# Compactness type properties and extensions of topological groups*

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**Abstract**

We study compact, countably compact, pseudocompact, and functionally bounded sets in extensions of topological groups. A property $\mathcal{P}$ is said to be a *three space property* if, for every topological group $G$ and a closed invariant subgroup $N$ of $G$, the fact that both groups $N$ and $G/N$ have $\mathcal{P}$ implies that $G$ also has $\mathcal{P}$. It is shown that if all compact (countably compact) subsets of the groups $N$ and $G/N$ are metrizable, then $G$ has the same property. However, the result cannot be extended to pseudocompact subsets, a counterexample exists under $CH$. Another example shows that extensions of groups do not preserve the classes of realcompact, Dieudonné complete and $\mu$-spaces: one can find a pseudocompact, non-compact Abelian topological group $G$ and an infinite, closed, realcompact subgroup $N$ of $G$

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such that $G/N$ is compact and all functionally bounded subsets of $N$ are finite. Several examples given in the article destroy a number of tempting conjectures about extensions of groups.

# 1 Introduction

Let $\mathcal{P}$ be a (topological, algebraic, or a mixed nature) property. We say that $\mathcal{P}$ is a *three space property* if the following holds: whenever $N$ is a closed invariant subgroup of a topological group $G$ and both $N$ and $G/N$ have $\mathcal{P}$, the group $G$ also has $\mathcal{P}$. Compactness, precompactness, pseudocompactness, completeness, connectedness and metrizability are three space properties (see [10, 22, 33]). However, having a countable network, $\sigma$-compactness, Lindelöfness, countable compactness, sequential compactness, sequential completeness and $\omega$-compactness are not three space properties (for the first three properties, this follows from an example given in [40], while for the last four properties, see [9]).

Here we study some properties of compact, countably compact, pseudocompact, and functionally bounded sets which are preserved or destroyed when taking extensions of topological groups. Extending the above terminology, we say that $\mathcal{P}$ is a *three space property for compact (countably compact) sets* if all compact (countably compact) subsets of a topological group $G$ have $\mathcal{P}$ whenever $G$ contains a closed invariant subgroup $N$ such that the compact (countably compact) subsets of both groups $N$ and $G/N$ have $\mathcal{P}$.

In many cases, a given topological property $\mathcal{P}$ is a three space property by a purely topological reason: $\mathcal{P}$ turns out to be an *inverse fiber property* (see Definition 2.1). In Section 2, we present several inverse fiber properties and finish the section with Theorem 2.16 which contains a list of seventeen three space properties.

We apply Theorem 2.16 in Section 3 to show that metrizability is a three space property for compact sets (see Theorem 3.2), which complements the fact that extensions of topological groups preserve metrizability. In Corollary 3.3 we establish that metrizability is a three space property for countably compact sets. However, one cannot extend the result to pseudocompact subsets: under $CH$, there exist an Abelian topological group $G$ and a closed subgroup $N$ of $G$ such that all pseudocompact subspaces of $N$ are finite, the quotient group $G/N$ is countable, but $G$ contains a pseudocompact subspace of uncountable character (see Example 4.8). In addition, according to Example 3.4, metrizability of separable subsets of compact sets (that is, $\aleph_0$-monolithicity of compact subsets) is not a three space property.

In Section 4, we present several examples that destroy a number of tempting conjectures about extensions of topological groups. A space $X$ is called *$P$-compact* (*$B$-compact*) if the closure of every pseudocompact (functionally bounded) subset of $X$ is compact. Similarly, one defines the classes of *$P$-closed* and *$B$-closed* spaces in which all pseudocompact (resp., functionally bounded) subsets are closed. It is worth mentioning that $P$-closed spaces are also known as spaces with the *Preiss–Simon property*, while $B$-compact spaces are usually called *$\mu$-spaces*. We adopt a distinct terminology here just to provide the reader with a simple mnemonic way of interpreting the terms.

We show in Examples 4.1 and 4.5 that the properties of being $P$-compact, $B$-compact or $B$-closed are not three space properties. Example 4.1 implies that realcompactness and Dieudonné completeness are not three space properties either. In Example 4.8 we construct, under the Continuum Hypothesis, an Abelian topological group $G$ and a closed subgroup $N$ of $G$ such that all pseudocompact subspaces of the groups $N$ and $G/N$ are compact and metrizable, but $G$ contains a pseudocompact subspace of uncountable character as well as a non-closed pseudocompact subspace. In particular, the class of $P$-closed topological groups is not stable under extensions (under $CH$).

The next diagram presents more information on the behavior of various properties under taking extensions of topological groups. For example, in the intersection of the second column (entitled "countably compact") and the third line (started with "closed") we consider the property "all *countably compact* subsets are *closed*". The conclusion is: Theorem 2.16 implies that this is a three space property. If no reference to a specific result is given, then the preservation of a property under extensions is trivial (or almost evident).

| | compact | countably compact | pseudocompact | functionally bounded |
|---|---|---|---|---|
| metrizable | Yes Th. 3.2 | Yes Coro. 3.3 | No, Ex. 4.8 (under CH) | No, Ex. 4.5 |
| first countable | Yes Th. 2.16 | Yes Th. 2.16 | No, Ex. 4.8 (under CH) | No, Ex. 4.5 |
| closed | Yes | Yes Th. 2.16 | No, Ex. 4.8 (under CH) | No, Ex. 4.5 |
| compact | Yes | Yes Th. 2.16 | No, Ex. 4.8 (under CH) | No, Ex. 4.5 |
| compact closure | Yes | No Ex. 4.10 | No, Ex. 4.1 | No, Ex. 4.5 |
| countably compact closure | Yes | No Ex. 4.11 | No, Ex. 4.8 (under CH) | No, Ex. 4.5 |
| pseudocompact closure | Yes | Yes | Yes | No, Ex. 4.5 |
| finite | Yes | Yes | Yes Th. 2.16 | No, Ex. 4.5 |

The diagram shows in a clear form how the stability of the properties decreases when moving from the “compact” column to the “functionally bounded” column.

## 1.1 Notation and terminology

All topological groups we consider are assumed to be Hausdorff, and topological spaces are Tychonoff. A *supersequence* in a space is an infinite compact subset with a single non-isolated point. Clearly, every supersequence is a one-point compactification of an infinite discrete set. A sequence $\{x_n : n \in \omega\}$ in a space $X$ is said to be *trivial* if it is eventually constant.

A space $X$ is *scattered* if every non-empty subset of $X$ contains an isolated point. Following [15] we say that $X$ is *left-separated* if there exists a well ordering $<$ on $X$ such that the set $X_{<x} = \{y \in X : y < x\}$ is closed in $X$ for each $x \in X$.

A subset $B$ of a space $X$ is called *functionally bounded* in $X$ if the image $f(B)$ is bounded in the real line $\mathbb{R}$, for every continuous function $f\colon X \to \mathbb{R}$. It is clear that pseudocompact subspaces of $X$ are functionally bounded, but not vice versa.

A subset $Y$ of a space $X$ is $G_\delta$*-dense in* $X$ if every non-empty $G_\delta$-set in $X$ intersects $Y$. It is known that $X$ is pseudocompact if and only if $X$ is $G_\delta$-dense in the Čech–Stone compactification $\beta X$ of the space $X$.

A space $X$ is called $\omega$*-bounded* if every closed separable subset of $X$ is compact or, equivalently, the closure of every countable set is compact. Every $\omega$-bounded space is countably compact, but the converse is clearly false.

The character and pseudocharacter of a point $x$ in a space $X$ is denoted by $\chi(x, X)$ and $\psi(x, X)$, respectively. Similarly, if $F \subseteq X$, then $\chi(F, X)$ and $\psi(F, X)$ are the character and pseudocharacter of $F$ in $X$. The weight, network weight, density and tightness of a space $X$ are denoted by $w(X)$, $nw(X)$, $d(X)$, and $t(X)$, respectively.

The neutral element of a group $G$ is usually denoted by $e_G$ or simply $e_G$. A group $G$ is *Boolean* if $x^2 = e$ for each $x \in G$. All Boolean groups are Abelian. We will use additive notation for Abelian groups. The subgroup of a group $G$ generated by a subset $A$ of $G$ is denoted by $\langle A\rangle$. Given $x \in G$, we use $\langle x\rangle$ for the minimal subgroup of $G$ which contains $x$. If $G$ is an abstract group and $\tau$ is a cardinal, $G^{(\tau)}$ denotes the direct sum of $\tau$ copies of the group $G$. The kernel of a homomorphism $\pi\colon G \to H$ is $\ker \pi$.

A topological group $G$ is *precompact* if $G$ can be covered by finitely many translates of an arbitrary neighborhood of the identity. Every pseudocompact group is precompact by [10, Theorem 1.1]. Similarly, a subset $P$ of

$G$ is *precompact* if, for every neighborhood $U$ of the identity in $G$, there exists a finite subset $F$ of $G$ such that $P \subseteq (F \cdot U) \cap (U \cdot F)$. A set $P$ is precompact in $G$ if and only if the closure of $P$ in the completion $\widetilde{G}$ of the group $G$ is compact. A topological group $G$ will be called *$\omega$-narrow* if, for every neighborhood $U$ of the identity in $G$, one can find a countable subset $C$ of $G$ such that $G = C \cdot U$ (earlier, the term *$\aleph_0$-bounded* was used for the groups with this property, which was misleading in occasions).

Given a set $X$ and an infinite cardinal $\tau$, we put $[X]^\tau = \{Y \subseteq X : |Y| = \tau\}$ and $[X]^{<\tau} = \{Y \subseteq X : |Y| < \tau\}$. The cardinality of the continuum is denoted by $\mathfrak{c}$, so $\mathfrak{c} = 2^\omega$.

# 2 Inverse fiber properties and extensions of groups

We start with the following definition which plays an important role in the article.

**Definition 2.1** A topological property $\mathcal{P}$ will be called an *inverse fiber property* if the following holds:

(IFP) If $f\colon X \to Y$ is a continuous onto mapping such that the space $Y$ and the fibers of $f$ have $\mathcal{P}$, then $X$ also has $\mathcal{P}$.

If the conclusion in (IFP) holds under the additional assumption that the domain $X$ is compact (countably compact), we say that $\mathcal{P}$ is an *inverse fiber property for compact (countably compact) sets.*

In the next two simple but important observations we establish a relation between inverse fiber properties and three space properties.

**Proposition 2.2** *Every inverse fiber property is a three space property.*

*Proof.* Suppose that $N$ is a closed invariant subgroup of a topological group $G$ and that both groups $N$ and $G/N$ have an inverse fiber property $\mathcal{P}$. Let $\pi\colon G \to G/N$ be the quotient homomorphism. If $y \in G/N$, take an element $x \in G$ with $\pi(x) = y$. Then $\pi^{-1}(y) = xN$ is a homeomorphic copy of $N$, so the fiber $\pi^{-1}(y)$ has $\mathcal{P}$ for each $y \in G/N$. Since $\mathcal{P}$ is an inverse fiber property, the space $G$ also has $\mathcal{P}$. □

**Proposition 2.3** *Let $\mathcal{P}$ be an inverse fiber property for compact (countably compact) sets. Then $\mathcal{P}$ is a three space property for compact (countably compact) sets.*

*Proof.* Apply the argument in the proof of Proposition 2.2 to a compact (countably compact) subset of a topological group $G$. □

Below we present several inverse fiber properties (for compact sets). Let us start with considering very simple compact sets.

**Proposition 2.4** *The following are inverse fiber properties:*

(1) *all compact (countably compact, pseudocompact) subsets are finite;*

(2) *all convergent sequences are trivial (that is, contrasequentiality);*

(3) *all supersequences have length less than a given cardinal $\tau \geq \omega$.*

*Proof.* Let $f\colon X \to Y$ be a continuous onto mapping. In the case of compactness and countable compactness, (1) is trivially an inverse fiber property. Suppose, therefore, that all pseudocompact subsets of $Y$ and of the fibers $f^{-1}(y)$ are finite. If $P$ is a pseudocompact subset of $X$, then $f(P) \subseteq Y$ is pseudocompact, hence finite. This implies that $\gamma = \{P \cap f^{-1}(y) : y \in f(P)\}$ is a finite partition of $P$ into disjoint clopen subsets, so that each element of $\gamma$ is pseudocompact and finite. We conclude that $P$ is finite as well.

Clearly, (2) is a special case of (3), so we only prove that (3) is an inverse fiber property. Suppose that all supersequences in $Y$ and in the fibers $f^{-1}(y)$ have length strictly less than an infinite cardinal $\tau$. Let $S$ be a supersequence in $X$ with a single non-isolated point $x_0 \in X$. Clearly, the image $T = \pi(S)$ is either finite or a supersequence in $Y$. In either case, $|T| < \tau$. The set $K = f^{-1}(f(x_0)) \cap S$ is closed in $S$, so $K$ is a supersequence (if infinite). Therefore, by our assumptions, the set $K \subseteq f^{-1}(f(x_0))$ has cardinality strictly less than $\tau$. In addition, if $y \in T \setminus \{f(x_0)\}$, then $f^{-1}(y) \cap S$ is a compact discrete set, so it must be finite. We conclude that $S$ is the union of at most $|T|$ finite sets and the set $K$ of cardinality less than $\tau$. This immediately implies that $|S| < \tau$. □

Let us say that a space $X$ is *path-free* if $X$ does not contain a copy of the closed unit interval $I = [0, 1]$. The next result complements Proposition 2.4.

**Proposition 2.5** *The property of being path-free is an inverse fiber property.*

*Proof.* Let $f\colon X \to Y$ be a continuous onto mapping. Suppose that $X$ contains a topological copy $C$ of the interval $I$. If the image $K = f(C)$ contains more than one point, then $K$ also contains a copy of the unit interval by [14, 6.3.12]. If $|K| = 1$, then $C$ lies in a fiber of $f$. This finishes the proof. □

Given an ordinal $\alpha$, we endow it with the topology generated by the well ordering $<$ of $\alpha$, that is, $\gamma < \beta$ iff $\gamma \in \beta$.

**Proposition 2.6** *Let $f: X \to Y$ be a continuous onto mapping. If $Y$ and the fibers of $f$ do not contain a copy of the space $\omega_1$, then neither does $X$. A similar conclusion remains valid with $\omega_1 + 1$ in place of $\omega_1$.*

*Proof.* It suffices to prove the proposition for $\omega_1$; the argument for $\omega_1 + 1$ is almost the same. Suppose to the contrary that $X$ contains a subspace $P$ homeomorphic to $\omega_1$. If the set $F = P \cap f^{-1}f(x)$ is uncountable for some $x \in P$, then $F$ is a homeomorphic copy of $\omega_1$ in the fiber $f^{-1}f(x)$, which is impossible. Hence the sets $P \cap f^{-1}f(x)$ are countable for all $x \in P$. Then one can find a closed unbounded subset $C$ of $P \cong \omega_1$ such that the restriction of $f$ to $C$ is a homeomorphism of $C$ onto $f(C)$, which is again contradictory. □

**Proposition 2.7** *The following are inverse fiber properties:*

(1) *being scattered;*

(2) *being left-separated.*

*Proof.* Both (1) and (2) follow directly from [37, Prop. 4]. □

The next result will be applied in Section 3 to show that metrizability is a three space property for compact and countably compact sets.

**Proposition 2.8** *The first axiom of countability is an inverse fiber property for compact and countably compact sets.*

*Proof.* Let $f: X \to Y$ be a continuous onto mapping. It suffices to deduce the conclusion for countably compact sets; the argument in the case of compact sets is even simpler. Suppose that all countably compact subsets of the space $Y$ and of the fibers $f^{-1}(y)$ are first countable. Let $C$ be a countably compact subspace of $X$. Then the image $K = f(C)$ is countably compact and, hence, satisfies $\chi(K) \leq \omega$. Take an arbitrary point $x \in C$ and put $y = f(x)$. Then $\chi(y, K) \leq \omega$. Denote by $g$ the restriction of $f$ to $C$. Since the space $g(C) = f(C)$ is first countable, $g$ is a closed mapping. The set $C_x = g^{-1}(y) = C \cap f^{-1}f(x)$ is countably compact as a closed subset of $C$, so $\chi(C_x) \leq \omega$. We have $\chi(g(x), K) \leq \omega$ and $\chi(x, g^{-1}g(x)) \leq \omega$, whence it follows that $\chi(x, C) \leq \omega$ by [14, 3.7.E]. This proves that $\chi(C) \leq \omega$. □

Let $Char(\omega)$ be the following property of a space $X$: for every non-empty closed subset $F$ of $X$, there exists a point $x \in F$ such that $\chi(x, F) \leq \omega$. An argument similar to that in the above proof implies the next result:

**Proposition 2.9** *$Char(\omega)$ is an inverse fiber property for compact and countably compact sets.*

A space $X$ is called *$C$-closed* if every countably compact subset of $X$ is closed [24]. It is easy to verify that the product of a finite family of $C$-closed spaces is $C$-closed, that is, the property of being $C$-closed is *finitely productive.* We refine this result in the proposition below.

**Proposition 2.10** *The property of being $C$-closed is an inverse fiber property.*

*Proof.* Let $f\colon X \to Y$ be a continuous onto mapping such that the space $Y$ and the fibers of $f$ are $C$-closed. Suppose to the contrary that $C$ is a countably compact non-closed subset of $X$ and take a point $x \in \overline{C} \setminus C$. The set $K = C \cap f^{-1}f(x)$ is countably compact as a closed subset of $C$ and, since the fiber $f^{-1}f(x)$ is $C$-closed, $K$ is closed in $X$. Since $x \notin K$, we can choose an open neighborhood $U$ of $x$ in $X$ such that $\overline{U} \cap K = \emptyset$. Then $D = \overline{U} \cap C$ is countably compact (as a closed subset of $C$) and $x \in \overline{D} \setminus D$. It follows from our choice of $U$ that $D \cap K = \emptyset$ and $f(x) \in \overline{f(D)} \setminus f(D)$, so $f(D)$ is a non-closed, countably compact subset of $Y$. This contradicts our assumption about $Y$. □

Another kind of an inverse fiber property for compact sets is given below. As usual, we use $BL$ to abbreviate Booth's Lemma (see [34, 25]).

**Corollary 2.11** *Suppose that $BL$ or $2^\omega < 2^{\omega_1}$ holds. Then sequentiality is an inverse fiber property for compact sets.*

*Proof.* By [24, Theorem 1.24], a compact space is sequential iff it is $C$-closed, under the assumption that $BL$ or $2^\omega < 2^{\omega_1}$ holds. [Formally, Martin's Axiom $MA$ is required in [24] instead of the weaker $BL$, but one can easily verify that only $BL$ is applied there.] The required conclusion now follows from Proposition 2.10. □

The next result concerns zero-dimensionality of compact subsets. Since the equalities $\dim X = 0$, $\operatorname{ind} X = 0$, and $\operatorname{Ind} X = 0$ are equivalent for a compact space $X$ (see [14, Th. 7.1.11]), we do not specify which dimension is meant below.

**Proposition 2.12** *Zero-dimensionality is an inverse fiber property for compact sets.*

*Proof.* Let $f: X \to Y$ be a continuous onto mapping of compact spaces and suppose that the space $Y$ and the fibers of $f$ are zero-dimensional. Then $f$ is a closed and zero-dimensional. Since $X$ and $Y$ are compact, the formula $\dim X \leq \dim Y + \dim f$ implies that $X$ is also zero-dimensional (see [27]). □

Let us say that $X$ is a *$CK$-space* if every countably compact subspace of $X$ is compact. It is known that many spaces of continuous real-valued functions endowed with the pointwise convergence topology are $CK$-spaces [7]. It turns out that the class of $CK$-spaces is stable when taking preimages with good fibers:

**Proposition 2.13** *The property of being a $CK$-space is an inverse fiber property.*

*Proof.* Let $f: X \to Y$ be a continuous onto mapping and suppose that the space $Y$ and the fibers of $f$ are $CK$-spaces. Consider an arbitrary countably compact subset $C$ of $X$. Then the image $D = f(C)$ is a countably compact subspace of $Y$, so $D$ is compact. In addition, if $y \in D$, then $C_y = C \cap f^{-1}(y)$ is countably compact as a closed subset of $C$. Hence $C_y \subseteq f^{-1}(y)$ is compact. So, $g = f{\restriction}_C$ is a continuous mapping with compact fibers. Note that the image $g(K) = f(K)$ is compact (hence closed in $Y$ and $D$) for every closed subset $K$ of $C$, so $g: C \to D$ is a perfect mapping. Since the image $D = g(C)$ is compact, we conclude that $C$ is also compact. □

In Theorem 2.15 below we present two profound results about inverse fiber properties for compact sets established by Arhangel'skii for the tightness and by Shapirovskiĭ for the *dyadicity index*, respectively. First, we recall the definition of the dyadicity index $\nabla\varrho$ introduced in [35]. Given a space $X$, one defines

$$\nabla\varrho\,(X) = \min\{\lambda \geq \omega : \text{ there is no continuous mapping of } X \text{ onto } I^\lambda\},$$

where $I = [0, 1]$ is the closed unit interval. The following fact is well known (see [3]).

**Lemma 2.14** *Let $X$ be a compact space. Then $X$ contains a topological copy of $\beta\omega$ if and only if $\nabla\varrho\,(X) > \mathfrak{c}$.*

The next result unifies Proposition 4.5 of [4] and Theorem 4° of [35].

**Theorem 2.15** *Let $g: X \to Y$ be a continuous onto mapping of compact spaces and $\tau$ be an infinite cardinal. Suppose that $\Phi$ is either the tightness* *or dyadicity index and that* $\Phi(Y) \leq \tau$ *and* $\Phi(g^{-1}(y)) \leq \tau$ *for each* $y \in Y$*. Then* $\Phi(X) \leq \tau$ *as well. In other words, the property* $\Phi(\cdot) \leq \tau$ *is an inverse fiber property for compact sets.*

Combining Proposition 2.2 with other results of this section, we obtain the list of seventeen three space properties in the next theorem:

**Theorem 2.16** *Each of the following is a three space property:*

(a) *all compact (countably compact, pseudocompact) subsets are finite;*

(b) *contrasequentiality;*

(c) *all supersequences have length less than a given infinite cardinal* $\tau$*;*

(d) *a group does not contain copies of the closed unit interval;*

(e) *a group does not contain a copy of the ordinal space* $\omega_1$;

(f) *a group does not contain a copy of the ordinal space* $\omega_1 + 1$;

(g) *the property of being scattered;*

(h) *the property of being left-separated;*

(i) *all compact (countably compact) subsets are first countable;*

(j) $Char(\omega)$;

(k) *the property of being* $C$*-closed;*

(l) *sequentiality of compact subsets* (*under* $BL$ *or* $2^{\omega} < 2^{\omega_1}$);

(m) *zero-dimensionality of compact subsets;*

(n) *the property of being a* $CK$*-space;*

(o) *every compact subset* $K$ *of a group satisfies* $t(K) \leq \tau$*;*

(p) *every compact subset* $K$ *of a group satisfies* $\nabla\varrho(K) \leq \tau$*;*

(q) *a group does not contain a copy of* $\beta\omega$*.*

In the next section, we present two results about preservation of certain properties when taking extensions of topological groups which fail to be inverse fiber properties (even for compact sets).

## 3 Compact sets in extensions of groups

The Birkhoff–Kakutani theorem establishes the equivalence of metrizability and the first axiom of countability for topological groups. Our first result is a "uniform" kind of the Birkhoff–Kakutani theorem for compact subsets of topological groups.

**Lemma 3.1** *The following conditions are equivalent for a topological group $G$:*

(a) *every compact subspace of $G$ is first countable;*

(b) *every compact subspace of $G$ is metrizable.*

*Proof.* It suffices to show that (a) implies (b). Suppose that $X$ is a non-empty compact subset of $G$. Consider the mapping $j: G \times G \to G$ defined by $j(x, y) = x^{-1}y$ for all $x, y \in G$. Clearly, $j$ is continuous, so the image $F = j(X \times X)$ is a compact subset of $G$ which contains the identity $e$ of $G$. By our assumption, the space $F$ is first countable, so $\chi(e, F) \leq \omega$. Denote by $f$ the restriction of $j$ to $X \times X$. Then $f^{-1}(e) = \Delta_X$ is the diagonal in $X \times X$. Since $f$ is a closed mapping, we have $\chi(\Delta, X \times X) = \chi(e, F) \leq \omega$. Therefore, the compact space $X$ is metrizable by [23, Coro. 7.6]. □

It is well known that metrizability is a three space property. It turns out that an analogous assertion remains valid for compact subsets of topological groups. This follows directly from Proposition 2.8 and Lemma 3.1:

**Theorem 3.2** *Let $N$ be a closed invariant subgroup of a topological group $G$ and suppose that all compact subsets of the groups $N$ and $G/N$ are metrizable. Then every compact subset of $G$ is also metrizable.*

It is worth noting that metrizability of compact sets is not an inverse fiber property. Indeed, the canonical projection $p$ of the two arrows space $X$ onto the closed unit interval is continuous and two-to-one, so the image of $X$ and the fibers of $p$ are compact and metrizable, while $X$ is compact and $w(X) = 2^\omega$. Hence Theorem 3.2 reflects a special behavior of compactness in topological groups. In the following result, we extend this statement to countable compactness.

**Corollary 3.3** *Suppose that all countably compact subsets of the groups $N$ and $G/N$ are metrizable. Then the same is true for the group $G$.*

*Proof.* Clearly, the compact subsets of $N$ and $G/N$ are metrizable, so Theorem 3.2 implies that all compact sets in $G$ are also metrizable. In addition, since every countably compact metrizable space is compact, it follows from Proposition 2.13 that all countably compact subspaces of $G$ are compact. The required conclusion is now immediate. $\square$

A space $X$ is called $\aleph_0$-*monolithic* if the closure of every countable subset of $X$ has a countable network [3]. Clearly, a compact space $X$ is $\aleph_0$-monolithic if and only if every closed separable subspace of $X$ has countable weight or, equivalently, is metrizable. It follows from Uspenskij's result in [40] that extensions of topological groups do not preserve $\aleph_0$-monolithicity. Let us show that the restriction of this property to compact sets is not a three space property either. In other words, the metrizability of separable compact sets fails to be a three space property. This requires the use of a *Franklin–Mrówka space* briefly described below (see also [14, 3.6.I]).

A family $\mathcal{D}$ is called *almost disjoint* if $D \cap E$ is finite for all distinct $D, E \in \mathcal{D}$. Given an almost disjoint family $\mathcal{D}$ of subsets of $\omega$, we say that $\mathcal{D}$ is *maximal almost disjoint* (or a *mad* family) if no almost disjoint family of subsets of $\omega$ contains $\mathcal{D}$ as a proper subfamily. Clearly, every almost disjoint family in $[\omega]^\omega$ is contained in a mad family. If $\mathcal{D}$ is a mad family in $[\omega]^\omega$, one can introduce a Hausdorff topology in the set $X = \omega \cup \mathcal{D}$ by declaring the points of $\omega$ isolated in $X$ and taking the sets $\{D\} \cup (D \setminus F)$ as basic open neighborhoods of an element $D \in \mathcal{D}$ in the space $X$, where $F$ is an arbitrary finite subset of $\omega$. The space $X$ is called a Franklin–Mrówka space. It is easy to verify that if $\mathcal{D}$ is infinite, then $X$ is a locally compact, pseudocompact, non-compact space [14, 3.6.I]. In particular, $X$ is not metrizable. Note that $\mathcal{D}$ is a closed discrete subset of $X$. Denote by $\alpha X$ the one-point compactification of $X$. Then $\alpha X$ is a non-metrizable separable compact space, so it is not $\aleph_0$-monolithic. We use this space in the next example.

**Example 3.4** The free Abelian topological group $A(\alpha X)$ over the compact space $\alpha X$ contains a closed subgroup $N$ such that the compact subsets of both groups $N$ and $A(\alpha X)/N$ are $\aleph_0$-monolithic. However, $\alpha X$ is a compact subset of $A(\alpha X)$ which is not $\aleph_0$-monolithic.

Indeed, let $\omega + 1$ be the one-point compactification of the discrete space $\omega$. Consider the mapping $f\colon \alpha X \to \omega+1$ defined by $f(n) = n$ for each $n \in \omega$ and $f(x) = \omega$ for each $x \in \alpha X \setminus \omega$. Clearly, $f$ is continuous. Hence $f$ admits an extension to a continuous homomorphism $\hat{f}\colon A(\alpha X) \to A(\omega + 1)$. Let $N = \ker \hat{f}$. Note that the homomorphism $\hat{f}$ is open since the mapping $f$ is closed (see [1, 5]). Therefore, the quotient group $A(\alpha X)/N$ is topologically isomorphic to the group $A(\omega + 1)$. It is easy to see that $N$ is contained in the subgroup $A(Y, \alpha X)$ of the group $A(\alpha X)$ generated by the compact set $Y = \alpha X \setminus \omega$. Evidently, $Y$ is a one-point compactification of the discrete set $\mathcal{D}$, so $Y$ is $\aleph_0$-monolithic. Since $Y$ is compact, the group $A(Y, \alpha X)$ is topologically isomorphic to the free Abelian topological group $A(Y)$ [26]. Hence all compact subsets of $A(Y, \alpha X) \cong A(Y)$ are $\aleph_0$-monolithic by a result in [5]. The same assertion for compact subsets of the countable group $A(\alpha X)/N \cong A(\omega + 1)$ is evident. This finishes our argument. □

**Remark 3.5** It is easy to see that all compact subsets of the groups $N$ and $G/N$ in Example 3.4 are Eberlein compacta and, hence, have the Preiss–Simon property. However, $\alpha X$ fails to be an Eberlein compact space: it contains the proper dense pseudocompact subspace $X$.

# 4 Some counterexamples

Here we show that several finitely productive topological properties fail to be stable with respect to taking extensions of topological groups.

Let us say that a space $X$ is *B-compact* if every functionally bounded subset of $X$ has compact closure. Similarly, we say that $X$ is *P-compact* if every pseudocompact subset of $X$ has compact closure. The space $X$ is *C-compact* if the closure of every countably compact subset of $X$ is compact. Clearly, we have:

$$B\text{-compact} \Rightarrow P\text{-compact} \Rightarrow C\text{-compact}.$$

Our first example shows that extensions of topological groups do not preserve $B$-compactness and $P$-compactness.

**Example 4.1** There exist a pseudocompact non-compact Abelian topological group $G$ and a closed subgroup $N$ of $G$ such that $G/N$ is compact and all functionally bounded subsets of $N$ are finite. Hence extensions of topological groups do not respect either $B$-compactness or $P$-compactness.

We will construct $G$ as a dense pseudocompact subgroup of the group $K \times K$, where $K = 2^{\mathfrak{c}}$ and $2 = \{0, 1\}$ is the discrete two-point group. First, let $H$ be the group $2^{(\omega)}$ endowed with the finest precompact group topology (that is, the Bohr topology generated by all homomorphisms of $2^{(\omega)}$ to 2). Then the completion $\widetilde{H}$ of $H$ is a compact Boolean topological group, so $\widetilde{H}$ is topologically isomorphic to the group $2^{\tau}$ for some cardinal $\tau$ with $\omega \le \tau \le \mathfrak{c}$ [22, 28]. Note that the group $K = 2^{\mathfrak{c}}$ is separable by the Hewitt-Marczewski-Pondiczery theorem, so $K$ contains a countable dense subgroup $S$. Take any homomorphism $\varphi$ of $H$ onto $S$. Since $H$ carries the Bohr topology and the group $S$ is precompact, the homomorphism $\varphi$ is continuous. Hence $\varphi$ extends to a continuous homomorphism $\tilde{\varphi}\colon \widetilde{H} \to K$. By the compactness argument, $\tilde{\varphi}(\widetilde{H}) = K$, so $\mathfrak{c} = w(K) \le w(\tilde{\varphi}(\widetilde{H})) = \tau$.

This shows that $\tau = \mathfrak{c}$ or, in other words, the group $\widetilde{H}$ is topologically isomorphic to $2^{\mathfrak{c}}$. Hence we can identify $H$ with a countable dense subgroup of $K$. Since $H$ is countable, the closure of every functionally bounded subset of $H$ is compact. In addition, every Abelian group endowed with the Bohr topology does not contain infinite compact sets. Indeed, this is a very special case of Glicksberg's theorem in [17] saying that a locally compact Abelian topological group $F$ and the group $F^+$ (the same group $F$ but endowed with the Bohr topology generated by all continuous homomorphisms to the circle group $\mathbb{T}$) have the same compact sets. An elementary proof of this theorem that does not require the methods of Functional Analysis can be found in [11, Theorem 3.4.3]. Summing up, all functionally bounded subsets of $H$ are finite.

Our second step is to define a homomorphism $g\colon K \to K$ such that the graph $P = \{(x, g(x)) : x \in K\}$ of $g$ is $G_\delta$-dense in $K \times K$. Denote by $[\mathfrak{c}]^{\leq\omega}$ the family of all non-empty countable subsets of $\mathfrak{c}$. It is clear that $|[\mathfrak{c}]^{\leq\omega}| = \mathfrak{c}$. For every non-empty $A \subseteq \mathfrak{c}$ and $u \in 2^A$, let $C(A, u) = \pi_A^{-1}(u)$, where $\pi_A\colon 2^{\mathfrak{c}} \to 2^A$ is the projection. The family $\mathcal{F} = \{C(A, u) : A \in [\mathfrak{c}]^{\leq\omega},\ u \in 2^A\}$ also has the cardinality $\mathfrak{c}$. Let $\{(D_\alpha, E_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathcal{F} \times \mathcal{F}$. By recursion on $\alpha < \mathfrak{c}$, one can define a family of pairs $\{(x_\alpha, y_\alpha) : \alpha < \mathfrak{c}\} \subseteq K \times K$ satisfying the following conditions for each $\alpha < \mathfrak{c}$:

(i) $x_\alpha \in D_\alpha$ and $y_\alpha \in E_\alpha$;

(ii) $x_\alpha \notin \langle X_\alpha \rangle$ and $y_\alpha \notin \langle Y_\alpha \rangle$, where $X_\alpha = \{x_\nu : \nu < \alpha\}$ and $Y_\alpha = \{y_\nu : \nu < \alpha\}$.

Condition (ii) is easy to fulfill since both sets $D_\alpha$ and $E_\alpha$ have the cardinality $2^{\mathfrak{c}}$. Let $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ and $Y = \{y_\alpha : \alpha < \mathfrak{c}\}$. Define a mapping $f\colon X \to Y$ by $f(x_\alpha) = y_\alpha$, for each $\alpha < \mathfrak{c}$. From (ii) it follows that each of the sets $X$ and $Y$ is linearly independent over $\mathbb{Z}(2)$ in $K$. Hence $f$ extends to a homomorphism of $\langle X \rangle$ onto $\langle Y \rangle$ which we denote by the same letter $f$. Since every subgroup of the Boolean group $K$ is a direct summand in $K$, we can extend $f$ to a homomorphism $g\colon K \to K$. Then the group

$$P = \{(x, g(x)) : x \in K\}$$

is $G_\delta$-dense in $K \times K$. Indeed, let $Z$ be a non-empty $G_\delta$-set in $K \times K$. There exist non-empty countable subsets $A$ and $B$ of $\mathfrak{c}$ and points $u \in 2^A$, $v \in 2^B$ such that $C(A, u) \times C(B, v) \subseteq Z$. Then $(C(A, u), C(B, v)) = (D_\alpha, E_\alpha)$ for some $\alpha < \mathfrak{c}$, so (i) implies that $(x_\alpha, y_\alpha) \in D_\alpha \times E_\alpha \subseteq Z$ and, hence, $P \cap Z \neq \emptyset$.

Let $\bar{0}$ be the neutral element of $K$. We define the group $G$ as the sum $G = P + N$, where $N = \{\bar{0}\} \times H$. Denote by $p$ the projection of $K \times K$

onto the first factor. Clearly, $p(G) = p(P) = K$. It is easy to verify that $G \cap (\{\bar{0}\} \times K) = N$, so $N$ is closed in $G$. Since $N$ is dense in $\{\bar{0}\} \times K = \ker p$, it follows from [19, Lemma 1.3] that the restriction of $p$ to $G$ is an open homomorphism of $G$ onto $K$. Hence the quotient group $G/N$ is topologically isomorphic to the compact group $K$.

It remains to note that the group $G$ is pseudocompact. First, $P$ is a $G_\delta$-dense subgroup of the compact group $K \times K$, so [10, Theorem 1.2] implies that $P$ is pseudocompact. Since $P \subseteq G \subseteq K \times K$, the group $G$ is also pseudocompact. Note that $G$ is a proper subgroup of $K \times K$ since the restriction of the projection $p\colon K \times K \to K$ to $G$ has countable fibers. As $G$ is dense in $K \times K$, it cannot be compact. Thus $G$ is a functionally bounded subset of itself whose closure (in $G$) is not compact. This shows that $B$-compactness is not a three space property. Since every pseudocompact subspace of $N \cong H$ is functionally bounded in $N$ (hence finite), we also conclude that extensions of topological groups do not respect $P$-compactness. □

**Remark 4.2** The countable subgroup $N$ of the group $G$ in Example 4.1 is evidently realcompact and, hence, Dieudonné complete. Since the quotient group $G/N$ is compact, we conclude that extensions of topological groups do not respect realcompactness or Dieudonné completeness.

We call a space $X$ *$B$-closed* if every functionally bounded subset of $X$ is closed. The next result describes $B$-closed topological groups.

**Lemma 4.3** *A topological group $G$ is $B$-closed if and only if all functionally bounded subsets of $G$ are finite.*

*Proof.* Suppose that $G$ contains an infinite functionally bounded subset $X$. We claim that the closure of the set

$$P = \{x^{-1}y : \; x, y \in X, \; x \neq y\}$$

contains the identity $e$ of $G$. If not, choose an open symmetric neighborhood $U$ of $e$ in $G$ such that $U^4 \cap P = \emptyset$. An easy verification shows that the family of open sets $\{xU : \; x \in X\}$ is discrete in $G$. Clearly, each element of this family intersects $X$. However, only finitely many elements of a discrete family of open sets in $G$ can meet a functionally bounded set, which gives a contradiction. Hence $e$ is in the closure of $P$ and $P$ is dense in $X^{-1}X = P \cup \{e\}$.

Evidently, $X^{-1}$ is functionally bounded in $G$, and so is the product $X^{-1}X$ by [38, Coro. 1]. Since $e \notin P \subseteq X^{-1}X$, we conclude that $P$ is a non-closed functionally bounded subset of $G$. This proves the necessity of the condition. The sufficiency is evident. □

Another way to prove Lemma 4.3 is to note that functionally bounded subsets of a topological group are precompact and apply [38, Coro. 1] together with Protasov's theorem in [32]: if $X$ is an infinite precompact subset of a topological group $G$, then the set $X^{-1}X$ contains a countable, discrete, non-closed subset of $G$.

Our next aim is to show that the class of $B$-closed groups is not stable under taking extensions. This requires an auxiliary fact given below.

**Lemma 4.4** *There exists a dense subgroup $H$ of $2^{\mathfrak{c}}$ satisfying the following conditions:*

(a) $|H| = \mathfrak{c}$*;*

(b) *all functionally bounded subsets of $H$ are finite;*

(c) $|\ker f| = \mathfrak{c}$*, for every continuous homomorphism $f\colon H \to P$ to a first countable group $P$.*

*Proof.* Let $X$ be a set satisfying $|X| = \mathfrak{c}$. Denote by $H_X$ the family of all finite subsets of $X$ with the binary operation $A \Delta B = (A \setminus B) \cup (B \setminus A)$, the symmetric difference of $A$ and $B$, where $A, B \in H_X$. Then $H_X$ is a Boolean group of the cardinality $\mathfrak{c}$ whose identity is the empty set. Given a homomorphism $\varphi\colon H_X \to 2$ and a subset $Y$ of $X$, we say that $\varphi$ *depends only on* $Y$ if $\varphi(\{x\}) = 0$ for each $x \in X \setminus Y$. Denote by $\mathcal{F}$ the family of all homomorphisms of $H_X$ to 2 that depend only on a countable subset of $X$. It is easy to see that $|\mathcal{F}| = \mathfrak{c}$. Let $\tau$ be the coarsest group topology on $H_X$ which makes continuous all homomorphisms from $\mathcal{F}$. Then $H_X = (H_X, \tau)$ is a precompact Hausdorff topological group and $w(H_X) \leq |\mathcal{F}| = \mathfrak{c}$. Hence its completion $\widetilde{H}_X$ is topologically isomorphic to the group $2^{\lambda}$, for some cardinal $\lambda$ satisfying $\omega \leq \lambda \leq \mathfrak{c}$. Since every element of $\mathcal{F}$ depends only on countably many indices, the pseudocharacter of the neutral element $\bar{0}$ of $H_X$ is equal to $\mathfrak{c}$. Therefore $\lambda = \mathfrak{c}$, that is, $\widetilde{H}_X$ is topologically isomorphic to $2^{\mathfrak{c}}$. This enables us to consider $H = H_X$ as a dense subgroup of $2^{\mathfrak{c}}$.

Clearly, $|H_X| = \mathfrak{c}$, thus implying (a) of the lemma. To deduce (b), we argue as follows. First, for every non-empty subset $Y$ of $X$, let $p_Y\colon H_X \to H_Y$ be defined by $p_Y(A) = A \cap Y$ for each $A \in H_X$, where $H_Y$ is the Boolean group of all finite subsets of $Y$. Clearly, $p_Y$ is a homomorphic retraction of $H_X$ onto its subgroup $H_Y$. Our definition of the topology $\tau$ on $H_X$ implies that $p_Y$ is continuous (when $H_Y$ is taken with the subspace topology) and, in particular, $H_Y$ is closed in $H_X$. Note also that, for countable $Y \subseteq$

$X$, the group $H_Y$ is countable and carries the Bohr topology. Hence all functionally bounded subsets of $H_Y$ are finite (as in Example 4.1, one can apply Glicksberg's theorem here). Let $K$ be an infinite subset of $H_X$. We can assume without loss of generality that $K$ is countable. Then there exists a countable subset $Y$ of $X$ such that $A \subseteq Y$ for each $A \in K$. Since $K$ is infinite, there exists a continuous real-valued function $f$ on $H_Y$ such that $f(K)$ is an unbounded subset of the reals. Then $g = f \circ p_Y$ is a continuous function on $H_X$ and $g(K) = f(K)$ is unbounded in $\mathbb{R}$. This implies (b).

Finally, let us verify (c). Let $f\colon H_X \to P$ be a continuous homomorphism to a first countable topological group $P$. Then the kernel of $f$ is of type $G_\delta$ in $H_X$ and, since the family $\mathcal{F}$ generates the topology $H_X$, we can find a countable family $\{f_n : n \in \omega\} \subseteq \mathcal{F}$ such that $\bigcap_{n\in\omega} \ker f_n \subseteq \ker f$. Each $f_n$ depends only on a countable subset of $X$, so there exists a countable set $C$ of $X$ such that $H_{X\setminus C} \subseteq \ker f$. Evidently, $|\ker f| \geq |H_{X\setminus C}| = |X \setminus C| = \mathfrak{c}$, whence (c) follows. □

**Example 4.5** There exist a precompact Boolean group $G$ and an infinite, closed, functionally bounded subgroup $N$ of $G$ such that both groups $N$ and $G/N \cong N$ are $B$-closed. In particular, $G$ fails to be $B$-closed, so $B$-closedness is not a three space property.

Similarly to that in Example 4.1, we shall define $G$ to be a dense subgroup of the product group $K \times K$, where $K = 2^{\mathfrak{c}}$. Let $H$ be a dense subgroup of $K$ as in Lemma 4.4. Put $N = \{\bar{0}\} \times H$, where $\bar{0}$ is the neutral element of $K$. Our aim is to define an algebraic homomorphism $\varphi\colon H \to K$ and to take $G = N + P$, where $P = \{(y, \varphi(y)) : y \in H\}$ is the graph of $\varphi$. For every $A \subseteq \mathfrak{c}$, let $\pi_A\colon 2^{\mathfrak{c}} \to 2^A$ be the projection and $\bar{0}_A$ be the neutral element of the group $2^A$. To guarantee the functional boundedness of $N$ in $G$, the homomorphism $\varphi$ has to satisfy the following condition:

(a) For every non-empty countable set $A \subseteq \mathfrak{c}$ and every $x \in 2^A$, there exists $y \in H$ such that $\pi_A(y) = \bar{0}_A$ and $\pi_A(\varphi(y)) = x$.

Consider the family

$$\gamma = \{(A, x) : \emptyset \neq A \subseteq \mathfrak{c},\ |A| \leq \omega,\ x \in 2^A\}.$$

It easy to see that $|\gamma| = \mathfrak{c}$, so we can write $\gamma = \{(A_\alpha, x_\alpha) : \alpha < \mathfrak{c}\}$. One can define by recursion two sets $Y = \{y_\alpha : \alpha < \mathfrak{c}\} \subseteq H$ and $Z = \{z_\alpha : \alpha < \mathfrak{c}\} \subseteq K$ satisfying the following conditions for each $\alpha < \mathfrak{c}$:

(i) $y_\alpha \notin \langle Y_\alpha \rangle$, where $Y_\alpha = \{y_\nu : \nu < \alpha\}$;

(ii) $\pi_{A_\alpha}(y_\alpha) = \bar{0}_{A_\alpha}$;

(iii) $\pi_{A_\alpha}(z_\alpha) = x_\alpha$.

Such a construction is possible since $|\pi_A^{-1}(\bar{0}_A) \cap H| = \mathfrak{c}$ for each countable set $A \subseteq \mathfrak{c}$ (see (c) of Lemma 4.4). It follows from (i) that the set $Y$ is algebraically independent in $K$. Hence there exists a homomorphism $p\colon \langle Y \rangle \to K$ such that $p(y_\alpha) = z_\alpha$ for each $\alpha < \mathfrak{c}$. Since the group $H$ is Boolean, $p$ admits an extension to a homomorphism $\varphi\colon H \to K$. Note that (a) immediately follows from (ii), (iii) and our definition of $\varphi$.

Let $P = \{(y, \varphi(y)) : \ y \in H\}$ be the graph of $\varphi$ and $G = N + P$. It is easy to verify that $G \cap (\{\bar{0}\} \times K) = N = \{\bar{0}\} \times H$, where $H$ is dense in $K$. Hence the quotient group $G/N$ is topologically isomorphic to the projection of $G$ onto the first factor of the product $K \times K$, that is, $G/N \cong H \cong N$. In particular, all functionally bounded subsets of $G/N$ are finite, so $G/N$ is $B$-closed. Note that the density of $H$ in $K$ implies that $G = N + P$ is dense in $K \times K$.

It remains to show that $N$ is functionally bounded in $G$. It follows from (a) that

$$(\pi_A \times \pi_A)(G) \supseteq (\pi_A \times \pi_A)(P) \supseteq \{\bar{0}_A\} \times 2^A,$$

for each countable set $A \subseteq \mathfrak{c}$. It is also clear that

$$(\pi_A \times \pi_A)(N) = \{\bar{0}_A\} \times \pi_A(H) \subseteq \{\bar{0}_A\} \times 2^A.$$

In other words, for every countable $A \subseteq \mathfrak{c}$, the set $(\pi_A \times \pi_A)(N)$ is contained in a compact subset of $(\pi_A \times \pi_A)(G)$. Let $f$ be a continuous real-valued function on $G$. Since $G$ is dense in $K \times K = 2^\mathfrak{c} \times 2^\mathfrak{c}$, it follows from [2, 6] that $f$ depends on at most countably many coordinates or, equivalently, one can find a non-empty countable set $A \subseteq \mathfrak{c}$ and a continuous real-valued function $g$ on $(\pi_A \times \pi_A)(G)$ such that $f = g \circ (\pi_A \times \pi_A)$. Since $(\pi_A \times \pi_A)(N)$ is contained in a compact subset of $(\pi_A \times \pi_A)(G)$, the image $f(N) = g((\pi_A \times \pi_A)(N))$ is a bounded subset of the reals.

Finally, since $N$ is infinite, it follows from Lemma 4.3 that $G$ fails to be $B$-closed. In fact, one can avoid the use of Lemma 4.3 by noting that $N \setminus \{(\bar{0}, \bar{0})\}$ is a non-closed functionally bounded subset of $G$. □

Our next step is to show that, under $CH$, extensions of topological groups can destroy metrizability of pseudocompact subspaces. In the following lemma, we construct a special Franklin–Mrówka space which will be used in Example 4.8. Let us recall that if $A$ and $B$ are sets, then $A \subseteq^* B$ means that $A \setminus B$ is finite.

**Lemma 4.6** *Under $CH$, there exists a mad family $\mathcal{D} \subseteq [\omega]^\omega$ such that the corresponding Franklin–Mrówka space $X = \omega \cup \mathcal{D}$ has the property that no* *sequence in* $(\mathcal{D}\times\mathcal{D})\setminus\Delta$ *converges to the diagonal* $\Delta$ *in* $X\times X$*. Furthermore, for every infinite set* $S=\{\,(x_n,y_n):\,n\in\omega\}\subseteq(\mathcal{D}\times\mathcal{D})\setminus\Delta$*, there exists a continuous function* $h\colon X\to\{0,1\}$ *such that the set* $\{n\in\omega:\,h(x_n)\neq h(y_n)\}$ *is infinite.*

*Proof.* Use $CH$ to enumerate $[\omega]^\omega=\{A_\alpha:\,\alpha<\omega_1\}$. For a set $B\subseteq\omega$, denote by $\omega_1^B$ the family of functions from $B$ to $\omega_1$. Let also

$$\mathcal{F}=\{(f,g):\,f,g\in\omega_1^B,\ B\in[\omega]^\omega,\ \text{ and }\ f(n)\neq g(n)\ \text{ for each }\ n\in B\}.$$

Then $|\mathcal{F}|=2^\omega=\omega_1$, so there exists an enumeration $\mathcal{F}=\{(f_\alpha,g_\alpha):\,\omega\le\alpha<\omega_1\}$ such that each pair $(f_\alpha,g_\alpha)$ satisfies

(i) $f_\alpha(n)<\alpha$ and $g_\alpha(n)<\alpha$ for each $n\in B_\alpha=dom(f_\alpha)=dom(g_\alpha)$.

We will define two families $\mathcal{D}=\{D_\alpha:\,\alpha<\omega_1\}$ and $\mathcal{P}=\{P_\alpha:\,\alpha<\omega_1\}$ in $[\omega]^\omega$ satisfying the following conditions for each $\alpha<\omega_1$:

(1) $A_\alpha\cap D_\beta$ is infinite for some $\beta\le\alpha$;

(2) $D_\beta\cap D_\alpha$ is finite for each $\beta<\alpha$;

(3) for every $\beta<\alpha$, either $D_\alpha\subseteq^* P_\beta$ or $D_\alpha\subseteq^*\omega\setminus P_\beta$;

(4) for every $\beta\le\alpha$, either $D_\beta\subseteq^* P_\alpha$ or $D_\beta\subseteq^*\omega\setminus P_\alpha$;

(5) the set $\{n\in B_\alpha:\,D_{f_\alpha(n)}\subseteq^* P_\alpha$ and $D_{g_\alpha(n)}\subseteq^*\omega\setminus P_\alpha\}$ is infinite for each $\alpha\ge\omega$.

Conditions (1) and (2) guarantee that $\mathcal{D}$ is a mad family in $[\omega]^\omega$. Conditions (3) and (4) imply that the function $h_\alpha\colon\omega\to\{0,1\}$ defined by $h_\alpha(n)=1$ iff $n\in P_\alpha$ extends to a continuous function $\tilde{h}_\alpha$ on $X$. This, together with (5), will imply that the set $\{n\in B_\alpha:\,\tilde{h}_\alpha(D_{f_\alpha(n)})\neq\tilde{h}_\alpha(D_{g_\alpha(n)})\}$ is infinite.

Let $D_0$ be an infinite subset of $\omega$ such that the complement $\omega\setminus D_0$ is also infinite. We set $P_0=D_0$. Suppose that, for some $\alpha<\omega_1$, we have defined two families $\{D_\beta:\,\beta<\alpha\}$ and $\{P_\beta:\,\beta<\alpha\}$ of infinite subsets of $\omega$ satisfying (1)–(5). Since (5) is vacuous if $\alpha$ is finite, we can assume that $\alpha\ge\omega$. First, let us define a set $D_\alpha$. If $A_\alpha\cap D_\beta$ is infinite for some $\beta<\alpha$, put $A=\omega$; otherwise put $A=A_\alpha$. Since the families $\{D_\beta:\,\beta<\alpha\}$ and $\{P_\beta:\,\beta<\alpha\}$ are countable, one can choose (enumerating both families in type $\omega$) an infinite subset $D_\alpha$ of $A$ satisfying (2) and (3). It is clear that (1) is fulfilled as well.

The choice of $P_\alpha$ requires more work. Let us consider two cases.

Case 1. One of the functions $f_\alpha$, $g_\alpha$ is constant on an infinite subset $B$ of $B_\alpha$. We may assume, using (i), that there exists $\theta<\alpha$ such that $f_\alpha(n)=\theta$

for each $n \in B$. Note that $g_\alpha(n) \neq \theta$ for each $n \in B$. Let us put $P_\alpha = D_\theta$. Then (2) implies that $D_\beta \subseteq^* \omega \setminus P_\alpha$ for each $\beta \leq \alpha$ with $\beta \neq \theta$, and $D_\theta = P_\alpha \subseteq^* P_\alpha$, which gives (4). Clearly, we also have $D_{f_\alpha(n)} = D_\theta \subseteq^* P_\alpha$ and $D_{g_\alpha(n)} \subseteq^* \omega \setminus P_\alpha$ for each $n \in B$, thus implying (5).

Case 2. The functions $f_\alpha$ and $g_\alpha$ are finite-to-one. By a standard argument, there exists an infinite subset $B$ of $B_\alpha$ such that the restrictions of both $f_\alpha$ and $g_\alpha$ to $B$ are one-to-one. Taking an infinite subset of $B$ once again, if necessary, we can additionally assume that the images $f_\alpha(B)$ and $g_\alpha(B)$ are disjoint (the fact that $f_\alpha(n) \neq g_\alpha(n)$ for each $n \in B_\alpha$ is used here). The family $\{D_\beta : \beta \leq \alpha\}$ is countable and infinite, so we can faithfully renumerate it as $\{G_n : n \in \omega\}$. Hence, for every $n \in B$, there exist distinct integers $k_n, l_n \in \omega$ such that $D_{f_\alpha(n)} = G_{k_n}$ and $D_{g_\alpha(n)} = G_{l_n}$. Clearly, we have $k_m \neq k_n \neq l_m \neq l_n$ whenever $m \neq n$. For every $n \in B$, put

$$F_n = \bigcup \{G_{k_n} \cap G_m : m \leq n,\ m \neq k_n\}.$$

The family $\{G_n : n \in \omega\}$ is almost disjoint by (2), so the sets $F_n$ are finite. Let

$$P_\alpha = \bigcup_{n \in B} (G_{k_n} \setminus F_n).$$

It is clear from the above definition that $G_{k_n} \subseteq^* P_\alpha$ for each $n \in B$. Let $K = \{k_n : n \in B\}$. We claim that $G_m \subseteq^* \omega \setminus P_\alpha$ for each $m \in \omega \setminus K$, which gives (4). Indeed, let $m \in \omega \setminus K$ be arbitrary. Then our definition of the sets $F_n$ and $P_\alpha$ implies that $G_m \cap P_\alpha \subseteq \bigcup \{G_{k_n} \setminus F_n : n \in B,\ n < m\}$. In addition, $G_m \cap G_{k_n}$ is finite for each $n \in B$ by (2), whence it follows that the intersection $G_m \cap P_\alpha$ is finite. Hence $G_m \subseteq^* \omega \setminus P_\alpha$, as claimed. Since $l_n \neq k_m$ for all $m, n \in B$, we conclude that $l_n \notin K$ and, therefore, $G_{l_n} \subseteq^* \omega \setminus P_\alpha$ for each $n \in B$.

Summing up, we have $G_{k_n} \subseteq^* P_\alpha$ and $G_{l_n} \subseteq^* \omega \setminus P_\alpha$ for each $n \in B$, which implies (5). This finishes our recursive construction of the families $\mathcal{D}$ and $\mathcal{P}$.

Let $X = \omega \cup \mathcal{D}$ be the Franklin–Mrówka space corresponding to the family $\mathcal{D}$. To finish the proof, take an arbitrary infinite subset $S = \{(x_n, y_n) : n \in \omega\}$ of the set $(\mathcal{D} \times \mathcal{D}) \setminus \Delta$. Choose $\alpha \in \omega_1 \setminus \omega$ such that $x_n = D_{f_\alpha(n)}$ and $y_n = D_{g_\alpha(n)}$ for each $n \in \omega$, where $\omega$ is the common domain of $f_\alpha$ and $g_\alpha$. By (5), the set

$$C = \{n \in \omega : D_{f_\alpha(n)} \subseteq^* P_\alpha \text{ and } D_{g_\alpha(n)} \subseteq^* \omega \setminus P_\alpha\}$$

is infinite. Denote by $h$ the characteristic function of $P_\alpha$, that is, $h(n) = 1$ if $n \in P_\alpha$ and $h(n) = 0$ if $n \in \omega \setminus P_\alpha$. It follows from (3) and (4) that $h$

extends to a continuous function $\tilde{h}\colon X \to \{0,1\}$ and it is easy to see that $\tilde{h}(x_n) = 1$ and $\tilde{h}(y_n) = 0$ for each $n \in C$. The lemma is proved. □

Given a space $X$ and a natural number $n$, we denote by $A_n(X)$ the subspace of the free Abelian topological group $A(X)$ which consists of all words of reduced length $\leq n$ with respect to the basis $X$. It is well known that $A_n(X)$ is closed in $A(X)$ for each $n \in \omega$ (see [21] or [5]). We need the following simple fact.

**Lemma 4.7** *If a space $X$ is scattered, then so is $A_n(X)$ for each $n \in \omega$.*

*Proof.* Let $P$ be an infinite subset of $A_n(X)$ for some $n \in \omega$. It suffices to show that $P$ contains an isolated point. Denote by $k$ the maximal integer such that $P \setminus A_k(X)$ is non-empty. Then $k < n$, $P \subseteq A_{k+1}(X)$ and $O = P \setminus A_k(X)$ is an open non-empty subspace of $P$. Let $A_1(X) = X \oplus \{\bar{0}\} \oplus (-X)$ be the subspace of $A(X)$ consisting of the words of length $\leq 1$, where $\bar{0}$ denotes the neutral element of $A(X)$. Clearly, $A_1(X)$ is scattered. The multiplication mapping $i_{k+1}\colon A_1(X)^{k+1} \to A_{k+1}(X)$ defined by $i_{k+1}(y_1, \ldots, y_{k+1}) = y_1 + \cdots + y_{k+1}$ is continuous and onto. Take an arbitrary element $g = \varepsilon_1 x_1 + \cdots + \varepsilon_{k+1} x_{k+1}$ in $O$, where $\varepsilon_i = \pm 1$ and $x_i \in X$ for each $i \leq k+1$. Choose open neighborhoods $U_1, \ldots, U_{k+1}$ of $x_1, \ldots, x_{k+1}$, respectively, such that $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Then $O_g = \varepsilon_1 U_1 + \cdots + \varepsilon_{k+1} U_{k+1}$ is an open neighborhood of $g$ in $A_{k+1}(X)$ disjoint from $A_k(X)$ and the restriction of $i_{k+1}$ to the open subspace $W = \varepsilon_1 U_1 \times \cdots \times \varepsilon_{k+1} U_{k+1}$ of $A_1(X)^{k+1}$ is an open finite-to-one mapping of $W$ onto $O_g$ (see [5]). Note that the space $A_1(X)^{k+1}$ and its subspace $W$ are scattered, and so is $O_g = i_{k+1}(W)$. Hence the set $O_g \cap O$ contains an isolated point which is clearly isolated in $P$. This finishes the proof of the lemma. □

**Example 4.8** Under $CH$, there exist a locally compact, pseudocompact, non-compact space $X$ and a closed subgroup $N$ of the free Abelian topological group $A(X)$ such that the quotient group $A(X)/N$ is countable and all pseudocompact subspaces of $N$ are finite. Therefore, $A(X)$ contains a non-metrizable pseudocompact subspace while $N$ and $A(X)/N$ do not. Furthermore, $A(X)$ contains a pseudocompact subspace of uncountable character as well as a non-closed pseudocompact subspace.

Indeed, choose a mad family $\mathcal{D}$ of infinite subsets of $\omega$ as in Lemma 4.6 and consider the Franklin–Mrówka space $X = \omega \cup \mathcal{D}$ (we use $CH$ here). Let us show that the subgroup $A(\mathcal{D}, X)$ of $A(X)$ generated by the set $\mathcal{D}$ does not contain infinite pseudocompact subspaces.

Suppose to the contrary that $P$ is an infinite pseudocompact subspace of $A(\mathcal{D}, X)$. Then $P \subseteq A_n(X)$ for some $n \in \omega$ (see [12, Lemma 2.4]), so

Lemma 4.7 implies that $A_n(X)$ and its subspace $P$ are scattered. Denote by $D$ the set of all isolated points in $P$. Then $D$ is an infinite dense subset of $P$ and the complement $P' = P \setminus D$ is not empty. Let $x$ be an isolated point of $P'$. Choose an open neighborhood $U$ of $x$ in $P$ such that the closure $K = cl_P(U)$ does not intersect the closed set $P' \setminus \{x\}$. The space $K$ is pseudocompact as a regular closed subset of the pseudocompact space $P$. We also have $K \cap D \subseteq U$, so the discrete set $U \setminus \{x\}$ has the single accumulation point $x$ in $K$. In view of the pseudocompactness of $K$, this implies that every neighborhood of $x$ in $K$ contains all but finitely many points of $U$. We have thus proved that $K = U \cup \{x\}$ is the one-point compactification of $U$. Since $U$ is infinite, $K$ contains non-trivial sequences converging to $x$. From [13, Theorem 3.1] (or the Abelian version of [12, Lemma 4.8]) it follows that the subspace $A_2(\mathcal{D}, X) = A_2(X) \cap A(\mathcal{D}, X)$ of the group $A(\mathcal{D}, X)$ contains a non-trivial sequence converging to the neutral element $\bar{0}$ of $A(X)$, where $A_2(X)$ is the set of all elements of $A(X)$ that have the reduced length less than or equal to 2. Let $T = \{x_n - y_n : n \in \omega\}$ be such a sequence, where $x_n, y_n \in \mathcal{D}$ and $x_n \neq y_n$ for each $n \in \omega$. Then the set $S = \{(x_n, y_n) : n \in \omega\} \subseteq (\mathcal{D} \times \mathcal{D}) \setminus \Delta$ is infinite, so we can apply Lemma 4.6 to find a continuous function $h\colon X \to \{0, 1\}$ such that $h(x_n) \neq h(y_n)$ for infinitely many $n$. Let $d$ be a pseudometric on $X$ defined by $d(x, y) = |h(x) - h(y)|$ for all $x, y \in X$. Then $d$ is continuous and, by Pestov's result in [31], the set

$$O = \{x - y : x, y \in X,\ d(x, y) < 1\}$$

is an open neighborhood of $\bar{0}$ in $A_2(X)$. Hence the set $T \setminus O$ is finite. Since the function $h$ takes only the values 0 and 1, we have $O = \{x - y : x, y \in X,\ h(x) = h(y)\}$. Therefore, the set $\{n \in \omega : h(x_n) \neq h(y_n)\}$ is finite, which is a contradiction. This proves that all pseudocompact subspaces of $A(\mathcal{D}, X)$ are finite.

Let $f\colon X \to \omega + 1$ be the mapping defined by $f(n) = n$ if $n \in \omega$ and $f(x) = \omega$ if $x \in \mathcal{D}$. It is easy to see that $f$ is continuous and closed, so it extends to a continuous open homomorphism $\hat{f}\colon A(X) \to A(\omega + 1)$ (see [18]). Clearly, $N = \ker \hat{f}$ is a closed subgroup of $A(X)$ and $N \subseteq A(\mathcal{D}, X)$. Therefore, all pseudocompact subspaces of $N$ are finite. Since $\hat{f}$ is open, the quotient group $A(X)/N$ is topologically isomorphic to the countable group $A(\omega + 1)$. So every pseudocompact subspace of $A(\omega + 1)$ is compact and metrizable. However, $X$ is a pseudocompact, non-compact, non-metrizable subspace of the group $A(X)$.

We claim that $A_2(X)$ is a pseudocompact subspace of $A(X)$ which has uncountable character. Indeed, let $i_2\colon A_1(X)^2 \to A_2(X)$ be the multiplication mapping, where $A_1(X) \cong X \oplus \{\bar{0}\} \oplus (-X)$. Clearly, $i_2$ is continuous and onto. Note that $A_1(X)^2$ is pseudocompact since $A_1(X)$ is locally compact and pseudocompact (see [16]), so $A_2(X)$ is pseudocompact. Therefore, if $A_2(X)$ had countable character, the space $X$ would be compact and metrizable by [41, Theorem 3.3].

Finally, the subspace $P = A_2(X) \setminus \{\bar{0}\}$ of $A_2(X)$ is pseudocompact and non-closed. Indeed, let $P'$ be the set of isolated points in $P$ and $D$ be a countable infinite subset of $P'$. It suffices to verify that $D$ has an accumulation point in $P$. Suppose not. Since $A_2(X)$ is closed and pseudocompact, $\bar{0}$ is the unique accumulation point of $D$ in $A_2(X)$. Hence $D^* = D \cup \{\bar{0}\}$ is pseudocompact as a regular closed subset of the pseudocompact space $A_2(X)$. Since $D^* \subseteq P' \subseteq A(\mathcal{D}, X)$, this contradicts the fact that all pseudocompact subspaces of $A(\mathcal{D}, X)$ are finite. □

In Example 4.10 below we show that extensions of topological groups do not preserve the property "every countably compact subset has compact closure". First, we need a lemma.

**Lemma 4.9** *Let $K$ be an arbitrary $\omega$-narrow (Abelian) topological group with $w(K) \leq \mathfrak{c}$. Then one can find a topological monomorphism $i\colon K \to H$ of $K$ to a direct product $H = \prod_{\alpha\in A} H_\alpha$ of second countable (Abelian) topological groups, where $|A| \leq \mathfrak{c}$, and a countable dense subgroup $S$ of $H$ such that the intersection $S \cap i(K)$ is trivial.*

*Proof.* Since the group $K$ is $\omega$-narrow, there exists a topological monomorphism $p$ of $K$ to a direct product $P = \prod_{\beta\in B} P_\beta$ of second countable topological groups (see [20] or [39, Theorem 3.4]). Since $w(K) \leq \mathfrak{c}$, we can assume without loss of generality that $|B| \leq \mathfrak{c}$. Put $H = P \times \mathbb{T}$, where $\mathbb{T}$ is the circle group. Clearly, $p(K) \times \{0\}$ is a copy the group $K$ in $H$, where $0$ is the neutral element of $\mathbb{T}$. In other words, the mapping $i\colon K \to H$ defined by $i(x) = (p(x), 0)$ for each $x \in K$ is a topological monomorphism and $i(K) = p(K) \times \{0\}$. Let $C$ be a countable dense subset of $\mathbb{T}$ which is algebraically independent. Modify the proof of the Hewitt–Marczewski–Pondiczery theorem (see [14, Theorem 2.3.15]) and define a countable dense subset $D$ of $H = P \times \mathbb{T}$ such that the restriction of the projection $\pi\colon P \times \mathbb{T} \to \mathbb{T}$ to the set $D$ is a bijection of $D$ onto $C$. Let $S = \langle D \rangle$ be the subgroup of $H$ generated by $D$. Then $S$ is countable and dense in $H$. Since the set $C = \pi(D)$ is algebraically independent, the restriction of $\pi$ to $S$ has trivial kernel. This together with the inclusion $i(K) \subseteq P \times \{0\}$ imply that the intersection $S \cap i(K)$ contains only the neutral element of $H$. The Abelian version of the lemma is obtained if we choose the factors $P_\beta$ to be Abelian. □

**Example 4.10** There exist an Abelian topological group $G$ and a closed subgroup $N$ of $G$ such that all functionally bounded subsets of both groups $N$ and $G/N$ have compact closures, but $G$ contains a closed copy of the ordinal space $\omega_1$.

Indeed, let $Y = \omega_1 + 1$ be the space of all ordinals $\leq \omega_1$ with the order topology. Clearly, $Y$ is compact. Let also $X = \omega_1$ be the corresponding subspace of $\omega_1 + 1$. Then $X$ is $\omega$-bounded and non-compact. Denote by $f$ the mapping of $Y$ onto itself defined by $f(0) = f(\omega_1) = \omega_1$, $f(n) = n-1$ for each $n \in \omega \setminus \{0\}$, and $f(\alpha) = \alpha$ for each $\alpha \in \omega_1 \setminus \omega$. It is easy to see that $f$ is continuous and that the restriction of $f$ to $X$ is a bijection of $X$ onto $Y$. Extend $f$ to a continuous homomorphism $\hat{f}\colon A(Y) \to A(Y)$ and denote by $A(X,Y)$ the subgroup of $A(Y)$ generated by $X$. Since $Y$ is closed in $A(Y)$, the set $X = Y \cap A(X,Y)$ is closed in $A(X,Y)$. It is easy to see that the restriction of $\hat{f}$ to $A(X,Y)$ is a continuous isomorphism of $A(X,Y)$ onto $A(Y)$ (but this restriction is not open).

Since the space $Y$ is compact, the group $A(Y)$ is $\sigma$-compact and, hence, $\omega$-narrow. Note that $w(Y) = \omega_1$, so we can apply [41, Theorem 2.1] to infer that $w(A(Y)) \leq (\aleph_1)^\omega = \mathfrak{c}$ (in fact, one can show that $w(A(Y)) = \mathfrak{d}$, see [29]). By Lemma 4.9, we can find a topological monomorphism $i$ of $A(Y)$ to a direct product $H = \prod_{\alpha \in A} H_\alpha$ of second countable Abelian topological groups, where $|A| \leq \mathfrak{c}$, and a countable dense subgroup $S$ of $H$ such that $S \cap i(A(Y))$ contains only the neutral element of $H$. Then the diagonal product $\varphi = i \triangle \hat{f}$ is a topological monomorphism of $A(Y)$ into the product $H \times A(Y)$. Denote by $\pi$ the projection of $H \times A(Y)$ onto the second factor. Then $\pi \circ \varphi = \hat{f}$. Put $N = S \times \{0_Y\}$ and $G = \varphi(A(X,Y)) + N$, where $0_Y$ is the neutral element of $A(Y)$. Then $G$ is a subgroup of $H \times A(Y)$ and $N$ is a closed subgroup of $G$, since $N = G \cap \ker \pi$. In addition, since $N$ is dense in $\ker \pi$, the quotient group $G/N$ is topologically isomorphic to the group

$$\pi(G) = \pi(\varphi(A(X,Y))) = \hat{f}(A(X,Y)) = A(Y).$$

By [12, Theorem 2.5], every functionally bounded subset $B$ of the group $A(Y)$ is contained in the compact set $A_n(Y)$ for some $n \in \omega$, so the closure of $B$ in $A(Y)$ is compact. The same holds true for bounded subsets of $N$, since $N$ is countable.

Since $Y$ is compact, the group $A(Y)$ is complete by Graev's theorem [18]. Hence $\varphi(A(Y)) \cong A(Y)$ is a closed subgroup of $H \times A(Y)$. In addition, our choice of $S$ and $N$ implies that

$$\varphi(A(Y)) \cap G = \varphi(A(X,Y)),$$

so $\varphi(A(X,Y))$ is a closed subgroup of $G$. Finally, since $X = \omega_1$ is a closed subset of $A(X,Y) \cong \varphi(A(X,Y))$, we conclude that $G$ contains a closed copy of $\omega_1$. $\square$

Our last example shows that the property "every countably compact subset has countably compact closure" is not a three space property either.

**Example 4.11** There exist an Abelian topological group $G$ and a closed subgroup $N$ of $G$ such that every functionally bounded subset of the groups $N$ and $G/N$ has countably compact closure, but the group $G$ contains an $\omega$-bounded (hence countably compact) subset whose closure is not countably compact.

Let $\bar{0}$ be the neutral element of the group $2^{\omega_1}$ and $\Sigma(\bar{0}) \subseteq 2^{\omega_1}$ be the $\Sigma$-product of $\omega_1$ copies of the group 2 with center at $\bar{0}$. Similarly, denote by $\Sigma(\bar{1})$ the $\Sigma$-product with center at the point $\bar{1} \in 2^{\omega_1}$ all whose coordinates are equal to 1. Choose a countable, infinite, discrete subset $D$ of $\Sigma(\bar{1})$ and put $X = \Sigma(\bar{0}) \cup D$. Since the closure of every countable subset of $\Sigma(\bar{1})$ is contained in $\Sigma(\bar{1})$, we conclude that $D$ is a closed discrete subset of $X$. Clearly, $X$ is pseudocompact subspace of $2^{\omega_1}$ and $\Sigma(\bar{0})$ is a dense countably compact subspace of $X$. In particular, the closure of $\Sigma(\bar{0})$ in $X$ is not countably compact.

Let $P$ be the partition of $X$ into disjoint closed sets with the only non-trivial element $D$. Denote by $Y$ the quotient set $X/P$ and let $f\colon X \to X/P$ be the natural projection. Then $f^{-1}f(x) = \{x\}$ for each $x \in X \setminus D$ and $D = f^{-1}f(x)$ for all $x \in D$. Let $Y$ carry the finest topology which makes the mapping $f$ $\mathbb{R}$-quotient [8]. In other words, a function $h\colon Y \to \mathbb{R}$ is continuous on $Y$ iff $h \circ f$ is continuous on $X$. It is easy to verify that the space $Y$ is Tychonoff and the mapping $f\colon X \to Y$ is continuous (see [8]). Pick a point $x_0 \in D$. The subspace $Z = \Sigma(\bar{0}) \cup \{x_0\}$ of $X$ is $\omega$-bounded, and so is the image $Y = f(Z)$.

Extend $f$ to a continuous homomorphism $\hat{f}$ of $A(X)$ onto $A(Y)$. Since $f$ is $\mathbb{R}$-quotient, the homomorphism $\hat{f}$ is open by [30]. From our definition of $f$ it follows that the kernel of $\hat{f}$ is contained in the subgroup $A(D,X)$ of $A(X)$ generated by the set $D$, so that $N = \ker \hat{f}$ is countable. Put $G = A(X)$. The quotient group $G/N$ is topologically isomorphic to the group $A(Y)$. Since the space $Y$ and all finite powers of $Y$ are $\omega$-bounded, so is $A_n(Y)$ for each $n \in \omega$. If $B$ is a bounded subset of $A(Y)$, then $B \subseteq A_n(Y)$ for some $n \in \omega$ [12, Lemma 2.4], so the closure of $B$ in $A(Y)$ is countably compact. Since the kernel $N$ of the homomorphism $\hat{f}$ is countable, the closure of every bounded subset of $N$ is compact. Finally, $X$ is a closed subset of $G = A(X)$, so the closure in $G$ of the dense $\omega$-bounded subset $\Sigma(\bar{0})$ of $X$ coincides with $X$ and, hence, fails to be countably compact. $\square$

## 5 Open problems

Evidently, every compact sequential space is sequentially compact. It is not clear whether item (l) of Proposition 2.11 can be extended to cover the case of sequential compactness:

**Problem 5.1** *Suppose that all compact subsets of the groups $N$ and $G/N$ are sequentially compact. Does then the same hold for the group $G$?*

Metrizability is a three space property for compact sets by Theorem 3.2. We do not know, however, if the result remains valid if one replaces metrizability by separability:

**Problem 5.2** *Suppose that all compact subsets of the groups $G$ and $G/N$ are separable. Are compact subsets of $G$ separable (or have countable cellularity)?*

It is known that the product of two compact Fréchet–Urysohn spaces can fail to be Fréchet–Urysohn [36]. This gives rise to the following problem:

**Problem 5.3** *Is the Fréchet–Urysohn property a three space property for compact sets?*

Extensions of topological groups respect pseudocompactness by a result of Comfort and Robertson [10]. On the other hand, by Example 4.8, it is consistent with $ZFC$ that extensions of topological groups do not preserve the property of being $P$-compact. This makes it interesting to answer the following question.

**Problem 5.4** *Suppose that all compact subsets of $N$ and $G/N$ are $P$-compact. Is the same true for compact subsets of $G$?*

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