, f_\xi)\}$ . It follows that there exists a maximal element  $(\overline{B}, \overline{f}) \in K$ . We claim that  $\overline{B} = \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$ . Suppose the contrary. Let  $k = \min\{n \in \mathbb{N} : \overline{C}_n(\overline{\mathbb{H}}^3) \not\subset \overline{B}\}$  and let  $\sigma$  be a simplex of  $\overline{C}_k(\overline{\mathbb{H}}^3) \setminus \overline{B}$ . If  $k = 0$ , set  $B_1$  the space spanned by  $\overline{B}$  and  $\bigcup_{\gamma \in \text{Isom}(\mathbb{H}^3)} \gamma_*(\sigma)$ , define  $\overline{f}(\sigma) = \sigma$ ,  $\overline{f}(\gamma_*(\sigma)) = \gamma_*(\overline{f}(\sigma))$  and extend  $\overline{f}$  on  $B_1$  by linearity. Then

$$(\overline{B}, \overline{f}) \prec (B_1, \overline{f})$$

contradicting the maximality of  $(\overline{B}, \overline{f})$ . If  $k > 0$ , then  $\overline{f}$  is defined on  $\partial\sigma$  and, as  $\overline{f}(\partial\sigma)$  is standard, it is not hard to show that it extends to a standard map  $\overline{f}(\sigma)$  defined on the whole  $\Delta^k$ . Then define  $B_1$  and extend  $\overline{f}$  to  $B_1$  as above. Again we have  $(\overline{B}, \overline{f}) \prec (B_1, \overline{f})$ , that contradicts the maximality of  $(\overline{B}, \overline{f})$ .

Thus  $\overline{B} = \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$  and  $\overline{f}$  is the requested chain map Str.

□

**Lemma 5.7** *There exists a homotopy operator  $H : \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3) \rightarrow \oplus_k \overline{C}_k(\overline{\mathbb{H}}^3)$  between Str and the identity such that  $H \circ \gamma_* = \gamma_* \circ H$  for any isometry  $\gamma$  of  $\mathbb{H}^3$ .*

*Proof.* A homotopy operator between Str and Id is a chain map of degree 1, *i.e.* a map  $H : \overline{C}_k(\overline{\mathbb{H}}^3) \rightarrow \overline{C}_{k+1}(\overline{\mathbb{H}}^3)$ , such that

$$\text{Str} - Id = \partial \circ H - H \circ \partial.$$

For any admissible  $\sigma : \Delta^k \rightarrow \overline{\mathbb{H}}^3$ , let  $h_\sigma(t, x)$  be the homotopy constructed as follows:  $h_\sigma(t, x)$  is the convex combination from  $\sigma(x)$  to  $\text{Str}(\sigma)(x)$  if

$\sigma(x) \notin \partial\mathbb{H}^3$  and  $h_\sigma(t, x) = \sigma(x)$  otherwise. Note that from the admissibility of  $\sigma$  it follows that  $h_\sigma(\infty, x) = \text{Str}(\sigma)(x)$  for any  $x$ . So the  $h_\sigma$  actually is a homotopy between  $\sigma$  and  $\text{Str}(\sigma)$ .

As  $h_\sigma$  is a map  $h_\sigma : \Delta^k \times [0, \infty] \rightarrow \overline{\mathbb{H}^3}$ , up to triangulating  $\Delta^k \times [0, \infty]$ , it is a chain in  $C_{k+1}(\overline{\mathbb{H}^3})$ . Fix a canonical triangulation of  $\Delta^k \times [0, \infty]$  and define  $H(\sigma)$  as  $h_\sigma$ . Since

$$\partial(\Delta^k \times [0, \infty]) = \Delta^k \times \{\infty\} - \Delta^k \times \{0\} + \partial\Delta^k \times [0, \infty]$$

then  $\partial \circ H = \text{Str} - \text{Id} + H \circ \partial$ .

Since  $h_\sigma$  is constructed using geodesic rays, then for every isometry  $\gamma$  we have  $h_{\gamma \circ \sigma} = \gamma \circ h_\sigma$ . It follows that  $H \circ \gamma_* = \gamma_* \circ H$ .

Finally, admissibility of  $h_\sigma$  follows from admissibility of  $\sigma$ . □

**Lemma 5.8** *Let  $c = \sum_i \lambda_i \sigma_i$  be a chain in  $C_k(\overline{M})$ . Let  $\{\gamma_j\}$  be a finite set of isometries and let  $A$  be the hyperbolic convex hull in  $\mathbb{H}^3$  of*

$$\bigcup_{i,j} \gamma_j(\text{Im}(\overline{\sigma}_i)).$$

*Then  $A$  has finite volume.*

*Proof.* Since  $D_\rho$  has the cone property on the cusps and since  $c$  is a finite sum of simplices, then  $A$  is contained in a geodesic polyhedron with a finite number of vertices, and such a polyhedron has finite volume. □

We are now ready to complete the proof of Theorem 5.1. Let  $c = \sum_i \lambda_i \sigma_i$  be a relative cycle in  $C_3(\overline{M})$  such that  $[c] = [\overline{M}]$  in  $H_3(\overline{M}, \partial\overline{M})$ . By Lemma 5.4 we can suppose that  $c$  is admissible. Assume  $\partial\tilde{c} = \sum_j l_j \eta_j$ . By Remark 5.2, there exists a finite set  $\{\alpha_j\} \subset \pi_1(M)$  such that  $\sum_j l_j \cdot \alpha_{j*} \eta_j \in C_2(\text{ideal points})$ .

Let  $A$  be as in Lemma 5.8, where we use  $\{\rho(\alpha_j)\} \cup \{\text{Id}\}$  as the set of isometries. Since  $A$  has finite volume, then the volume form  $\omega$  of  $\mathbb{H}^3$  is an element of  $H^3(A)$ . Moreover, the straightening of any admissible simplex in  $\overline{C}_k(A)$  is contained in  $\overline{C}_k(A)$  and, since the homotopy operator  $H$  is constructed using convex combinations, then  $H$  is well-defined on  $\oplus_k \overline{C}_k(A)$ . Called  $v_i$  the volume of the straight version of  $\sigma_i$ , we have

$$\begin{aligned} \text{vol}(\rho) &= \langle D_\rho^* \omega, \tilde{c} \rangle = \langle \omega, (D_\rho)_*(\tilde{c}) \rangle = \langle \omega, \overline{c} \rangle \\ &= \langle \omega, \text{Str}\overline{c} \rangle + \langle \omega, H\partial\overline{c} \rangle - \langle \omega, \partial H\overline{c} \rangle \\ &= \sum_i \lambda_i v_i + \langle \omega, H\partial\overline{c} \rangle - \langle d\omega, H\overline{c} \rangle = \sum_i \lambda_i v_i + \langle \omega, H\partial\overline{c} \rangle \end{aligned}$$

By Lemma 5.7 we have  $\rho(\alpha_j)_*H = H\rho(\alpha_j)_*$ . Moreover, the volume form is invariant by isometries. It follows that

$$\begin{aligned}
\langle \omega, H\partial\bar{c} \rangle &= \langle \omega, H \sum_j l_j (D_\rho)_* \eta_j \rangle = \sum_j l_j \langle \omega, H (D_\rho)_* \eta_j \rangle \\
&= \sum_j l_j \langle \rho(\alpha_j)_* \omega, H (D_\rho)_* \eta_j \rangle = \sum_j l_j \langle \omega, \rho(\alpha_j)_* H (D_\rho)_* \eta_j \rangle \\
&= \sum_j l_j \langle \omega, H \rho(\alpha_j)_* (D_\rho)_* \eta_j \rangle = \sum_j l_j \langle \omega, H (D_\rho)_* \alpha_{j*} \eta_j \rangle \\
&= \langle \omega, H (D_\rho)_* \sum_j l_j \alpha_{j*} \eta_j \rangle.
\end{aligned}$$

The last product is zero because  $D_{\rho*} \sum_j l_j \alpha_{j*} \eta_j$  lies on the ideal points of  $A$ , where  $H$  is fixed and  $\omega$  vanishes.

This completes the proof of Theorem 5.1. □

**Remark 5.9** Theorem 5.1 applies when the cycle  $c$  is an ideal triangulation. So it implies Theorem 4.10.

**Corollary 5.10** *Let  $M$  be a graph 3-manifold. Then for all representations  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  we have  $\text{vol}(\rho) = 0$ .*

*Proof.* This is because for each graph manifold  $M$  we have  $\|(M, \partial M)\| = 0$  (see [G], [Ku]). □

**Corollary 5.11** *Let  $M$  be a complete hyperbolic 3-manifold of finite volume. Then for all representations  $\rho : \pi_1(M) \rightarrow \text{Isom}^+(\mathbb{H}^3)$  we have  $|\text{vol}(\rho)| \leq \text{vol}(M)$ .*

*Proof.* This follows from the fact that for complete hyperbolic 3-manifold we have  $\text{vol}(M) = v_3 \|(M, \partial M)\|$  (see [G], [Ku]). □

In [D] it is proved that, for compact manifolds, equality holds if and only if  $\rho$  is discrete and faithful. In the next section we show that this is true in general for manifolds of finite volume.

## 6 Rigidity of representations

This section is completely devoted to proving the following:

**Theorem 6.1** *Let  $M$  be a non-compact, complete, orientable hyperbolic 3-manifold of finite volume. Let  $\Gamma \cong \pi_1(M)$  be the sub-group of  $\text{PSL}(2, \mathbb{C})$*

such that  $M = \mathbb{H}^3/\Gamma$ . Let  $\rho : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$  be a representation. If  $|\mathrm{vol}(\rho)| = \mathrm{vol}(M)$  then  $\rho$  is discrete and faithful. More precisely there exists  $\varphi \in \mathrm{PSL}(2, \mathbb{C})$  such that for any  $\gamma \in \Gamma$

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}.$$

**Remark 6.2** It is well-known that, in the hypotheses of Theorem 6.1, the manifold  $M$  is the interior of a compact manifold  $\overline{M}$  whose boundary consists of tori. Thus  $M$  is a cusped manifold and, by Remark 2.10, all the definitions and results we gave for ideally triangulated manifold apply.

As product structure on the cusps we fix the horospherical one, having the arc-length as cone parameter. For this section  $D_\rho$  will denote a fixed pseudo-developing map for  $\rho$ .

**Remark 6.3** By Proposition 4.18 we can suppose  $\mathrm{vol}(\rho) \geq 0$ .

**Remark 6.4** A subgroup of  $\mathrm{PSL}(2, \mathbb{C})$  is said to be *elementary* if it has an invariant set of at most two points in  $\partial\mathbb{H}^3$ . If the image of  $\rho$  is elementary, then one can construct a pseudo-developing map as in Lemma 2.7 in such a way that all the tetrahedra of any ideal triangulation of  $M$  collapse in the straightening. Thus, by Theorem 4.10,  $\mathrm{vol}(\rho) = 0$ .

This remark implies that, in the present case, since  $\mathrm{vol}(\rho) = \mathrm{vol}(M) \neq 0$ , the image of  $\rho$  is non-elementary.

The idea for proving Theorem 6.1 is to rewrite the Gromov-Thurston-Goldman-Dunfield proof of Mostow's rigidity, valid in the compact case.

We will follow the lead-line of [D], with the difference that we will use classical chains instead of measure-chains. The technique for constructing classical chains representing smear-cycles is that used in [BP] for the proof of Mostow's rigidity for compact manifolds. As an effect of non-compactness we will work with infinite chains. Therefore, we have to prove that some usual homological arguments actually work for our chains.

The core of the proof is to deduce from the equality  $\mathrm{vol}(\rho) = \mathrm{vol}(M)$  that  $D_\rho$  "does not shrink the volume." This allows us to construct a measurable extension of  $D_\rho$  to the whole  $\overline{\mathbb{H}^3}$ , whose restriction to  $\partial\mathbb{H}^3$  is almost everywhere a Möbius transformation. Such a Möbius transformation will be the  $\varphi$  of Theorem 6.1.

The key fact is the following proposition, whose proof can be found in [D] (claim 3 of Theorem 6.1).

**Proposition 6.5** *Let  $f : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$  be a measurable map that maps the vertices of almost all regular ideal tetrahedra to vertices of regular ideal tetrahedra. Then  $f$  coincides almost everywhere with the trace of an isometry  $\varphi$ .*

We want to apply Proposition 6.5 to  $D_\rho$ , and we do it in two steps. Let  $M_0$  be  $M$  minus the cusps and let  $\pi : \mathbb{H}^3 \rightarrow M$  be the universal cover.

**Proposition 6.6** *The map  $D_\rho$  extends to  $\overline{\mathbb{H}^3}$ . More precisely, there exists a measurable map  $\overline{D}_\rho : \partial\mathbb{H}^3 \rightarrow \partial\mathbb{H}^3$  such that for almost all  $x \in \partial\mathbb{H}^3$ , for any geodesic  $\gamma^x$  ending at  $x$ , for any sequence  $t_n \rightarrow \infty$  such that  $\pi(\gamma^x(t_n)) \in M_0$ , we have*

$$\lim_{n \rightarrow \infty} D_\rho(\gamma^x(t_n)) = \overline{D}_\rho(x).$$

**Proposition 6.7** *The map  $\overline{D}_\rho$  satisfies the hypothesis of Proposition 6.5.*

Before proving Propositions 6.6, and 6.7 we show how they imply Theorem 6.1

*Proof of 6.1.* By Proposition 6.7, Proposition 6.5 applies. By Proposition 6.6 the equivariance of  $D_\rho$  implies the equivariance of  $\overline{D}_\rho$ , getting for any  $\gamma \in \Gamma$

$$\rho(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}. \quad \square$$

**Remark 6.8** Both Propositions 6.6, and 6.7 will follow from Lemma 6.22 and 6.23 below. We notice that Lemma 6.22 is a restatement of Lemmas 6.2 of [D]. While Proposition 6.7 corresponds to *Claim 2* of [D]. Proposition 6.6 follows from Lemmas 6.22 and 6.23 exactly as in [D]. We will give a complete proof of Proposition 6.7 because the proof of *Claim 2* in [D] seems to be incomplete.

From now until Lemma 6.11 we will describe how to construct a simplicial version of the *smearing* process of measure-homology (see [Th1] or [Ra]). Then we will prove Lemma 6.23. Finally we will complete the proof of Theorem 6.1 by proving Propositions 6.6 and 6.7.

Let  $\mu$  be the Haar measure on  $\text{Isom}(\mathbb{H}^3)$  such that for each  $x \in \mathbb{H}^3$  and  $A \subset \mathbb{H}^3$  we have

$$\mu\{g \in \text{Isom}(\mathbb{H}^3) : g(x) \in A\} = \text{vol}(A)$$

where  $\text{vol}(A)$  is the hyperbolic volume of  $A$ .

In the following by a tetrahedron of  $\overline{\mathbb{H}^3}$  we mean an ordered 4-tuple of points (the vertices). The volume of a tetrahedron is the hyperbolic volume with sign of the convex hull of its vertices.

Let  $S$  be the set of all genuine (non-ideal, non-degenerate) tetrahedra:

$$S = \{(y_0, \dots, y_3) \in (\mathbb{H}^3)^4 : \text{vol}(y_0, \dots, y_3) \neq 0\}.$$

For any  $Y \in S$  let  $S(R)$  be the set of all isometric copies of  $Y$ :

$$S(Y) = \{X \in S : \exists g \in \text{Isom}(\mathbb{H}^3), X = g(Y)\}.$$

Then a natural bijection  $f_Y : \text{Isom}(\mathbb{H}^3) \rightarrow S(Y)$  is well-defined by

$$f_Y(g) = g(Y).$$

Thus  $\mu$  induces a measure, which we still call  $\mu$ , on  $S(Y)$  defined by

$$\mu(A) = \mu(f_Y^{-1}(A)).$$

We consider the sets  $S_{\pm}(Y) = f_Y^{-1}(\text{Isom}^{\pm}(\mathbb{H}^3))$  of tetrahedra respectively positively and negatively isometric to  $Y$ . Note that  $S_+(Y)$  and  $S(Y)_-$  are both measurable.

Set  $\mathfrak{G} = \Gamma^4/\Gamma$  where  $\Gamma$  acts on  $\Gamma^4$  by left multiplication. Each element  $\sigma = [(\gamma_0, \dots, \gamma_3)] \in \mathfrak{G}$  has a unique representative with  $\gamma_0 = \text{Id}$ . When we write  $\sigma \in \mathfrak{G}$  we tacitly assume that the representative of the form  $(\gamma_0, \dots, \gamma_3)$  with  $\gamma_0 = \text{Id}$  has been chosen. So  $\gamma_0$  is always the identity.

For the rest of the section we fix a fundamental polyhedron  $F \subset \mathbb{H}^3$  for  $M$ . For all  $\varepsilon > 0$  let  $\mathcal{F}^\varepsilon$  be a locally finite  $\varepsilon$ -net in  $F$ . For any  $\xi \in \mathcal{F}^\varepsilon$  let

$$F_\xi = \{x \in F : d(x, \xi) = d(x, \mathcal{F}^\varepsilon)\}.$$

Each  $F_\xi$  is a geodesic polyhedron of diameter less than  $\varepsilon$ . From the cone-property of  $D_\rho$  it follows that the diameters of  $D_\rho(F_\xi)$  are bounded by a constant  $\delta$  that depends on  $\varepsilon$ . Moreover, by removing some boundary face from some  $F_\xi$ , we get that  $F$  is the disjoint union of the  $F_\xi$ 's. We define now a family of special simplices. Let

$$\mathfrak{N} = \{(\gamma_0, \dots, \gamma_3, \xi_0, \dots, \xi_3) : (\gamma_0, \dots, \gamma_3) \in \mathfrak{G}, \xi_i \in \mathcal{F}^\varepsilon \text{ for all } i\}.$$

For each  $\eta \in \mathfrak{N}$  define  $\Delta_\eta$  as the straight geodesic singular 3-simplex whose vertices are the points  $\xi_0, \gamma_1(\xi_1), \gamma_2(\xi_2), \gamma_3(\xi_3)$ , more precisely

$$\Delta_\eta : \Delta^3 \ni t \mapsto \pi\left(\sum_{i=0}^3 t_i \gamma_i(\xi_i)\right).$$

For each tetrahedron  $X = (x_0, \dots, x_3) \in S(Y)$  there exists a unique  $\eta = (\gamma_0, \dots, \gamma_3, \xi_0, \dots, \xi_3) \in \mathfrak{N}$  such that  $x_i \in \gamma_i(F_{\xi_i})$  for  $i = 0, \dots, 3$ . This defines a function

$$s_Y : S(Y) \rightarrow \mathfrak{N}.$$

Roughly speaking,  $\mathfrak{N}$  is a locally finite  $\varepsilon$ -net in the space of 3-simplices of  $M$  and  $s_Y$  is the ‘‘closest point’’-projection.

For any  $\eta \in \mathfrak{N}$  define

$$a_Y^\pm(\eta) = \mu\{s_Y^{-1}(\eta) \cap S_\pm(Y)\} = \mu\{X \in S_\pm(Y) : x_i \in \gamma_i(F_{\xi_i})\}$$

and

$$a_Y(\eta) = a_Y^+(\eta) - a_Y^-(\eta).$$

In the language of measures, one can think of  $a_Y^\pm$  as the push-forward of the measure  $\mu$  under the map  $s_Y : S_\pm(Y) \rightarrow \mathfrak{N}$ . This is the key for the passage from measure-chains to classical ones.

The smearing of the tetrahedron  $Y$  is the cycle:

$$Z_Y = \sum_{\eta \in \mathfrak{N}} a_Y(\eta) \Delta_\eta.$$

We notice that, as  $\mathfrak{N}$  depends on the family  $\mathcal{F}^\varepsilon$ , the cycle  $Z_Y$  actually depends on  $\varepsilon$ .

**Remark 6.9** The smearing of a tetrahedron in general is not a finite sum. Nevertheless, as the following lemma shows, it has bounded  $l^1$ -norm.

**Lemma 6.10** *For any  $Y \in S$ , we have  $\sum_{\eta} |a_Y(\eta)| < \text{vol}(M)$ .*

*Proof.* If  $Y = (y_0, \dots, y_3)$  then

$$\begin{aligned} \sum_{\eta} |a_Y(\eta)| &\leq \sum_{\eta} (a_Y^+(\eta) + a_Y^-(\eta)) = \sum_{\eta} \mu\{s_Y^{-1}(\eta)\} = \mu\left\{\bigcup_{\eta} s_Y^{-1}(\eta)\right\} \\ &= \mu\{s_Y^{-1}(\mathfrak{N})\} = \mu\{f_Y^{-1}s_Y^{-1}(\mathfrak{N})\} = \mu\{g : g(y_0) \in F\} = \text{vol}(F) = \text{vol}(M). \end{aligned}$$

□

**Lemma 6.11** *The infinite chain  $Z_Y$  is a cycle, i.e.  $\partial Z_Y = 0$ .*

*Proof.* First note that the  $l^1$ -norm of  $\partial Z_Y$  is bounded by 4 times the  $l^1$ -norm of  $Z_Y$ . Thus all the sums we will consider make sense.

Let  $v$  be a simplex of  $\partial Z_Y$ . By construction  $v$  is obtained as the projection of an  $(n-1)$ -simplex having vertices in  $F_{\xi_0}, \gamma_1(F_{\xi_1}), \gamma_2(F_{\xi_2})$  for some  $\gamma_1, \gamma_2 \in \Gamma$  and  $\xi_0, \xi_1, \xi_2 \in \mathcal{F}^\varepsilon$ . Let  $A_v$  be the set of the elements of  $\mathfrak{N}$  of the form  $\eta = (\gamma_0, \gamma_1, \gamma_2, \gamma, \xi_0, \xi_1, \xi_2, \xi)$  with  $\gamma \in \Gamma$  and  $\xi \in \mathcal{F}^\varepsilon$ . The simplices  $\Delta_\eta$  of  $Z_Y$  having  $v$  as the last face contribute to the coefficient of  $v$  in  $\partial Z_Y$  by

$$\begin{aligned} \sum_{\eta \in A_v} a_Y(\eta) &= \sum_{\eta \in A_v} \mu(s_Y^{-1}(\eta) \cap S_+(Y)) - \sum_{\eta \in A_v} \mu(s_Y^{-1}(\eta) \cap S_-(Y)) \\ &= \mu(s_Y^{-1}(A_v) \cap S_+(Y)) - \mu(s_Y^{-1}(A_v) \cap S_-(Y)) = 0. \end{aligned}$$

The same calculation, made with the simplices having  $v$  as the  $i$ th face, shows that the coefficient of  $v$  in  $\partial Z_Y$  is zero.  $\square$

For any ideal, non-flat, tetrahedron  $Y = (y_0, \dots, y_3)$  let  $t \mapsto y_i(t)$  be the geodesic ray from the center of mass of  $Y$  to  $y_i$ ,  $i = 0, \dots, 3$ . For any  $R > 0$  let  $Y_R$  be the following element of  $S$ :

$$Y_R = (y_0(R), \dots, y_3(R)).$$

**Remark 6.12** From now on we fix a positively oriented regular ideal tetrahedron  $Y$ , and we write  $S_{\pm}(R)$ ,  $f_R$ ,  $s_R$ ,  $a_R(\eta)$  and  $Z_R$  for  $S_{\pm}(Y_R)$ ,  $f_{Y_R}$ ,  $s_{Y_R}$ ,  $a_{Y_R}(\eta)$  and  $Z_{Y_R}$ .

We say that a 3-simplex  $\Delta$  is  $\varepsilon$ -close to a tetrahedron  $X$  if the vertices of  $\Delta$  are  $\varepsilon$ -close to  $X$ . We define

$$\epsilon(R, \varepsilon) = \sup\{v_3 - \text{vol}(\Delta) : \Delta \text{ is } \varepsilon\text{-close to an element of } S(R)\}$$

**Lemma 6.13** *For any fixed  $\varepsilon$ , for large  $R$  the function  $\epsilon(R, \varepsilon)$  goes to zero exponentially in  $R$ .*

This is because  $v_3 - \text{vol}(Y_R)$  goes to zero like  $e^{-R}$  and the volume of any  $\Delta$  which is  $\varepsilon$ -close to  $Y_R$  is close to the volume of  $Y_R$ . See [BP], [D], [Th1] for details.

**Remark 6.14** What we actually need to prove our claims is a restatement for  $Z_R$  of the *Step 2* of Theorem 5.1. From now until Proposition 6.20, we prove facts that are standard for finite chains, but need a proof for  $Z_R$ .

For  $\eta \in \mathfrak{N}$ , we set  $v_\eta = \text{vol}(\Delta_\eta)$ . Using the fact that all the  $F_\xi$ 's have diameter less than  $\varepsilon$ , one can prove the following lemma (see [BP] for details). Recall that  $\mathfrak{N}$  depends on  $\mathcal{F}^\varepsilon$  and so it depends on  $\varepsilon$ .

**Lemma 6.15** *For any  $\varepsilon > 0$ , for large enough  $R$  we have that for any  $\eta \in \mathfrak{N}$*

- $a_R^+(\eta) \cdot a_R^-(\eta) = 0$ .
- $a_R(\eta) \neq 0 \implies a_R(\eta) \cdot v_\eta \geq 0$ .

**Lemma 6.16** *There exists a constant  $c$  such that  $|D_\rho^* \omega| < c|\omega|$ , where  $\omega$  is the volume-form of  $\mathbb{H}^3$ .*

*Proof.* Let  $M_0$  be  $M$  minus the cusps. The function  $|D_\rho^*\omega|/|\omega|$  is continuous and hence bounded on  $M_0$ . In the cusps, by direct calculation and using the cone property of  $D_\rho$ , one can show that the same bound holds.  $\square$

**Lemma 6.17** *The integrals  $\langle \omega, Z_R \rangle$  and  $\langle D_\rho^*\omega, Z_R \rangle$  are well-defined.*

*Proof.* As  $\sum |a_R(\eta)| < +\infty$ , since  $|\langle \omega, \Delta_\eta \rangle|$  is bounded by  $v_3$ , then  $\langle \omega, Z_R \rangle$  is well-defined. Consider now  $D_\rho^*\omega$ . From Lemma 6.16 it follows that the integral of  $|D_\rho^*|$  over straight geodesic simplices is bounded by  $cv_3$ . Hence also  $\langle D_\rho^*\omega, Z_R \rangle$  is well-defined.  $\square$

As above, let  $M_0$  denote  $M$  minus the cusps and, for  $k \in \mathbb{N}^*$  let

$$M_k = \bigcup_{T \subset \partial M_0} T \times [k-1, k].$$

Let  $\mathcal{F}_k^\varepsilon = \mathcal{F}^\varepsilon \cap \pi^{-1}(M_k)$  and  $\mathfrak{N}_k = \{\eta \in \mathfrak{N} : \xi_0 \in \mathcal{F}_k^\varepsilon\}$ . We have

$$Z_R = \sum_{k \in \mathbb{N}} \sum_{\eta \in \mathfrak{N}_k} a_R(\eta) \Delta_\eta.$$

**Lemma 6.18** *For any  $k$  the chain  $\sum_{\eta \in \mathfrak{N}_k} a_R(\eta) \Delta_\eta$  is a finite sum.*

*Proof.* If  $a_R(\eta) \neq 0$  and  $\eta \in \mathfrak{N}_k$  then  $\Delta_\eta$  is  $\varepsilon$ -close to an element  $X \in S(R)$  having first vertex in  $F_{\xi_0}$  with  $\xi_0 \in \mathcal{F}_k^\varepsilon$ . Since  $\mathcal{F}^\varepsilon$  is locally finite and  $\overline{M}_k$  is compact,  $\mathcal{F}_k^\varepsilon$  is finite, so there is only a finite number of possibilities for  $\xi_0$ . Since  $\overline{F}_{\xi_0}$  is compact, any  $X \in S(R)$  with first vertex in  $F_{\xi_0}$  lies on a compact ball  $B$  of  $\mathbb{H}^3$ . Since  $F$  is a fundamental domain, then there exists only a finite number of elements  $\gamma \in \Gamma$  so that  $\gamma(F)$  intersects  $B$ . Then for any  $\xi_0$  there is only a finite number of possibilities for  $\xi_1, \xi_2$  and  $\xi_3$ . It follows that there exists only a finite number of  $\eta \in \mathfrak{N}_k$  such that  $a_R(\eta) \neq 0$ .  $\square$

**Lemma 6.19** *For any  $R$ , if  $k$  is large enough, then for any  $\eta \in \mathfrak{N}_k$  with  $a_R(\eta) \neq 0$ , the simplex  $\Delta_\eta$  is completely contained in a cusp of  $M$ .*

*Proof.* If  $X = (x_0, \dots, x_3) \in S(R)$  then  $X$  lies in the ball  $B(x_0, 2R)$ . Since  $M$  has a finite number of cusps, for any  $R$  there exists  $m \in \mathbb{N}$  such that for  $k \geq m$  if  $x_0 \in M_k$  then the whole ball  $B(x_0, 2R + \varepsilon)$  is contained in the cusp

containing  $x_0$ . If  $\eta \in \mathfrak{N}_k$  and  $a_R(\eta) \neq 0$ , then there exists  $X \in S(R)$  with  $x_0 \in \pi^{-1}(M_k) \cap F$  hence  $\Delta_\eta$  is  $\varepsilon$ -close to  $X$ . Thus  $\Delta_\eta \subset B(x_0, 2R + \varepsilon)$  is contained in the cusp that contains  $x_0$ .  $\square$

Now for  $k \in \mathbb{N}$  define

$$Z_{R,k} = \sum_{j < k} \sum_{\eta \in \mathfrak{N}_j} a_R(\eta) \Delta_\eta.$$

$Z_{R,k}$  is a finite chain by Lemma 6.18. Moreover, since  $\partial Z_R = 0$ , then each simplex  $v$  of  $\partial Z_{R,k}$  appears as a face of a simplex  $\Delta_\eta$  with  $a_R(\eta) \neq 0$  and  $\eta \in \mathfrak{N}_j$  for some  $j \geq k$ . Therefore, by Lemma 6.19, for  $k$  large enough each simplex  $v$  of  $\partial Z_{R,k}$  is contained in a cusp of  $M$ . Thus to each  $v$  there corresponds an ideal point of  $\widehat{M}$ . For each  $v \in \partial Z_{R,k}$  let  $\lambda_{R,k}(v)$  be the coefficient of  $v$  in  $\partial Z_{R,k}$  and let  $C_v$  be the cone from  $v$  to the corresponding ideal point.

Let  $\widehat{Z}_{R,k}$  be the chain obtained by adding to  $Z_{R,k}$  the cones  $C_v$ :

$$\widehat{Z}_{R,k} = Z_{R,k} + \sum_{v \in \partial Z_{R,k}} \lambda_{R,k}(v) C_v.$$

The chain  $\widehat{Z}_{R,k}$  is a finite sum and it is easily checked that it is a cycle.

For any 3-simplex  $\Delta$  let  $\text{Strvol}(\Delta)$  denote the volume of the convex hull of the vertices of  $D_\rho(\Delta)$ . For any  $\eta \in \mathfrak{N}$  set  $w_\eta = \text{Strvol}(\Delta_\eta)$ .

**Proposition 6.20** *For any  $R > 0$  We have:*

$$\sum_{\eta} a_R(\eta) v_\eta = \langle \omega, Z_R \rangle = \langle D_\rho^* \omega, Z_R \rangle = \sum_{\eta} a_R(\eta) w_\eta.$$

*Proof.* The first equality is tautological. We use now the cycles  $\widehat{Z}_{R,k}$  to approximate  $Z_R$ . Since  $\text{vol}(\rho) = \text{vol}(M)$ , then  $[\omega] = [D_\rho^* \omega]$  as elements of  $H^3(\widehat{M})$ . Thus for any  $k \in \mathbb{N}$  we have  $\langle \omega, \widehat{Z}_{R,k} \rangle = \langle D_\rho^* \omega, \widehat{Z}_{R,k} \rangle$ . As in *Step 2* of Theorem 5.1, we can straighten the finite cycle  $\widehat{Z}_{R,k}$ , getting:

$$\langle \omega, \widehat{Z}_{R,k} \rangle = \langle D_\rho^* \omega, \widehat{Z}_{R,k} \rangle = \sum_{j < k} \sum_{\eta \in \mathfrak{N}_k} a_R(\eta) w_\eta + \sum_{v \in \partial Z_{R,k}} \lambda_{R,k}(v) \text{Strvol}(C_v).$$

For each simplex  $\alpha$  of  $\widehat{Z}_{R,k}$  we have  $|\text{vol}(\alpha)| \leq v_3$ ,  $|\text{Strvol}(\alpha)| \leq v_3$  and, by Lemma 6.16,  $|\langle D_\rho^* \omega, \alpha \rangle| \leq cv_3$ . It follows that to get the remaining inequalities it suffices to show that

$$\lim_{k \rightarrow \infty} \sum_{v \in \partial Z_{R,k}} |\lambda_{R,k}(v)| = 0.$$

Since  $\partial Z_R = 0$ , if  $v \in \partial Z_{R,k}$  then  $v \in \partial \Delta_\eta$  with  $a_R(\eta) \neq 0$  and  $\eta \in \mathfrak{N}_j$  for some  $j \geq k$ . So we have

$$\begin{aligned} \sum_{v \in \partial Z_{R,k}} |\lambda_{R,k}(v)| &\leq 4 \sum_{j \geq k} \sum_{\eta \in \mathfrak{N}_j} a_R^+(\eta) + a_R^-(\eta) = 4 \sum_{j \geq k} \sum_{\eta \in \mathfrak{N}_j} \mu\{s_R^{-1}(\eta)\} \\ &= 4 \sum_{j \geq k} \mu\{s_R^{-1}(\mathfrak{N}_j)\} = 4 \sum_{j \geq k} \mu\{Y \in S(R) : \exists \xi \in \mathcal{F}_j^\varepsilon, y_0 \in F_\xi\} \\ &= 4 \sum_{j \geq k} \sum_{\xi \in \mathcal{F}_j^\varepsilon} \text{vol}(F_\xi) \leq 4 \text{vol}\left(\bigcup_{j \geq k-\varepsilon} M_j\right) \end{aligned}$$

The last term goes to zero as  $k \rightarrow \infty$  because  $M$  has finite volume and the desired equality follows.  $\square$

Now that we have Proposition 6.20, forget about the cycles  $\widehat{Z}_{R,k}$ .

From triangular inequality, Proposition 6.20 and Lemma 6.15 we have

$$\begin{aligned} \sum_{\eta} |a_R(\eta)| \cdot |w_\eta| &\geq \left| \sum_{\eta} a_R(\eta) w_\eta \right| = \left| \sum_{\eta} a_R(\eta) v_\eta \right| \\ &= \sum_{\eta} |a_R(\eta)| \cdot |v_\eta| \geq \sum_{\eta} |a_R(\eta)| (v_3 - \epsilon(R, \varepsilon)) \end{aligned}$$

from which and Lemma 6.10 we get:

**Proposition 6.21** *For  $R$  large enough we have*

$$\sum_{\eta \in \mathfrak{N}} |a_R(\eta)| (v_3 - |w_\eta|) \leq \sum_{\eta \in \mathfrak{N}} |a_R(\eta)| \epsilon(R, \varepsilon) \leq \text{vol}(M) \epsilon(R, \varepsilon).$$

For any  $R > 0$  let  $A_R \subset \mathfrak{N}$  be the set of tetrahedra with “small” straight volume:

$$A_R = \{\eta \in \mathfrak{N} : v_3 - |w_\eta| > R^2 \cdot \text{vol}(M) \cdot \epsilon(R, \varepsilon)\}.$$

**Lemma 6.22** *For  $R$  large enough we have*

$$\sum_{\eta \in A_R} |a_R(\eta)| \leq \frac{1}{R^2}.$$

*Proof.* From Proposition 6.21 we get

$$\begin{aligned} R^2 \text{vol}(M) \epsilon(R, \varepsilon) \cdot \sum_{\eta \in A_R} |a_R(\eta)| &\leq \sum_{\eta \in A_R} |a_R(\eta)| (v_3 - |w_\eta|) \\ &\leq \sum_{\eta \in \mathfrak{N}} |a_R(\eta)| (v_3 - |w_\eta|) \leq \text{vol}(M) \epsilon(R, \varepsilon) \end{aligned}$$

The claimed inequality follows.  $\square$

**Lemma 6.23** *For almost all isometries  $g$  we have*

$$\lim_{n \rightarrow \infty} \text{Strvol}(g(Y_n)) = v_3.$$

*Proof.* Since  $a_R^+ \cdot a_R^- = 0$ , then  $\sum_{\eta \in A_R} |a_R(\eta)| = \mu(s_R^{-1}(A_R))$ . Thus for any fixed  $R > 0$  we have

$$\mu\left(\bigcup_{\mathbb{N} \ni n > R} s_R^{-1}(A_n)\right) \leq \sum_{n > R} \frac{1}{n^2} < \frac{1}{R}.$$

Recalling that for any set  $A \subset \mathfrak{N}$  we have  $\mu(s_R^{-1}(A)) = \mu(f_R^{-1}s_R^{-1}(A))$ , we get

$$\mu\{g \in \text{Isom}(\mathbb{H}^3) : \exists n > R, w_{s_n(g(Y_n))} < v_3 - n^2 \cdot \text{vol}(M) \cdot \epsilon(n, \varepsilon)\} < \frac{1}{R}.$$

From Lemma 6.13 it follows that  $\lim_{n \rightarrow \infty} n^2 \epsilon(n, \varepsilon) = 0$ . As  $R \rightarrow \infty$ , this implies that for any  $\varepsilon > 0$ , for almost any isometry  $g$  we have

$$\lim_{n \rightarrow \infty} w_{s_n(g(Y_n))} = v_3.$$

Let  $g$  be one of such maps. Since the diameters of the  $D_\rho(F_\xi)$  are bounded by  $\delta$ , then  $D_\rho(\Delta_{s_R(g(Y_R))})$  is  $\delta$ -close to  $D_\rho(g(Y_R))$ . Recalling that  $w_{s_R(g(Y_R))} = \text{Strvol}(\Delta_{s_R(g(Y_R))})$ , we have that

$$\lim_{n \rightarrow \infty} \text{Strvol}(\Delta_{s_n(g(Y_n))}) = v_3$$

and, since  $D_\rho(g(Y_R))$  is  $\delta$ -close to  $D_\rho(\Delta_{s_R(g(Y_R))})$ , then also

$$\lim_{n \rightarrow \infty} \text{Strvol}(g(Y_n)) = v_3.$$

□

We sketch here the proof of Proposition 6.6, referring to [D] for details.

*Proof of Proposition 6.6.* In the disc model let  $\gamma$  be a geodesic from 0 to a point in  $\partial\mathbb{H}^3$ . Let  $X_R$  be a family of regular tetrahedra of edge  $R$  with first vertex in 0 and second in  $\gamma(R)$ . All the claims from Lemma 6.10 to Lemma 6.23 hold for  $\{X_R\}$ . It follows that for almost all isometries  $g$  we have

$$\lim_{n \rightarrow \infty} \text{Strvol}(g(X_n)) = v_3.$$

Then  $D_\rho(g(\gamma(n)))$  must reach the boundary of  $\mathbb{H}^3$ . Using again the above property of the limit, one can estimate the angle  $\alpha(n)$  between the geodesic from  $D_\rho(g(0))$  to  $D_\rho(g(\gamma(n)))$  and the geodesic from  $D_\rho(g(0))$  and

$D_\rho(g(\gamma(n+1)))$ . Such estimate shows that  $\sum \alpha(n) < \infty$ , which implies that  $D_\rho(g(\gamma(n)))$  converges. The claim follows because  $D_\rho$  is locally Lipschitz outside the cusps. Measurability follows because the extension can be viewed as a point-wise limit of measurable functions.  $\square$

**Remark 6.24** In general  $D_\rho$  is not uniformly continuous in the cusps. So it cannot be locally Lipschitz on the whole  $\mathbb{H}^3$ .

We come now to the proof of Proposition 6.7.

**Lemma 6.25** *Let  $X = (x_0, x_1, x_2, x_3)$  be an ideal tetrahedron in  $\overline{\mathbb{H}^3}$ . Suppose that no three vertices of  $X$  coincide. Then for any  $\varepsilon > 0$  there exist neighborhoods  $U_i$  of  $x_i$  in  $\overline{\mathbb{H}^3}$  such that for any tetrahedron  $Y = (y_0, \dots, y_3)$  with  $y_i \in U_i$  we have  $|\text{vol}(Y) - \text{vol}(X)| < \varepsilon$ .*

This follows from the formula of the volume for ideal tetrahedra, see [BP] for details.

**Remark 6.26** Lemma 6.25 does not hold if three vertices of  $X$  coincide. To see this, let  $Y$  be a regular ideal tetrahedron and let  $\gamma$  be a parabolic or hyperbolic isometry. Then  $\gamma^n(Y)$  is a family of tetrahedra with maximal volume, but at least three of the vertices of  $\gamma^n(Y)$  converge to the same point.

**Lemma 6.27** *For almost all regular ideal tetrahedra  $Y$ , the ideal tetrahedron  $\overline{D}_\rho(Y)$  is defined. Moreover, for almost all  $Y$  either  $\overline{D}_\rho(Y)$  is regular (whence  $\text{vol}(\overline{D}_\rho(Y)) = v_3$ ) or at least three of its vertices coincide (whence  $\text{vol}(\overline{D}_\rho(Y)) = 0$ ).*

*Proof.* Without loss of generality, we can restrict the first claim to the space of positive regular ideal tetrahedra. We parametrize such a space with

$$\{(a, b, c) \in S_\infty^2 \times S_\infty^2 \times S_\infty^2 : a \neq b \neq c\}$$

where  $S_\infty^2 = \partial\mathbb{H}^3$ , by mapping  $(a, b, c)$  to the unique positive regular ideal tetrahedron with  $(a, b, c)$  as the first three vertices. We denote by  $Q(a, b, c)$  the fourth vertex of such tetrahedron. Since  $\overline{D}_\rho$  is defined almost everywhere, the first claim follows from Fubini's theorem. The second claim follows from Lemmas 6.23 and 6.25.  $\square$

With the above notation, by Lemma 6.27 we can restate Proposition 6.7 as follows.

**Proposition 6.28** *The set  $\{Y \in S_\infty^2 \times S_\infty^2 \times S_\infty^2 : \text{vol}(\overline{D}_\rho(Y)) = 0\}$  has zero measure.*

The proof of this result will follow from the next:

**Lemma 6.29** *If the set*

$$\{Y \in S_\infty^2 \times S_\infty^2 \times S_\infty^2 : \text{vol}(\overline{D}_\rho(Y)) = 0\}$$

*has positive measure, then the map  $\overline{D}_\rho$  is constant almost everywhere.*

Before proving Lemma 6.29 we show how it implies Proposition 6.28.

*Proof of 6.28.* By contradiction, we apply Lemma 6.29 deducing that  $\overline{D}_\rho$  is a.e. a constant  $p$ . From the equivariance of  $\overline{D}_\rho$  it follows that for any  $\gamma \in \Gamma$  and  $x \in \partial\mathbb{H}^3$  we have

$$p = \overline{D}_\rho \gamma(x) = \rho(\gamma)(\overline{D}_\rho(x)) = \rho(\gamma)(p).$$

Thus  $p$  is a fixed point of any element of  $\Gamma$ . This implies that the image of  $\rho$  is elementary, but this cannot happen because of Remark 6.4.  $\square$

We now prove Lemma 6.29.

**Lemma 6.30** *In the hypothesis of Lemma 6.29 there exists a positive-measure set  $A \subset S_\infty^2$  such that  $\overline{D}_\rho$  is constant on  $A$ .*

*Proof.* By Lemma 6.27 it is not restrictive to suppose that the set

$$\{(a, b, c) \in S_\infty^2 \times S_\infty^2 \times S_\infty^2 : \overline{D}_\rho(a) = \overline{D}_\rho(b) = \overline{D}_\rho(c)\}$$

has positive measure. Then by Fubini's theorem there exists a positive-measure set  $A_0 \subset S_\infty^2$  such that for all  $a_0 \in A_0$  the set

$$\{(b, c) \in S_\infty^2 \times S_\infty^2 : \overline{D}_\rho(a_0) = \overline{D}_\rho(b) = \overline{D}_\rho(c)\}$$

has positive measure in  $S_\infty^2 \times S_\infty^2$ . Again by Fubini's theorem for all  $a_0 \in A_0$  there exists a positive-measure set  $A_1 \in S_\infty^2$  such that for any  $a_1 \in A_1$  the set

$$\{c \in S_\infty^2 : \overline{D}_\rho(a_0) = \overline{D}_\rho(a_1) = \overline{D}_\rho(c)\}$$

has positive measure. In particular  $\overline{D}_\rho$  is constant on  $A_1$ .  $\square$

We set  $p = \overline{D}_\rho(A_1)$  and  $A = \overline{D}_\rho^{-1}(p)$ .

**Remark 6.31** In the sequel we use the symbol  $\tilde{\forall}$  to mean “for almost all.”

By Lemma 6.27 the set  $A$  has the following property

$$\tilde{\forall}(a_0, a_1, x) \in A \times A \times A^c, \quad Q(a_0, a_2, x) \in A.$$

We work now in the half space model  $\mathbb{C} \times \mathbb{R}^+$  of  $\mathbb{H}^3$ . So  $S_\infty^2 = \mathbb{C} \cup \{\infty\}$ . In that model

$$Q(\infty, a, z) = \alpha(z - a) + a$$

where  $\alpha = (1 + i\sqrt{3})/2$ . Again by Fubini’s theorem  $\tilde{\forall}a_0 \in A$ ,  $\tilde{\forall}(a_1, x) \in A \times A^c$ , we have  $Q(a_0, a_1, x) \in A$  and we can suppose that this holds for  $a_0 = \infty$ .

In other words, for almost all  $(a, x) \in A \times A^c$  the third vertex of the equilateral triangle with the first two vertices in  $a$  and  $x$  is in  $A$ . For any  $a, x \in \mathbb{C}$  we call  $E_x(a)$  the set of the vertices of the regular hexagon centered at  $x$  and with a vertex in  $a$ . Then we have

$$\tilde{\forall}(a, x) \in A \times A^c, \quad E_x(a) \subset A \tag{1}$$

and in particular  $\tilde{\forall}(a, x) \in A \times A^c$ ,  $2x - a \in A$ . Note that  $x$  is the middle-point of the segment between  $a$  and  $2x - a$ .

**Lemma 6.32** For any open set  $B \subset \mathbb{C}$  we have  $\mu(A \cap B) > 0$ .

*Proof.* Suppose the contrary. Then there exists an open set  $B$  such that  $\mu(A \cap B) = 0$ . That is, almost all the points of  $B$  are in  $A^c$ . Moreover, from (1) and Fubini’s theorem it follows that  $\tilde{\forall}x \in A^c, \tilde{\forall}a \in A, E_x(a) \in A$ . Therefore there exists a point  $x_0 \in B$  such that a small ball  $B_0 = B(x_0, r_0)$  is contained in  $B$  and

$$\tilde{\forall}a \in A, \quad E_{x_0}(a) \in A. \tag{2}$$

Since  $\mu(A) > 0$  then there exists a small ball  $B_1 = B(x_1, r_1)$  such that  $\mu(A \cap B_1) > 0$ . Let  $x_2 = (x_1 + x_0)/2$ . If there exists  $r > 0$  such that  $\mu(A \cap B(x_2, r)) = 0$ , then applying the same argument we can find a point  $y$  arbitrarily close to  $x_2$  such that (2) holds for  $y$ . In particular we get that almost all the points of the set  $C = \{2y - a : a \in B_1 \cap A\}$  are in  $A$ . But if  $y$  is close enough to  $x_2$  then  $C \cap B_0$  has positive measure, contradicting that  $\mu(A \cap B) = 0$ .

It follows that for all  $r_2 > 0$  we have  $\mu(A \cap B(x_2, r_2)) > 0$ , in particular we choose  $r_2 < r_0/2$ . By iterating this construction, we find a sequence of points  $x_n \rightarrow x_0$  and radii  $r_0/2 > r_n > 0$  such that  $\mu(A \cap B(x_n, r_n)) > 0$ . For  $n$  large enough this contradicts the fact that  $\mu(A \cap B) = 0$ . □

**Lemma 6.33** For all  $z \in \mathbb{C}$  we have

$$\forall r > 0 \quad \mu(B(z, r) \cap A) \geq \frac{1}{2}\mu(B(z, r)). \quad (3)$$

*Proof.* From Fubini's theorem, and condition (1), it follows that for almost all  $a \in A$  we have

$$\tilde{\forall} x \in A^c, \quad E_x(a) \subset A. \quad (4)$$

Note that if (4) holds for  $a$ , then (3) holds for  $a$ .

Let  $z \in \mathbb{C}$ . From Lemma 6.32 it follows that there exists a sequence  $x_n \rightarrow z$  such that (4) (and hence (3)) holds for  $x_n$ . As the function  $x \mapsto \mu(A \cap B(x, r))$  is continuous, then the claim holds for  $z$ . □

**Lemma 6.34** Let  $X \subset \mathbb{R}^2$  be a measurable set. If there exists  $\alpha > 0$  such that for any ball  $B$

$$\mu(B \cap X) \geq \alpha\mu(B)$$

then  $\mu(\mathbb{R}^2 \setminus X) = 0$ .

This is a standard fact of integration theory and it follows from Lebesgue's differentiation theorem (see for example [Ru]).

From this lemma and Lemma 6.33 it follows that the set  $A$  has full measure. Since  $A = \overline{D}_\rho^{-1}(p)$  then  $\overline{D}_\rho$  is constant almost everywhere and Lemma 6.29 is proved. □

This completes the proof of Theorem 6.1.

## 7 Corollaries

In this section we prove some corollaries that can be useful for studying hyperbolic 3-manifolds.

First we show how from Theorem 6.1 one gets a proof of Mostow's rigidity for non-compact manifolds (see [P] and [BCS] for a more general statement and a different proof).

**Theorem 7.1 (Mostow's rigidity for non-compact manifold)** *Let  $f : M \rightarrow N$  be a proper map between two orientable non-compact, complete hyperbolic 3-manifolds of finite volume. Suppose that  $\text{vol}(M) = \text{deg}(f)\text{vol}(N)$ . Then  $f$  is properly homotopic to a locally isometric covering with the same degree as  $f$ .*

*Proof.* Let  $\omega$  be the volume form of  $N$ . For  $X = M, N$  let  $\Gamma_X \cong \pi_1(X)$  be the subgroup of  $\mathrm{PSL}(2, \mathbb{C})$  such that  $X = \mathbb{H}^3/\Gamma_X$ . Let  $f_*$  denote both the map induced in homology and the representation  $f_* : \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ .

First assume that the lift  $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$  has the cone-property on the cusps. This implies that  $\tilde{f}$  is a pseudo-developing map for  $f_*$ . Since  $f_*[M] = \mathrm{deg}(f)[N]$  we have

$$\begin{aligned} \mathrm{vol}(M) &= \mathrm{deg}(f) \cdot \mathrm{vol}(N) = \langle \omega, \mathrm{deg}(f)[N] \rangle \\ &= \langle \omega, f_*[M] \rangle = \langle f^*\omega, [M] \rangle = \mathrm{vol}(f_*). \end{aligned}$$

Thus, by Theorem 6.1 there exists an isometry  $\varphi$  such that for any  $\gamma \in \Gamma_M$

$$f_*(\gamma) = \varphi \circ \gamma \circ \varphi^{-1}.$$

As  $\tilde{M} \cong \mathbb{H}^3$ , we consider the isometry  $\varphi$  as an  $f_*$ -equivariant map from  $\tilde{M}$  to  $\mathbb{H}^3$ . Namely, for any  $x \in \mathbb{H}^3$  and  $\gamma \in \Gamma_M$

$$\varphi(\gamma(x)) = f_*(\gamma)(\varphi(x)).$$

It follows that  $\varphi$  projects to a locally isometric covering  $\varphi : M \rightarrow N$  and the convex combination from  $\tilde{f}$  to  $\varphi$  projects to a proper homotopy from  $f$  to  $\varphi$ . Since the degree of a map is invariant under proper homotopies, then  $\mathrm{deg}(\varphi) = \mathrm{deg}(f)$ .

We prove now that  $f$  is always properly homotopic to a map whose lift has the cone property on the cusps. Let  $\tilde{f}$  be a lift of  $f$ . For each cusp  $N_p = P_p \times [0, \infty)$  let  $f_p = \tilde{f}|_{P_p \times \{0\}}$ . Since  $f$  is proper it follows that  $\tilde{f}(N_p \times \{\infty\})$  is well-defined. Let  $F_p : N_p \times [0, \infty) \rightarrow \mathbb{H}^3$  be the map obtained by coning  $f_p$  to  $\tilde{f}(N_p \times \{\infty\})$  along geodesic rays. Let  $\tilde{f}'$  be the map obtained by replacing, on each cusp  $N_p$ , the map  $\tilde{f}|_{N_p}$  with the map  $F_p$ . The map  $\tilde{f}'$  obviously has the cone-property on the cusps, and projects to a map  $f' : M \rightarrow N$ . Moreover, the convex combination from  $\tilde{f}$  to  $\tilde{f}'$  projects to a proper homotopy between  $f$  and  $f'$ .  $\square$

From Theorem 6.1, Theorem 7.1, Corollary 5.11 and the corresponding statements for compact manifolds, we get the following statement.

**Theorem 7.2** *Let  $M$  be a complete, oriented hyperbolic 3-manifold of finite volume. Let  $\Gamma \cong \pi_1(M)$  be the sub-group of  $\mathrm{PSL}(2, \mathbb{C})$  such that  $M = \mathbb{H}^3/\Gamma$ . Let  $\rho : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{C})$  be a representation. Then  $|\mathrm{vol}(\rho)| \leq |\mathrm{vol}(M)|$  and equality holds if and only if  $\rho$  is discrete and faithful.*

**Corollary 7.3** *Let  $M$  be an atoroidal, irreducible, ideally triangulated 3-manifold. Let  $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$  be a solution of the hyperbolicity equations such that  $\mathrm{vol}(\mathbf{z}) \neq 0$ . Then  $M$  is hyperbolic.*

*Proof.* This immediately follows from Corollary 5.10 and Thurston's Hyperbolization Theorem ([Th2]).  $\square$

In [F] the notion of *geometric* solution of the hyperbolicity equations is introduced. Roughly speaking, a geometric solution of the hyperbolicity equations for a given ideal triangulation  $\tau$  is a choice of moduli which is compatible with a global hyperbolic structure on  $M$ . In [F] it is shown that not each solution of hyperbolicity equations is geometric (see [F] for more details on algebraic and geometric solutions of hyperbolicity equations).

**Corollary 7.4** *Let  $M$  be a complete hyperbolic 3-manifold of finite volume and let  $\tau$  be an ideal triangulation of  $M$ . If there exists a solution  $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$  of the hyperbolicity equations for  $\tau$ , then there exists a solution  $\mathbf{z}'$  of hyperbolicity equations that is geometric. Moreover such a solution is the one of maximal volume.*

*Proof.* Consider a natural straightening of  $\tau$ , and let  $\mathbf{z}'$  be the moduli induced on  $\tau$ . By Proposition 3.7, we have only to prove that the moduli are not in  $\{0, 1, \infty\}$ . Suppose that there is a degenerate tetrahedron  $\Delta_i$ . Then at least two vertices, say  $v$  and  $w$ , of  $\Delta_i$  coincide.

Let  $\rho(\mathbf{z})$  be the holonomy relative to  $\mathbf{z}$  and let  $D_{\mathbf{z}}$  be a developing map that is also a pseudo-developing map for  $\rho(\mathbf{z})$ . Then  $D_{\mathbf{z}}$  maps  $\Delta_i$  into a tetrahedron of modulus  $z_i$ . But by hypothesis,  $\mathbf{z}$  is in  $\{\mathbb{C} \setminus \{0, 1\}\}^n$  and so the vertices of  $\Delta_i$  are four distinct points. The last assertion follows from Corollary 5.11 and Theorem 6.1  $\square$

Corollary 7.4 tells that, once one has a solution  $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$  of the hyperbolicity equations for a triangulation  $\tau$  of a cusped manifold  $M$ , in order to know if  $M$  admits a complete hyperbolic structure of finite volume, it suffices to study the solution of maximal volume. Namely, if one succeeds to prove that the solution of maximal volume is geometric, then  $M$  is hyperbolic. Conversely, if one proves that such a solution is not geometric (for example if its holonomy is not discrete) then  $M$  cannot be hyperbolic, and this does not depend on the chosen triangulation.

As an example of application of Corollary 7.4 we give the following:

**Corollary 7.5** *Let  $M$  be a cusped 3-manifold equipped with an ideal triangulation  $\tau$ . If there exists a solution  $\mathbf{z} \in \{\mathbb{C} \setminus \{0, 1\}\}^n$  of the hyperbolicity equations for  $\tau$ , and all the solutions have zero volume, then  $M$  is not hyperbolic.*

We notice that the hypothesis that all the solutions have zero volume can be replaced by requiring that the volumes are *too small*. This is because the set of the volumes of the hyperbolic manifolds is bounded from below by a positive constant.

Finally, we obtain another proof of the well-know fact that no Dehn filling of a Seifert manifold is hyperbolic.

**Corollary 7.6** *Let  $M$  be a 3-manifold such that  $|(M, \partial M)| = 0$  and let  $N$  be a Dehn filling of  $M$ . Then  $N$  is not hyperbolic.*

*Proof.* Suppose the contrary. Let  $\rho$  be the holonomy of the hyperbolic structure of  $N$ . From Theorem 5.1 it follows that  $\text{vol}(\rho) = 0$ , but from Proposition 4.20 and Corollary 4.15 it follows that  $\text{vol}(\rho) = \text{vol}(N) > 0$ . □

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