Adaptive backstepping control of a class of uncertain nonlinear systems. Application to Bouc–Wen hysteretic oscillators*

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Abstract

A backstepping-based adaptive controller is designed for a class of uncertain nonlinear systems under the strict-feedback form, which, in particular, includes models of base-isolation hysteretic structures. Our contribution is two-fold. First, we develop a control strategy which in one hand allows to deal with unbounded uncertainties, and in the other hand does not assume the exact knowledge of the term multiplying the control (unlike most adaptive backstepping control strategies for strict-feedback systems). We show that the closed loop is globally uniformly ultimately bounded and we give explicit bounds on both the asymptotic and transient performance. Second, the control strategy is applied to a system typically found in base isolation schemes for seismic active protection of building structures. This system exhibits a hysteretic nonlinear behavior which is described analytically by the so-called Bouc–Wen model. Unlike other control schemes, the developed backstepping control does not require an exact knowledge of the model parameters. They are only defined within

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known intervals. An analytical study of the interval Bouc-Wen model is done to ensure bounded solutions and to allow the control tuning and implementation. The practical effectiveness of the controller is illustrated by numerical simulations.

1 Introduction

During the last few years, backstepping-based designs have emerged as powerful tools for stabilizing nonlinear systems both for tracking and regulation purposes (Krstic et al. 1995). The main advantage of these designs is the *systematic* construction of a Lyapunov function for the closed loop, allowing the analysis of its stability properties. The adaptive version of these designs, especially the tuning functions design, offers the possibility to synthetize in a systematic way controllers for a wide class of nonlinear systems (those under the strict-feedback form) whose structure is known but with *unknown* parameters (Krstic et al. 1995, Chapter 4). They also offer the possibility to analyze the transient behavior of the closed loop *in the absence of uncertainties*. Despite the fact that the robustness of the tuning functions design has been studied extensively in the case of linear systems (Ioannou and Sun 1996; Wen et al. 1976; Ikhouane and Krstic 1998a, 1998b; Zhang and Ioannou 1998; Nikiforov and Voronov 2001), much more is to be done in the case of nonlinear systems (Aloliwi and Khalil 1997; Stoev et al. 2002). In (Jiang and Hill 1999) a robust adaptive scheme for nonlinear systems with globally exponentially stable unmodeled dynamics has been developed for the *regulation* case. For the class of nonlinearities studied in (Jiang and Hill 1999) the unmodeled dynamics enter to the system state equations as functions which can be unbounded with respect to the state, but bounded with respect to the time. Despite the fact that the scheme in (Jiang and Hill 1999) ensures arbitrary asymptotic performance, it does not allow the quantification of the transient performance as an *explicit* function of the design parameters.

In this paper, we propose a simple backstepping-based adaptive scheme for a class of strict-feedback nonlinear systems for the *tracking* problem. The systems studied in the present paper arise from a class of nonlinear second order oscillators, which are common in structural engineering models of base isolation devices for seismic protection of buildings (Soong and Dargush 1997).

The proposed adaptive scheme uses the switching σ-modification (Ioannou and Sun 1996; Ikhouane and Krstic 1998a) and new terms that incorporate part of the information on the uncertainties. The adaptive algorithm allows the quantification of both transient and asymptotic performance as explicit functions of the design parameters. The uncertain nonlinear part of the open loop is written as the sum of the scalar product of -possibly-unknown coefficients with known functions, plus a residual which may be unbounded with respect to the state, but is bounded with respect to the time. This representation has the practical advantage of giving an estimation of the uncertain part by an open loop identification. For structural systems, which are stable in open loop, this is often possible. The reduction of the size of the uncertainty often results in a reduction of the amplitude of the control signal since the nonlinear terms which counteract the effect of the uncertainty are smaller.

In many situations, the control may be multiplied in the state equations by an unknown coefficient (for instance, the inverse of the mass for structural systems). Despite the fact that this problem has been solved for linear systems (Krstic et al. 1995, Chapter 10), most schemes using the tuning function design for strict-feedback systems assume an exact knowledge of the function (or the coefficient) that multiplies the control (Krstic et al. 1995, Chapter 4). We cope with this problem by estimating on-line the mass of the structural system. Our scheme allows the designer to tune the design coefficients in an explicit way to obtain the closed loop desired behavior both for the transient and the asymptotic tracking.

In order to test the practical potential of the proposed control scheme, it is applied to design an active controller for a seismic base isolation scheme which has a nonlinear hysteretic behavior. This behavior is described by the so-called Bouc–Wen model (Wen 1976), which is well accepted in the context of structural mechanics for its ability to describe analytically a wide spectrum of hysteretic loops (Barbat and Bozzo 1997).

Hysteresis is encountered in a wide variety of processes in which the input-output dynamic relations between variables involve memory effects. Examples are found in biology, optics, electronics, ferroelectricity, magnetism, mechanics, structures, among other areas. This paper is primarily concerned with hysteresis in mechanical and structural systems. In these systems, hysteresis appears as a natural mechanism of materials to supply restoring forces against movements and dissipate energy (Kayvani and Barzegar 1996). This mechanism has been exploited in recent years in building damping devices and vibration isolation schemes (Battaini and Casciati 1996; Kikuchi and Aiken 1997). In a near context, mechanical and structural hysteresis is also encountered when using new “smart” materials and actuators for vibration control, as the cases of shape memory alloys (Seelecke 2002) and electro/magnetorheological fluids (Spencer et al. 1997).

While there is an extensive literature about physical characterization and mathematical modelling of hysteretic systems in different areas, only a few references are found reporting feedback controllers in the general literature on control systems (Tao and Kokotovic 1995, Hsiao and Hwang 1997, Belbas and Mayergoz 2000, Su et al. 2000). In structural systems, feedback controllers in the presence of hysteretic components have been primarily encountered when dealing with smart actuators and base isolation schemes. A passivity based control strategy has been presented in (Gorbert et al. 2001) along with a hysteretic Preisach model. In base isolated structures, feedback control problems arise when hysteretic isolators are coupled with active feedback controllers. In this case, the Bouc-Wen model (Wen 1976) has been extensively used to describe the hysteretic behavior. In (Yang et al. 1994) a stochastic linearization of the model is used in conjunction with a linear optimal control. A robust sliding mode control strategy has been proposed in (Luo et al. 2000) considering that the output of the hysteresis model can be bounded by an uncertain function with linear bounds.

A contribution of this paper to the problem of controlling base isolation schemes is in the use of the Bouc-Wen model with uncertain parameters without relying on any linearization. We consider that all the model parameters are defined within known intervals, without the need of knowing the exact values of the parameters. In practical problems, these intervals can be obtained through identification of real structures (Smyth et al. 1999). An analytical study of the Bouc-Wen model with interval parameters is performed to allow the use and the tuning of the backstepping controller. The effectiveness of the controller is shown by means of numerical simulations.

The paper is organized as follows: Section 2 states the control problem including the assumptions on the system and the uncertainties. Section 3 presents the details of the controller design together with the stability and performance results. The proofs of these results are given in Section 4. Section 5 describes the application of the controller to the hysteretic base isolation system, including the implementation issues and numerical results. This Section is complemented with the study of the Bouc-Wen model in Appendix A.

2 Problem statement

We are interested in controlling the second order system

$$m\ddot{x} + c\dot{x} + \Phi(x,t) = f(t) + u(t), \tag{1}$$

where m and c are real parameters and Φ represents a nonlinear component. $f(t)$ represents an external disturbance and $u(t)$ a control input. Although many systems can be represented by the above model, in this paper we will consider mechanical systems, so that m and c are the mass and the damping coefficient, respectively, and Φ characterizes a nonlinear restoring force. x gives the position, $f(t)$ is an exciting force and $u(t)$ is an active control force supplied by appropriate actuators.

We assume that the nonlinear function has the following structure:

$$\Phi(x,t) = \phi_1\psi_1\left(\frac{x}{d},t\right) + \phi_2\psi_2\left(\frac{x}{d},t\right) + \cdots + \phi_n\psi_n\left(\frac{x}{d},t\right) + R\left(x,t\right), \quad (2)$$

where $\psi_1,\ \psi_2,\ldots,\ \psi_n$ are known (possibly unbounded) locally Lipschitz functions with respect to x, piecewise continuous and bounded with respect to the time. The known constant d is a positive scaling factor which has the same dimension as the displacement x. As we shall see later, we can take

$$d \triangleq \sqrt{\frac{1}{T_0}\int_0^{T_0} x_{ol}^2(t)dt}, \quad (3)$$

that is the root mean-square of the open loop displacement response x_{ol} to some "standard" excitation $f(t)$ during some given period of time T_0.

The constant uncertain parameters $\phi_1,\ \phi_2,\ \ldots,\ \phi_n$ have the same physical dimension (that of a force). The nonlinear restoring force Φ may not be available for on-line measurement.

The following assumptions complete the description of system (1) - (2).

Assumption 1. *There exists a known (not necessarily bounded) function* $r\left(x,t\right)$ *which is locally Lipschitz with respect to* x*, piecewise continuous and bounded with respect to* t*, such that* $\left|R\left(x,t\right)\right| \leq r\left(x,t\right)$.

Assumption 2. *The unknown constant vector* $\theta_\phi = \left(\phi_1,\phi_2,\ldots,\phi_n\right)^T$ *lies inside a known sphere. That is, we know a positive constant* M_ϕ *such that* $\|\theta_\phi\| \leq M_\phi$.

Assumption 3. *The uncertain parameters* m *and* c *lie in known intervals, that is there exist known positive constants* $m_{\max}$ *and* $c_{\max}$ *such that* $0 < m \leq m_{\max}$ *and* $0 \leq c \leq c_{\max}$.

Assumption 4. *A known bound* F *on the unknown disturbance* $f(t)$ *is available. That is* $|f(t)| \leq F$ *for all* $t \geq 0$.

Assumption 5. *The displacement* x *and velocity* $\dot{x}$ *are available for on-line measurement.*

Control objective: To design a backstepping-based adaptive control law such that

- The closed loop is globally uniformly ultimately bounded.
- Let $y_r(t)$ be a known bounded reference signal such that $\dot{y}_r$ and $\ddot{y}_r$ are known, bounded and piecewise continuous. Our objective is also that the tracking error $x(t) - y_r(t)$ can be made arbitrarily small both in the transient and asymptotically by an explicit choice of the design parameters.

3 Controller design and main results

3.1 Controller design

We first rewrite equations (1)-(2) in the following state space form:

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{1}{m}\left(-cv\frac{x_2}{v} - \phi_1\Psi_1(\frac{x_1}{d},t) - \cdots - \phi_n\Psi_n(\frac{x_1}{d},t) - R(x_1,t) + f(t) + u(t)\right) \\
&= \frac{1}{m}\left(\theta^T\varphi\left(\frac{x_1}{d},\frac{x_2}{v},t\right) - R(x_1,t) + f(t) + u(t)\right). \qquad (4)
\end{aligned}$$

where $x_1 = x$, $x_2 = \dot{x}$, $\theta = (cv, \phi_1, \ldots, \phi_n)^T$ is the (constant) vector of uncertain parameters and $\varphi = \left(-\frac{x_2}{v}, -\Psi_1(\frac{x_1}{d},t), \ldots, -\Psi_n(\frac{x_1}{d},t)\right)^T$. The known positive constant v is introduced to have dimensionless components in the regression vector φ and (force) dimension-like terms in the parameter vector θ. As in (3), we take

$$v \triangleq \sqrt{\frac{1}{T_0}\int_0^{T_0} \dot{x}_{ol}^2(t)dt}\,, \qquad (5)$$

that is the root mean-square of the open loop velocity response $\dot{x}_{ol}$ to the "standard" excitation $f(t)$ during the period of time T_0.

From Assumptions 2 and 3, it follows that

$$\|\theta\| \leq \sqrt{(c_{\max}v)^2 + M_\phi^2} \triangleq M_\theta. \qquad (6)$$

It is worth noting that in equation (4) the control $u(t)$ is multiplied by an unknown term. Although solved for linear systems, to the best of our knowledge, this case has not been treated in most backstepping adaptive designs for nonlinear parametric strict-feedback systems (Krstic et al. 1995, Chapter 4). Thus we need to construct an estimator $\hat{m}(t)$ of the parameter m.

Consider now the standard variables:

$$z_1 = x_1 - y_r, \tag{7}$$
$$\alpha_1 = -c_1 \frac{v}{d} z_1, \tag{8}$$
$$z_2 = x_2 - \dot{y}_r - \alpha_1, \tag{9}$$

The control law and parameter update laws are given in equations (10) and (11) below.

Adaptive control law:

$$\begin{aligned} u(t) \quad = \quad & -\hat{\theta}^T \varphi - c_1 \frac{v}{d} \left(x_2 - \dot{y}_r \right) \hat{m} - \frac{v^2}{d^2} \hat{m} z_1 + \hat{m} \ddot{y}_r - \frac{d}{v^3 m_{\max}} d_2 z_2 r^2 \\ & - \frac{v m_{\max}}{d} c_2 z_2 - \mathbf{sg}\left(\frac{z_2}{v} \right) \mathbf{cf}\left(\frac{|z_2|}{v} \right) gF. \end{aligned} \tag{10}$$

Parameter estimate laws:

$$\begin{cases} \dot{\hat{\theta}} = \frac{M_\theta^2}{m_{\max} v^2} \Gamma \varphi z_2 - \frac{v}{d} \Gamma \sigma_\theta \left(\frac{\|\hat{\theta}\|}{M_\theta} \right) \hat{\theta}, \\ \dot{\hat{m}} = \gamma m_{\max} \left(\frac{c_1}{dv} x_2 + \frac{1}{d^2} z_1 - \frac{1}{v^2} \ddot{y}_r - \frac{c_1}{dv} \dot{y}_r \right) z_2 - \gamma \frac{v}{d} \sigma_m \left(\frac{|\hat{m}|}{m_{\max}} \right) \hat{m}. \end{cases} \tag{11}$$

In the above expressions, c_1, c_2, d_2 are dimensionless positive design parameters and $0 \leq g \leq 1$ adjusts the part of the information on the perturbation f to be included in the control law; Γ is a (dimensionless) positive definite design matrix, $\sigma_\theta(y) = \bar{\sigma}_\theta \sigma(y)$, $\sigma_m(y) = \bar{\sigma}_m \sigma(y)$, and $\mathbf{cf}(y) = \sigma(y/\varepsilon_1)$, where

$$\sigma(y) = \begin{cases} 0 & y \leq 1, \\ y - 1 & y \in [1, 2], \\ 1 & y \geq 2. \end{cases}$$

In the above expression $\bar{\sigma}_\theta$, $\bar{\sigma}_m$ and ε_1 are (dimensionless) positive design parameters. The function **sg** is defined as follows:

$$\mathbf{sg}(y) = \begin{cases} -1 & y \leq -\varepsilon_2, \\ \frac{1}{\varepsilon_2} y & y \in [-\varepsilon_2, \varepsilon_2], \\ 1 & y \geq \varepsilon_2, \end{cases}$$

where ε_2 is a (dimensionless) positive design parameter.

3.2 Main results

In this section we state the main stability and performance (transient and asymptotic) results of the paper. The tracking error both of the closed loop displacement and velocity is measured by the root mean-square norm defined in the form (Boyd and Barrat 1991):

$$\|y\|_{rms,[0,T]} \triangleq \sqrt{\frac{1}{T}\int_0^T y(t)^2 dt}, \tag{12}$$

for some time interval $[0,T]$. We introduce also the following notation:

$$f_{dv} \triangleq \frac{\frac{1}{2}mv^2}{d}, \tag{13}$$

which is the ratio of the mean of the open loop kinetic energy of the mass m during the time interval $[0,T_0]$ (indeed, $\frac{1}{2}mv^2 = \frac{1}{T_0}\int_0^{T_0}\frac{1}{2}m\dot{x}_{ol}^2(t)dt$) to the root mean-square d of the open loop displacement response to some "standard" excitation during the time interval $[0,T_0]$. Note that f_{dv} has the dimension of a force.

Theorem 1. *The closed loop consisting of the system (4) under Assumptions 1-5 along with the control law given by (10) and (11) is globally uniformly ultimately bounded. Moreover, the control signal is bounded.*

Theorem 2. *Consider system (4) under Assumptions 1-5 along with the control law given by (10) and (11). Then the following statements hold:*

(a) The transient displacement tracking error performance is given by

$$\begin{aligned}\left(\frac{\|z_1\|_{rms,[0,T]}}{\|x_{ol}\|_{rms,[0,T_0]}}\right)^2 \leq & \left(\frac{m}{m_{\max}}+\frac{m_{\max}}{m}\right)\left(\frac{1}{\gamma}\cdot\left(\frac{\tilde{m}(0)}{m_{\max}}\right)^2+\left\|\frac{\tilde{\theta}(0)}{M_\theta}\right\|_{\Gamma^{-1}}^2+\right.\\ & \left.+\frac{\bar{\sigma}_m}{2c_1}+\frac{\bar{\sigma}_\theta}{2c_1}\cdot\frac{\|\theta\|^2}{M_\theta^2}+\frac{1}{2c_1d_2}+\frac{(1-g)^2}{4c_1c_2}\cdot\frac{\|f\|_{rms,[0,T]}^2}{f_{dv}^2}\right)+\\ & +\frac{g\,(4\varepsilon_1+2\varepsilon_2)}{c_1}\cdot\frac{F}{f_{dv}},\end{aligned} \tag{14}$$

for all $T \geq 0$, and with the notation $\|X\|_P \triangleq \sqrt{X^TPX}$ for any vector X and positive definite matrix P.

(b) The asymptotic displacement tracking error performance is given by

$$\left(\frac{\|z_1\|_{rms,[t_0,\infty]}}{\|x_{ol}\|_{rms,[0,T_0]}}\right)^2 \le \left(\frac{m}{m_{\max}} + \frac{m_{\max}}{m}\right)\cdot$$
$$\cdot\left(\frac{1}{4c_1d_2} + \frac{(1-g)^2}{8c_1c_2}\cdot\frac{\|f\|^2_{rms,[t_0,\infty]}}{f_{dv}^2}\right) + \frac{1}{c_1}g(2\varepsilon_1+\varepsilon_2)\frac{F}{f_{dv}}, \qquad (15)$$

for all $t_0 \ge 0$*.*

(c) The transient velocity tracking error performance is given by

$$\left(\frac{\|\dot{x}-\dot{y}_r\|_{rms,[0,T]}}{\|\dot{x}_{ol}\|_{rms,[0,T_0]}}\right)^2 \le 2\left(\frac{m}{m_{\max}} + \frac{m_{\max}}{m}\right)\left(\frac{1+c_1^2}{\gamma}\cdot\left(\frac{\tilde{m}(0)}{m_{\max}}\right)^2 + \right.$$
$$+\left(1+c_1^2\right)\left\|\frac{\tilde{\theta}(0)}{M_\theta}\right\|^2_{\Gamma^{-1}} + \bar{\sigma}_m\left(\frac{c_1}{2} + \frac{m}{c_2m_{\max}}\right) + \bar{\sigma}_\theta\cdot\frac{\|\theta\|^2}{M_\theta^2}\left(\frac{c_1}{2}+\frac{1}{c_2}\right) +$$
$$\left. +\frac{1}{d_2}\left(\frac{1}{c_2}+\frac{c_1}{2}\right) + \left(\frac{c_1}{c_2} + \frac{m}{m_{\max}c_2^2}\right)(1-g)^2\cdot\frac{\|f\|^2_{rms,[0,T]}}{f_{dv}^2}\right) +$$
$$+ g\left(\frac{2}{c_2}+c_1\right)\left(1+\frac{m}{m_{\max}}\right)(8\varepsilon_1+4\varepsilon_2)\cdot\frac{F}{f_{dv}}, \qquad (16)$$

for all $T \ge 0$*.*

(d) The asymptotic velocity tracking error performance is given by

$$\left(\frac{\|\dot{x}-\dot{y}_r\|_{rms,[t_0,\infty]}}{\|\dot{x}_{ol}\|_{rms,[0,T_0]}}\right)^2 \le 2\left(\frac{m}{m_{\max}} + \frac{m_{\max}}{m}\right)\left(c_1+\frac{2}{c_2}\right)\cdot$$
$$\cdot\left(\frac{1}{4d_2} + \frac{(1-g)^2}{8c_2}\cdot\frac{\|f\|^2_{rms,[t_0,\infty]}}{f_{dv}^2}\right) +$$
$$+ g\left(c_1+\frac{2}{c_2}\right)(4\varepsilon_1+2\varepsilon_2)\left(1+\frac{m}{m_{\max}}\right)\frac{F}{f_{dv}}, \qquad (17)$$

for all $t_0 \ge 0$*.*

Remarks. From Theorem 2 the following conclusions can be drawn:

- The displacement and velocity tracking performances (both transient and asymptotic) are characterized by **explicit** functions of the design parameters.

- The transient performance depends on the initial estimate errors $\tilde{m}(0)$ and $\tilde{\theta}(0)$. The closest the initial estimates $\hat{m}(0)$ and $\hat{\theta}(0)$ to the true values m and θ, the better the transient performance. The asymptotic behavior is not affected by the initial estimate errors.

- We may decrease the effect of the initial error estimates on the transient performance by increasing the adaptation gains γ and Γ. This increase has no effect on the asymptotic performance.

- Over–estimating the mass leads to a poor performance (both transient and asymptotic). Indeed, the term $\frac{m}{m_{\max}} + \frac{m_{\max}}{m}$ which appears in inequalities (14)-(17) is minimal if $m_{\max} = m$ and increases as $m_{\max}$ grows.

- To improve the displacement tracking performance (transient and asymptotic) we may also increase the gains c_1, c_2, d_2 or decrease ε_1, ε_2, $\bar{\sigma}_\theta$ and $\bar{\sigma}_m$. However, increasing the gain c_1 increases also the root mean-square norm of the velocity tracking error (both in the transient and asymptotically). Since the latter enters directly into the control law, this means that improving the closed loop displacement behavior may be done at the expense of an increase in the control signal amplitude. This suggests to fix the gain c_1 to some acceptable value and adjust the other gains.

- The formulae in statements (c) and (d) show that by fixing the gain c_1, increasing the gains c_2, d_2 and decreasing ε_1, ε_2, we may achieve a velocity tracking mean-square error as small as desired both in the transient and asymptotically.

- The gain g may be used for a trade-off between the desired tracking performance and an acceptable control amplitude.

4 Proof of the main results

4.1 Proof of Theorem 1: Stability analysis

We choose as a Lyapunov function candidate:

$$V = \frac{1}{2}\left(\frac{z_1}{a}\right)^2 + \frac{1}{2}\left(\frac{z_2}{v}\right)^2 + \frac{1}{2\gamma} \cdot \frac{m}{m_{\max}}\left(\frac{\tilde{m}}{m}\right)^2 + \frac{1}{2 \cdot \frac{m}{m_{\max}}}\left(\frac{\tilde{\theta}}{M_\theta}\right)^T \Gamma^{-1}\left(\frac{\tilde{\theta}}{M_\theta}\right), \tag{18}$$

where $\hat{m}(t) = m - \tilde{m}(t)$ and $\hat{\theta}(t) = \theta - \tilde{\theta}(t)$ are estimates respectively of m and θ. To simplify the notations, we introduce

$$\bar{\Gamma}^{-1} = \frac{m_{\max}}{M_\theta^2}\Gamma^{-1}, \tag{19}$$

$$\bar{\gamma} = \gamma m_{\max}. \tag{20}$$

From (4), (7), (8) and (9) we obtain:

$$\dot{z}_2 = \frac{1}{m}\left(\theta^T\varphi - R + f + u - m\frac{\partial\alpha_1}{\partial x_1}\dot{x}_1 - m\frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - m\ddot{y}_r\right). \tag{21}$$

Taking into account that

$$\theta^T\varphi = \tilde{\theta}^T\varphi + \hat{\theta}^T\varphi, \tag{22}$$

and

$$-m\frac{\partial\alpha_1}{\partial x_1}\dot{x}_1 - m\frac{\partial\alpha_1}{\partial y_r}\dot{y}_r - m\ddot{y}_r = (\tilde{m} + \hat{m})\left(c_1\frac{v}{a}x_2 - c_1\frac{v}{a}\dot{y}_r - \ddot{y}_r\right), \tag{23}$$

and from equations (7)–(9), and (18)–(23), lengthy but straightforward computations (involving some adequate terms arrangements) lead to the following expression:

$$\begin{aligned} \dot{V} &= -c_1\frac{v}{d^3}z_1^2 + \frac{\tilde{\theta}^T}{m}\bar{\Gamma}^{-1}\left(\frac{1}{v^2}\bar{\Gamma}\varphi z_2 - \dot{\hat{\theta}}\right) \\ &+ \frac{\tilde{m}}{\bar{\gamma}m}\left(\bar{\gamma}\frac{c_1}{dv}x_2 z_2 + \frac{\bar{\gamma}}{d^2}z_1 z_2 - \frac{\bar{\gamma}}{v^2}\ddot{y}_r z_2 - \bar{\gamma}\frac{c_1}{dv}z_2\dot{y}_r - \dot{\hat{m}}\right) \\ &+ \frac{z_2}{mv^2}\left(\frac{v^2}{d^2}\hat{m}z_1 - \hat{m}\ddot{y}_r + \hat{\theta}^T\varphi - R + f + u + \hat{m}c_1\frac{v}{d}x_2 - c_1\frac{v}{d}\hat{m}\dot{y}_r\right). \end{aligned} \tag{24}$$

Choosing the control law and the parameters update laws as in (10) - (11), we get

$$\begin{aligned} \dot{V} = &-c_1\frac{v}{d^3}z_1^2 - \frac{m_{\max}c_2}{dmv}z_2^2 + \frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\sigma_\theta}{m}\tilde{\theta}^T\hat{\theta} + \frac{v}{dm_{\max}}\cdot\frac{\sigma_m}{m}\tilde{m}\hat{m} \\ &- \frac{Rz_2}{mv^2} + \frac{fz_2}{mv^2} - \frac{dd_2}{mm_{\max}v^5}z_2^2 r^2 - \frac{z_2}{mv^2}\mathbf{sg}\left(\frac{z_2}{v}\right)\mathbf{cf}\left(\frac{|z_2|}{v}\right)gF. \end{aligned} \tag{25}$$

For the sake of simplicity, in the above expression and in the following we will use the following notation: σ_m denotes $\sigma_m\left(\frac{|\hat{m}|}{m_{\max}}\right)$ and σ_θ denotes $\sigma_\theta\left(\frac{||\hat{\theta}||}{M_\theta}\right)$.

The right-hand part of inequality (25) contains nonlinear terms that reduce the negativity of $\dot{V}$. In the following we analyze the effect of these terms. For computational purposes we split $\frac{fz_2}{mv^2} = g\frac{fz_2}{mv^2} + (1-g)\frac{fz_2}{mv^2}$, and first, we focus on the term

$$D_f \triangleq g\frac{fz_2}{mv^2} - \frac{z_2}{mv^2}\mathbf{sg}(\frac{z_2}{v})\mathbf{cf}\left(\frac{|z_2|}{v}\right)gF. \tag{26}$$

We will consider the following two cases:

- Case 1: $\frac{|z_2|}{v} \geq 2\varepsilon_1 + \varepsilon_2$
- Case 2: $\frac{|z_2|}{v} \leq 2\varepsilon_1 + \varepsilon_2$

Using the definition of the functions **cf** and **sg**, it follows that in Case 1: $\mathbf{cf}\left(\frac{|z_2|}{v}\right) = 1$ and $\mathbf{sg}\left(\frac{z_2}{v}\right) = \text{sign}(z_2)$. This implies that

$$D_f = g\frac{fz_2}{mv^2} - \frac{|z_2|}{mv^2}gF \leq 0, \tag{27}$$

since $|f| \leq F$ by Assumption 4.

In Case 2, $\left|\mathbf{sg}\left(\frac{z_2}{v}\right)\right| \leq 1$ and $\left|\mathbf{cf}\left(\frac{|z_2|}{v}\right)\right| \leq 1$, hence

$$mv|D_f| \;\leq\; g|f|\frac{|z_2|}{v} + gF\frac{|z_2|}{v} \;\leq\; 2gF\frac{|z_2|}{v} \;\leq\; g\left(4\varepsilon_1 + 2\varepsilon_2\right)F \tag{28}$$

This means that in all cases we have

$$D_f \leq \frac{1}{mv}g(4\varepsilon_1 + 2\varepsilon_2)F. \tag{29}$$

We now turn to the term

$$D_R \triangleq -\frac{Rz_2}{mv^2} - \frac{d\,d_2}{mm_{\max}v^5}z_2^2r^2. \tag{30}$$

Using Assumption 1, it follows that

$$\begin{aligned} D_R &\leq \frac{1}{mv^2}\left(r|z_2| - \frac{d\,d_2}{m_{\max}v^3}r^2z_2^2\right) = \\ &= \frac{1}{mv^2}\left(-\frac{d\,d_2}{m_{\max}v^3}\cdot\left(r|z_2| - \frac{m_{max}v^3}{2d\,d_2}\right)^2 + \frac{m_{\max}v^3}{4d\,d_2}\right) \\ &\leq \frac{m_{max}v}{4dmd_2}. \end{aligned} \tag{31}$$

Combining (25), (29) and (31) we obtain

$$\begin{aligned}\dot{V} &\leq -c_1\frac{v}{d^3}z_1^2 - \frac{m_{\max}c_2}{dmv}z_2^2 + \frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\sigma_\theta}{m}\tilde{\theta}^T\hat{\theta} + \frac{v}{dm_{\max}}\cdot\frac{\sigma_m}{m}\tilde{m}\hat{m} \\ &\quad +\frac{1}{mv}g(4\varepsilon_1+2\varepsilon_2)F + \frac{m_{max}v}{4dmd_2} + (1-g)\,\frac{fz_2}{mv^2}. \end{aligned} \tag{32}$$

On the other hand, it is easy to see that

$$\begin{aligned} &-\frac{m_{\max}c_2}{2dmv}z_2^2 + (1-g)\,\frac{fz_2}{mv^2} = \\ &\qquad = -\frac{m_{\max}c_2}{2dmv}\left(\left(z_2 - \frac{(1-g)\,df}{m_{\max}vc_2}\right)^2 - \frac{(1-g)^2\,d^2f^2}{m_{\max}^2v^2c_2^2}\right), \\ &\qquad \leq \frac{d\,(1-g)^2\,f^2}{2mv^3c_2m_{\max}}, \end{aligned} \tag{33}$$

so that we can rewrite equation (32) as

$$\begin{aligned}\dot{V} &\leq -c_1\frac{v}{d^3}z_1^2 - \frac{m_{\max}c_2}{2dmv}z_2^2 + \frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\sigma_\theta}{m}\tilde{\theta}^T\hat{\theta} + \frac{v}{dm_{\max}}\cdot\frac{\sigma_m}{m}\tilde{m}\hat{m} \\ &\quad +\frac{1}{mv}g(4\varepsilon_1+2\varepsilon_2)F + \frac{m_{max}v}{4dmd_2} + \frac{d(1-g)^2f^2}{2mv^3c_2m_{\max}}. \end{aligned} \tag{34}$$

Now, we take into account the standard properties of the switching σ-modification (Ioannou and Sun 1996, page 588)

$$\sigma_\theta\tilde{\theta}^T\hat{\theta} \leq -\frac{\bar{\sigma}_\theta}{2}\|\tilde{\theta}\|^2 + \frac{\bar{\sigma}_\theta}{2}\|\theta\|^2, \tag{35}$$

$$\sigma_m\tilde{m}\hat{m} \leq -\frac{\bar{\sigma}_m}{2}\tilde{m}^2 + \frac{\bar{\sigma}_m}{2}m^2\,. \tag{36}$$

Combining (34), (35) and (36) we get

$$\begin{aligned}\dot{V} &\leq -c_1\frac{v}{d^3}z_1^2 - \frac{m_{\max}c_2}{2dmv}z_2^2 - \frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\bar{\sigma}_\theta}{2m}\|\tilde{\theta}\|^2 - \frac{v}{dm_{\max}}\cdot\frac{\bar{\sigma}_m}{2m}\tilde{m}^2 + \\ &\quad + \frac{v}{dm_{\max}}\cdot\frac{\bar{\sigma}_m}{2}m + \frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\bar{\sigma}_\theta}{2m}\|\theta\|^2 + \frac{1}{mv}g(4\varepsilon_1+2\varepsilon_2)F + \\ &\quad + \frac{m_{max}v}{4dmd_2} + \frac{d\,(1-g)^2\,f^2}{2mv^3c_2m_{\max}}. \end{aligned} \tag{37}$$

Notice that

$$-c_1\frac{v}{d^3}z_1^2 - \frac{m_{\max}c_2}{2dmv}z_2^2 - \frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\bar{\sigma}_\theta}{2m}\|\tilde{\theta}\|^2 - \frac{v}{dm_{\max}}\cdot\frac{\bar{\sigma}_m}{2m}\tilde{m}^2 \leq -\underline{a}\bar{V} \tag{38}$$

and

$$\bar{a}\bar{V} \geq \frac{1}{2}\left(\frac{z_1}{d}\right)^2 + \frac{1}{2}\left(\frac{z_2}{v}\right)^2 + \frac{1}{2\gamma}\cdot\frac{\tilde{m}^2}{m m_{\max}} + \frac{m_{\max}}{2m}\left(\frac{\tilde{\theta}}{M_\theta}\right)^T \Gamma^{-1}\left(\frac{\tilde{\theta}}{M_\theta}\right) \;=\; V \tag{39}$$

where

$$\bar{V} = \left(\frac{z_1}{d}\right)^2 + \left(\frac{z_2}{v}\right)^2 + \left(\frac{\|\tilde{\theta}\|}{M_\theta}\right)^2 + \frac{\tilde{m}^2}{m m_{\max}} \tag{40}$$

$$\bar{a} = \max\left(\frac{1}{2}, \frac{1}{2\gamma}, \frac{m_{\max}}{2m}\cdot\lambda_{\max}\left(\Gamma^{-1}\right)\right) \tag{41}$$

$$\underline{a} = \frac{v}{d}\min\left(c_1, \frac{m_{\max}c_2}{2m}, \frac{m_{\max}\bar{\sigma}_\theta}{2m}, \frac{\bar{\sigma}_m}{2}\right) \tag{42}$$

and $\lambda_{\max}(\Gamma^{-1})$ is the maximum eigenvalue of Γ^{-1}. Therefore, from equation (37) we obtain

$$\dot{V} \;\leq\; -a^\star V + b + \frac{d\,(1-g)^2 f^2}{2mv^3 c_2 m_{\max}} \tag{43}$$

$$\dot{V} \;\leq\; -a^\star V + b^\star\,, \tag{44}$$

where

$$a^\star = \underline{a}/\bar{a} \tag{45}$$

$$b = \frac{v}{d m_{\max}}\cdot\frac{\bar{\sigma}_m}{2}m + \frac{v m_{\max}}{d M_\theta^2}\cdot\frac{\bar{\sigma}_\theta}{2m}\|\theta\|^2 + \frac{1}{mv}g(4\varepsilon_1 + 2\varepsilon_2)F + \frac{m_{max}v}{4dmd_2} \tag{46}$$

$$b^\star = b + \frac{d\,(1-g)^2 F^2}{2mv^3 c_2 m_{\max}}\,. \tag{47}$$

By direct integration of the differential inequality (44), we have

$$V(t) \;\leq\; V(0)e^{-a^\star t} + \frac{b^\star}{a^\star}\left(1 - e^{-a^\star t}\right) \;\leq\; V(0) + \frac{b^\star}{a^\star}\,. \tag{48}$$

Equation (48) shows that $V(t)$ is uniformly bounded. This implies that z_1, z_2, $\tilde{\theta}$ and $\tilde{m}$ are bounded. Thus, the state variables x_1, x_2, and the parameter estimates $\hat{\theta}$, $\hat{m}$ are also bounded. These facts, along with Assumption 1, lead to the boundedness of the control $u(t)$, which ends the proof of Theorem 1. ∎

4.2 Proof of Theorem 2: Performance analysis

From equation (34), is easy to see that

$$\begin{aligned}\dot{V} &\le -c_1\frac{v}{d^3}z_1^2+\frac{vm_{\max}}{dM_\theta^2}\cdot\frac{\sigma_\theta}{m}\tilde{\theta}^T\hat{\theta}+\frac{v}{dm_{\max}}\cdot\frac{\sigma_m}{m}\tilde{m}\hat{m}\\ &\quad+\frac{1}{mv}g(4\varepsilon_1+2\varepsilon_2)F+\frac{m_{max}v}{4dmd_2}+\frac{d(1-g)^2f^2}{2mv^3c_2m_{\max}}.\end{aligned}\tag{49}$$

Using again the standard properties of the switching σ-modification in (Ioannou and Sun 1996, page 588), we have $\sigma_\theta\tilde{\theta}^T\hat{\theta}\le 0$ and $\sigma_m\tilde{m}\hat{m}\le 0$. This, combined with (49), gives

$$z_1^2\le\frac{d^3}{c_1v}\left(-\dot{V}+\frac{1}{mv}g(4\varepsilon_1+2\varepsilon_2)F+\frac{m_{max}v}{4dmd_2}+\frac{d\,(1-g)^2f^2}{2mv^3c_2m_{\max}}\right).\tag{50}$$

Integrating both parts of (50) we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T z_1^2(t)dt &\le \frac{d^3}{c_1v}\left(\frac{|V(0)-V(T)|}{T}+\frac{1}{mv}g(4\varepsilon_1+2\varepsilon_2)F+\frac{m_{max}v}{4dmd_2}+\right.\\ &\quad\left.+\frac{d\,(1-g)^2}{2mv^3c_2m_{\max}}\cdot\frac{1}{T}\int_0^T f^2(\tau)d\tau\right).\end{aligned}\tag{51}$$

On the other hand, from (43), we have

$$\begin{aligned}\frac{|V(0)-V(T)|}{T} &\le \frac{1-e^{-a^\star T}}{T}\left(\frac{b}{a^\star}+V(0)\right)+\\ &\quad+\frac{d\,(1-g)^2}{2mv^3c_2m_{\max}T}\int_0^T e^{-a^\star(T-\tau)}f^2(\tau)d\tau\\ &\le b+a^\star V(0)+\frac{d\,(1-g)^2}{2mv^3c_2m_{\max}T}\int_0^T f^2(\tau)\,d\tau,\ \text{for all } T\ge 0,\end{aligned}\tag{52}$$

where we have used the fact that $e^{-a^\star(T-\tau)}\le 1$, and that $(1-e^{-a^\star T})/T\le a^\star$ for all $T\ge 0$.

Initializing appropriately the reference trajectories (Krstic et al. 1995, Chapter 4) by taking $y_r(0)=x_1(0)$ and $\dot{y}_r(0)=x_2(0)$ we obtain

$$V(0)=\frac{\tilde{m}(0)^2}{2\gamma mm_{\max}}+\frac{m_{\max}}{2mM_\theta^2}\tilde{\theta}^T(0)\Gamma^{-1}\tilde{\theta}(0).\tag{53}$$

Using (45)-(46), the fact that $a^\star/c_1 \leq 2v/d$, as well as equations (52) and (53), we obtain

$$\begin{aligned}\frac{1}{T}\int_0^T z_1^2(t)dt \;\; &\leq \;\; \frac{d^2\tilde{m}(0)^2}{\gamma m m_{\max}} + \frac{d^2 m_{\max}}{m}\left\|\frac{\tilde{\theta}(0)}{M_\theta}\right\|^2_{\Gamma^{-1}} + \frac{4d^3}{c_1 v^2 m} g(2\varepsilon_1+\varepsilon_2)F \\ &+\frac{d^2 m_{\max}}{2mc_1d_2} + \frac{d^2 m\bar{\sigma}_m}{2c_1 m_{\max}} + \frac{d^2 m_{\max}\bar{\sigma}_\theta}{2c_1 m}\cdot\frac{\|\theta\|^2}{M_\theta^2} \\ &+\frac{d^4\left(1-g\right)^2}{mv^4c_2c_1m_{max}}\cdot\frac{1}{T}\int_0^T f^2(\tau)d\tau \qquad (54)\end{aligned}$$

for all $T \geq 0$. Dividing both terms of equation (54) by $||x_{ol}||^2_{rms,[0,T_0]} = d^2$, using the definition of f_{dv} given by equation (13), and the fact that $\frac{m_{max}}{m} < \left(\frac{m_{max}}{m} + \frac{m}{m_{max}}\right)$, $\frac{m}{m_{max}} < \left(\frac{m_{max}}{m} + \frac{m}{m_{max}}\right)$, and $2 \leq \left(\frac{m_{max}}{m} + \frac{m}{m_{max}}\right)$, we obtain equation (14), which proves statement (a) of Theorem 2.

Integrating both parts of (50), we obtain

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} z_1^2(t)dt \;\; &\leq \;\; \frac{d^3}{mv^2c_1}g(4\varepsilon_1+2\varepsilon_2)F + \frac{d^2 m_{\max}}{4mc_1d_2} \qquad (55) \\ &+ \;\; \frac{d^4\left(1-g\right)^2}{2mv^4c_1c_2m_{\max}}\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T} f^2(t)dt \quad (56)\end{aligned}$$

for all $t_0 \geq 0$. Indeed, $\lim\limits_{T\to\infty} \frac{|V(t_0+T)-V(t_0)|}{T} = 0$, as $V(t)$ is bounded (Theorem 1). Dividing both terms of equation (55) by $||x_{ol}||^2_{rms,[0,T_0]}$, and using the same arguments as above, we get equation (15). This proves statement (b) of Theorem 2.

Statements (c) and (d) of Theorem 2 follow in a similar way using equations (8)–(9).

5 Application to a hysteretic base-isolation system

5.1 The system

In this section, the backstepping controller designed in Section 3 is applied to a system within the class described in Section 2. It is illustrated in Figure 1. This system is the main component in base isolation schemes installed to supply passive and active protection of structures (like buildings) against earthquakes. The passive resistance relies on the physical design of the isolator between the base and the foundation. The active protection is added by a controller that supplies forces generated by a feedback control law. Hybrid schemes by combining passive and active schemes against seismic excitations have been proposed in recent years, attracting the interest of researchers both from structural and control engineering (Kelly et al. 1987, Yang et al. 1994, Luo et al. 2000).

In this work, the isolation scheme, as illustrated in Figure 1(b), is modelled as 1 degree-of-freedom system with mass m and damping c plus a restoring force Φ characterizing a hysteretic behavior of the isolator material, which is usually made with inelastic rubber bearings. The excitation force is $f(t) = -ma(t)$, where $a(t)$ is the earthquake ground acceleration. The hysteretic force Φ is described by the so-called Bouc–Wen model (Wen 1976) in the following form:

$$\Phi(x,t) = \alpha k x(t) + (1-\alpha) D k z(t), \tag{57}$$

$$\dot{z} = D^{-1}\left[A\dot{x} - \beta|\dot{x}|\,|z|^{n-1}z - \lambda\dot{x}|z|^{n}\right]. \tag{58}$$

This model represents the restoring force $\Phi(x,t)$ by the superposition of an elastic component $\alpha k x$ and a hysteretic component $(1-\alpha)kDz$, in which $D > 0$ is the yield constant displacement and $\alpha \in [0,1]$ is the post to pre-yielding stiffness ratio. The hysteretic part involves a nondimensional auxiliary variable z which is the solution of the nonlinear first order differential equation (58). In this equation, A, β and γ are nondimensional parameters which control the shape and the size of the hysteresis loop, while n is an integer that governs the smoothness of the transition from elastic to plastic response. The Bouc-Wen model is able to capture, in an analytical form, a range of shapes of hysteretic cycles which match the behavior of a wide class of nonlinear structures. This is why it is widely used in structural dynamics, particularly to describe rubber bearing base isolation schemes (Barbat and Bozzo 1997).

5.2 Control design

The first issue to design the backstepping controller for the base isolation system is to verify that the restoring force Φ given by the Bouc-Wen model can be written in the form assumed in (2) and that the Assumptions 1-5 are satisfied.

Assumptions 3 and 4 are easily satisfied since we consider that the seismic acceleration $a(t)$ is unknown but is bounded by a known constant, while the unknown mass m and damping c have known maximum values. In practice, for a real base isolation system, the parameters $\alpha, D, A, \beta, \lambda$ and n are usually obtained by identification procedures (Smyth et al. 1999), trying to approximate experimental data. To cope with discrepancies between the identified model and the real hysteretic behavior, it is reasonable to assume that the identification process leads to the knowledge of an interval for each parameter. Thus, denoting anyone of these parameters by p, we consider $p \in [p_{min}, p_{max}]$, p_{min} and p_{max} being known, and define $p^* = (p_{\min} + p_{max})/2$. With these notations, we write (57) in the form

$$\begin{aligned} \Phi(x,t) &= (\alpha k - \delta)\, x + (1-\alpha)Dkz + \delta x \\ &= \phi_1 \frac{x}{d} + (1-\alpha)Dkz + \delta x, \end{aligned} \tag{59}$$

where $\delta = \alpha^* k^*$.

Equation (59) is under the form (2) with $\phi_1 = d\,(\alpha k - \delta)$, $\psi_1(x) = \dfrac{x}{d}$, and $R(x,t) = (1-\alpha)Dkz$. Since the term δx is known, it will be incorporated into the control u.

A bound on ϕ_1 may be determined as follows:

$$|\phi_1| \leq d \max\left(\alpha_{\max} k_{\max} - \alpha^* k^*, \alpha^* k^* - \alpha_{\min} k_{\min}\right) \triangleq M_\phi, \tag{60}$$

so that Assumption 2 is verified.

For the residual term R we have the following inequality:

$$|R(x,t)| \leq (1-\alpha_{\min})\, D_{\max} k_{\max} \max_{t \geq 0} |z(t)| \triangleq r. \tag{61}$$

The remaining question is now the boundedness of $|z(t)|$, so that r exists and consequently Assumption 1 can be verified. Appendix A presents a study of the Bouc-Wen model, which proves that, for any piecewise continuos signal $\dot{x}(t)$ (not necessarily bounded), the solution $z(t)$ of the differential equation (58) is bounded provided that the lower and upper bounds of the model parameters and the initial condition $z(0)$ are chosen within a specified range of values. The bound on the solution $z(t)$ is explicitly given in terms of these values and summarized in Table 1.

5.3 Simulation results

For the simulations, the following values are selected as "true" nominal parameters for the system and the hysteresis model: $m_e = 156 \cdot 10^3 Kg$, $k_e = 6 \cdot 10^6 N/m$, $c_e = 2 \cdot 10^4 Ns/m$, $\alpha_e = 0.6$, $D_e = 0.6m$, $A_e = 1$, $\beta_e = 0.1$, $\lambda_e = 0.5$, $n_e = 3$. In fact, it is not required to know the exact values of these parameters to implement the controller, only their $[min, max]$ intervals are needed. We assume that the following intervals are known:

$$\begin{aligned}
0 < \quad & m && \leq 2m_e \triangleq m_{\max}, \\
\frac{k_e}{2} \triangleq k_{\min} \leq \quad & k && \leq 2k_e \triangleq k_{\max}, \\
0 \leq \quad & c && \leq 2c_e \triangleq c_{\max}, \\
\frac{\alpha_e}{2} \triangleq \alpha_{\min} \leq \quad & \alpha && \leq \min(2\alpha_e, 1) \triangleq \alpha_{\max}, \\
\frac{D_e}{2} \triangleq D_{\min} \leq \quad & D && \leq 2D_e \triangleq D_{\max}, \\
\frac{A_e}{2} \triangleq A_{\min} \leq \quad & A && \leq 2A_e \triangleq A_{\max}, \\
\frac{\beta_e}{2} \triangleq \beta_{\min} \leq \quad & \beta && \leq 2\beta_e \triangleq \beta_{\max}, \\
\frac{\lambda_e}{2} \triangleq \lambda_{\min} \leq \quad & \lambda && \leq 2\lambda_e \triangleq \lambda_{\max}, \\
\frac{n_e}{2} \triangleq n_{\min} \leq \quad & n && \leq 2n_e \triangleq n_{\max}.
\end{aligned} \tag{62}$$

With these values we are in the case $A_{\min} > 0$, $\beta_{\min} + \lambda_{\min} > 0$, $\beta_{\max} - \lambda_{\min} < 0$ and $\beta_{\min} > 0$ in Table 1. Thus we can obtain from this table an upper bound on $z(t)$ as $\max_{t \geq 0} |z(t, z(0))| \leq \max(|z(0)|, \bar{z}_0) = \bar{z}_0$, taking $z(0) = 0$ and $\bar{z}_0$ given by (72) in Appendix A. In this way, we determine the constant r from equation (61).

The control law is obtained from equation (10):

$$\begin{aligned}
u(t) \quad = \quad & -\hat{\theta}^T \varphi - c_1 \frac{v}{d} (x_2 - \dot{y}_r) \hat{m} - \frac{v^2}{d^2} \hat{m} z_1 + \hat{m} \ddot{y}_r - \frac{d}{v^3 m_{\max}} d_2 z_2 r^2 \\
& - \frac{v m_{\max}}{d} c_2 z_2 - \mathbf{sg}\left(\frac{z_2}{v}\right) \mathbf{cf}\left(\frac{|z_2|}{v}\right) gF + \delta x_1,
\end{aligned} \tag{63}$$

where the known term δx_1 has been incorporated to the control law. To compute the control we need to choose the scaling factors d, v and the design parameters. To choose the scaling factors we determine the open loop response of the hysteretic system to the excitation of the Taft's earthquake acceleration $a(t)$ shown in Figure 2. An upper bound of its infinity norm is

$F = 1.2m_e$ [m/s^2]. Then we take d and v as the root mean-square of the open loop displacement and velocity respectively during the time period $T_0 = 20$ seconds as defined in (3) and (5).

The performance analysis of Section 3.2 gave explicit formal bounds on the displacement and velocity tracking error both in the transient and asymptotically. These bounds give a precise indication on how to increase or decrease the design parameters to improve performance. We take the following set of design parameters: $\gamma = 0.1$, $\Gamma = 0.1 \times I_2$, $\varepsilon_1 = 0.1$, $\varepsilon_2 = 0.1$, $\bar{\sigma}_\theta = 0.1$, $\bar{\sigma}_m = 0.1$, $c_1 = 1$, $c_2 = 0.02$, $d_2 = 0.00045$, $g = 0.333$. The reference trajectory $y_r(t)$ is set to zero. The initial values of $\hat{\theta}$ and $\hat{m}$ are set to $\hat{\theta}(0) = (c_{\max} v, M_\phi)^T$ and $\hat{m}(0) = m_{\max}$.

The comparison of the open loop and closed loop behaviors is shown in Figures 3–6. Figures 3 and 4 show the time history of the state variables x_1 (displacement) and x_2 (velocity). A significant reduction in both displacement and velocity can be observed in the controlled case. After $t = 20$ seconds, the excitation disappears and the uncontrolled case corresponds to free vibration response. The open loop system exhibits a low damping behavior. On the contrary, the control drives the response fast to zero, thus introducing a significant damping effect into the system. These features are also observed in Figure 5, which depicts the solution the open (dashed line) and closed loop (solid line, in the center) system in the phase space, including the auxiliary variable z that describes the hysteretic behavior. The control acceleration signal $u(t)/m_e$ is shown in Figure 6. Its magnitude seems reasonable in comparison with the seismic excitation acceleration $a(t)$ plotted in Figure 2.

6 Conclusion

This paper has presented a backstepping-based adaptive controller for a class of strict-feedback uncertain nonlinear one input one output systems. Hysteretic mechanical systems arising, for instance, in active control of base isolation vibration schemes have been adopted as a prototype. The system has a linear part and a nonlinear part, which is represented as the sum of the scalar product of a parameter vector and a function vector plus a residual function. All these functions may be unbounded with respect to the output and are bounded with respect to the time. The function vector is known, while the residual function is unknown but its absolute value is less than a known function. All the system parameters are unknown but lie within known intervals. One of these parameters appears in the model as multiplying the control input. In the mechanical prototype, this parameter is the inverse of the mass. Unlike most adaptive backstepping control strategies for nonlinear strict-feedback systems, the controller presented in this paper has been able to cope with the uncertainty in this parameter. It has been proved that the closed loop is globally uniformly ultimately bounded and explicit bounds on both the asymptotic and transient performance have been given allowing the designer to tune the control parameters to obtain desired closed loop behaviors.

The control strategy has been numerically tested on a one degree of freedom mechanical system commonly found in base isolation schemes for seismic active protection of building structures. This system exhibits a hysteretic nonlinear behavior which is described analytically by the so-called Bouc–Wen model. Unlike other active control approaches for base isolation systems, the developed backstepping control does not require an exact knowledge of the model parameters. They are only defined within known intervals. The applicability of the backstepping controller is guaranteed by an analytical study of the interval Bouc-Wen model, which supplies regions for the parameter intervals such that the hysteretic behavior predicted by the model lies under the class of nonlinearity assumed in the control design. The numerical results showed that the combination of this description of the hysteresis and the backstepping adaptive control law is satisfactory in that the response induced by the seismic action is significantly reduced. These results are encouraging towards the applicability of the control scheme proposed in this paper.

A Appendix: Boundedness of the Bouc–Wen model

Let us consider the differential equation (58). A solution $z(t)$ is said to be *ultimately bounded* (Khalil 1992, Definition 4.4) if there exist positive constants b_0 and c_0, and for every $\alpha_0 \in (0, c_0)$ we have

$$|z(0)| < \alpha_0 \Rightarrow |z(t)| < b_0, \ \forall t \geq 0.$$

We define the set $\Omega_{\dot{x}}$ as *the largest set of initial conditions* $z(0)$ *for which* $z(t, z(0))$ *is ultimately bounded for a given piecewise continuous signal* $\dot{x}$. This set is the region of ultimate boundedness corresponding to the differential equation (58) for a particular piecewise continuous signal $\dot{x}$ (not necessarily bounded). This motivates the definition of the set $\Omega \triangleq \bigcap_{\dot{x}} \Omega_{\dot{x}}$ as the set of initial conditions $z(0)$ for which $z(t, z(0))$ is ultimately bounded *for every piecewise continuous signal* $\dot{x}$. The following analysis determines the set Ω as an explicit function of the Bouc–Wen model parameters. It also determines the bounds on the output $z(t)$. The study is performed in two steps: first nominal parameter values are assumed in three different cases: $A > 0$, $A < 0$ and $A = 0$; second, the Bouc-Wen model parameters are given by interval bounds as it is considered for the controller design and implementation.

A.1 Nominal parameters. Case $A > 0$

Consider the following four possibilities:

- $P_1 : \beta + \lambda > 0$ and $\beta - \lambda \geq 0$.
- $P_2 : \beta + \lambda > 0$ and $\beta - \lambda < 0$.
- $P_3 : \beta + \lambda \leq 0$ and $\beta - \lambda \geq 0$.
- $P_4 : \beta + \lambda \leq 0$ and $\beta - \lambda < 0$.

Let's focus on the case P_1. We consider the Lyapunov function candidate $V(t) = z(t)^2/2$. Its derivative takes different forms depending on the signs of $\dot{x}$ and z. Indeed, setting $Q_1 := \{\dot{x} \geq 0 \text{ and } z \geq 0\}$, and denoting $\dot{V}_{|Q_1}$ as the expression of the derivative of the Lyapunov function V over the set Q_1, we have $\dot{V}_{|Q_1} = z\dot{x}D^{-1}\left(A - (\beta+\lambda)\, z^n\right)$. Thus $\dot{V}_{|Q_1} \leq 0$ for

$$z \geq \sqrt[n]{\frac{A}{\beta+\lambda}} \triangleq z_0. \tag{64}$$

Also, if we set $Q_2 := \{\dot{x} \geq 0 \text{ and } z \leq 0\}$, we have $\dot{V}_{|Q_2} = z\dot{x}D^{-1}(A + (\beta - \lambda)|z|^n)$. In this case, $\dot{V}_{|Q_2} \leq 0$ for all values of z. The same conclusion is drawn in the case of $Q_3 := \{\dot{x} \leq 0 \text{ and } z \geq 0\}$, since $\dot{V}_{|Q_3} = z\dot{x}D^{-1}\left(A + (\beta-\lambda)\, z^n\right)$. Finally, taking $Q_4 := \{\dot{x} \leq 0 \text{ and } z \leq 0\}$, we get $\dot{V}_{|Q_4} = z\dot{x}D^{-1}\left(A - (\beta+\lambda)\,|z|^n\right)$. Thus, $\dot{V}_{|Q_4} \leq 0$ for $|z| \geq z_0$. We then conclude that, for all the possibilities of the signs of $\dot{x}$ and z, we have $\dot{V} \leq 0$ for all $|z| \geq z_0$. By Theorem 4.10 in (Khalil 1992) we conclude that $z(t)$ is bounded for every piecewise function $\dot{x}(t)$ and every initial condition $z(0)$. The bounds on $z(t)$ can be derived from (Khalil 1992, equations 4.22 and 4.23) as follows:

- If the initial condition of z is such that $|z(0)| \leq z_0$, then $|z(t)| \leq z_0$ for all $t \geq 0$
- If the initial condition of z is such that $|z(0)| \geq z_0$, then $|z(t)| \leq |z(0)|$ for all $t \geq 0$

We now consider the case P_2. Again, the derivative of $V(t)$ depends on the signs of $\dot{x}$ and z. Indeed, $\dot{V} \leq 0$ in the following regions:

$$\{\dot{x} \geq 0 \text{ and } z \geq 0 \text{ and } z \geq z_0\}, \tag{65}$$

$$\{\dot{x} \geq 0 \text{ and } z \leq 0 \text{ and } |z| \leq z_1\}, \tag{66}$$

$$\{\dot{x} \leq 0 \text{ and } z \geq 0 \text{ and } z \leq z_1\}, \tag{67}$$

$$\{\dot{x} \leq 0 \text{ and } z \leq 0 \text{ and } |z| \geq z_0\}. \tag{68}$$

where

$$z_1 \triangleq \sqrt[n]{\frac{A}{\lambda - \beta}}.$$

Then, from (65)-(68) we conclude that: if $z_1 > z_0$ (that is, when $\beta > 0$), then $\dot{V} \leq 0$ for every $z_0 \leq |z| \leq z_1$ independently of the sign of $\dot{x}$. By Theorem 4.10 in (Khalil 1992) we conclude that $z(t)$ is bounded for every piecewise continuous function $\dot{x}(t)$ and any initial state $z(0)$ such that $|z(0)| < z_1$. Using the same theorem we can obtain the following bound : $|z(t)| \leq \max(|z(0)|, z_0)$.

If we turn now to the cases P_3 and P_4, then we can see following the same analysis that $z(t)$ may be unbounded for some functions $\dot{x}$. This implies that the region of ultimate boundedness is empty.

The following table summarizes the results for all cases with $A > 0$.

Case $A > 0$	Region of UB, Ω	Bound on $z(t)$
$\beta + \lambda > 0$ and $\beta - \lambda \geq 0$	$z(0)$ arbitrary	$\lvert z(t)\rvert \leq \max(\lvert z(0)\rvert, z_0)$
$\beta + \lambda > 0$, $\beta - \lambda < 0$ and $\beta > 0$	$\lvert z(0)\rvert < z_1$	$\lvert z(t)\rvert \leq \max(\lvert z(0)\rvert, z_0)$
$\beta + \lambda > 0$, $\beta - \lambda < 0$ and $\beta \leq 0$	$\emptyset$	
$\beta + \lambda \leq 0$ and $\beta - \lambda$ arbitrary	$\emptyset$	

A.2 Nominal parameters. Cases $A < 0$ and $A = 0$

Following analogous arguments, the results are summarized in the following tables.

Case $A < 0$	Region of UB, Ω	Bound on $z(t)$
$\beta - \lambda > 0$ and $\beta + \lambda \geq 0$	$z(0)$ arbitrary	$\lvert z(t)\rvert \leq \max(\lvert z(0)\rvert, z_1)$
$\beta - \lambda > 0$, $\beta + \lambda < 0$ and $\beta > 0$	$\lvert z(0)\rvert < z_0$	$\lvert z(t)\rvert \leq \max(\lvert z(0)\rvert, z_1)$
$\beta - \lambda > 0$, $\beta + \lambda < 0$ and $\beta \leq 0$	$\emptyset$	
$\beta - \lambda \leq 0$ and $\beta + \lambda$ arbitrary	$\emptyset$	

Case $A=0$	Region of UB, Ω	Bound on $z(t)$
$\beta+\lambda\geq 0$ and $\beta-\lambda\geq 0$	$z(0)$ arbitrary	$\lvert z(t)\rvert \leq \lvert z(0)\rvert$
$\beta+\lambda\geq 0$, $\beta-\lambda<0$	$\emptyset$	
$\beta+\lambda<0$ and $\beta-\lambda$ arbitrary	$\emptyset$	

A.3 Interval parameters

Let us consider now that the parameters of the Bouc-Wen model are not known exactly. Instead, known interval bounds $[min, max]$ are available, for instance, from an identification procedure. The purpose is to determine a computable interval contained in the region of ultimate boundedness and a computable bound on the solution $z(t)$ of the differential equation (58). We define the following constants:

$$\underline{z}_1 = \min\left(\sqrt[n_{\max}]{\left|\frac{A_{\min}}{\lambda_{\max}-\beta_{\min}}\right|},\ \sqrt[n_{\min}]{\left|\frac{A_{\min}}{\lambda_{\max}-\beta_{\min}}\right|}\right), \tag{69}$$

$$\bar{z}_1 = \max\left(\sqrt[n_{\max}]{\left|\frac{A_{\min}}{\lambda_{\max}-\beta_{\min}}\right|},\ \sqrt[n_{\min}]{\left|\frac{A_{\min}}{\lambda_{\max}-\beta_{\min}}\right|}\right), \tag{70}$$

$$\underline{z}_0 = \min\left(\sqrt[n_{\max}]{\left|\frac{A_{\max}}{\lambda_{\min}+\beta_{\min}}\right|},\ \sqrt[n_{\min}]{\left|\frac{A_{\max}}{\lambda_{\min}+\beta_{\min}}\right|}\right), \tag{71}$$

$$\bar{z}_0 = \max\left(\sqrt[n_{\max}]{\left|\frac{A_{\max}}{\lambda_{\min}+\beta_{\min}}\right|},\ \sqrt[n_{\min}]{\left|\frac{A_{\max}}{\lambda_{\min}+\beta_{\min}}\right|}\right). \tag{72}$$

The calculations (lengthy but straightforward) are summarized in Table 1. For example, if $A_{\min}>0$, $\beta_{\min}+\lambda_{\min}>0$ and $\beta_{\max}-\lambda_{\min}\geq 0$, then each trajectory $z(t,z(0))$ is ultimately bounded and a computable bound on $|z(t)|$ is $\max(|z(0)|,\bar{z}_0)$. Consider now the case $A_{\min}>0$, $\beta_{\min}+\lambda_{\min}>0$ and $\beta_{\max}-\lambda_{\min}<0$. For $\beta_{\max}\leq 0$ we have an empty region of ultimate boundedness. From a practical point of view, this case should not be really expected from an identification procedure. Indeed one should expect to have a bounded solution $z(t)$ to have a resulting bounded function $\Phi(x,t)$ able to represent a real hysteretic behavior. Therefore, we consider such an unbounded case without a physical meaning. When $\beta_{\min}\leq 0<\beta_{\max}$, we cannot know from the available information if the region of ultimate boundedness is empty or not. This case may suggest to repeat the identification of the parameters with more care to look for a physical meaning.

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CASE				Comput. Reg. of UB	Computable bound
$A_{\min} > 0$	$\beta_{\min} + \lambda_{\min} > 0$	$\beta_{\min} - \lambda_{\max} \geq 0$		$z(0)$ arbitrary	$\lvert z(t)\rvert \leq \max\left(\lvert z(0)\rvert, \bar{z}_0\right)$
		$\beta_{\max} - \lambda_{\min} < 0$	$\beta_{\min} > 0$	$\lvert z(0)\rvert < \underline{z}_1$	$\lvert z(t)\rvert \leq \max\left(\lvert z(0)\rvert, \bar{z}_0\right)$
			$\beta_{\max} \leq 0$	Not a physical model	
			$\beta_{\min} \leq 0 < \beta_{\max}$	Refine the identification procedure	
		$\beta_{\min} - \lambda_{\max} < 0 \leq \beta_{\max} - \lambda_{\min}$		Refine the identification procedure	
	$\beta_{\max} + \lambda_{\max} \leq 0$	Not a physical model			
	$\beta_{\min} + \lambda_{\min} \leq 0 <$ $< \beta_{\max} + \lambda_{\max}$	Refine the identification procedure			
$A_{\max} < 0$	$\beta_{\min} - \lambda_{\max} > 0$	$\beta_{\min} + \lambda_{\min} \geq 0$		$z(0)$ arbitrary	$\lvert z(t)\rvert \leq \max\left(\lvert z(0)\rvert, \bar{z}_1\right)$
		$\beta_{\max} + \lambda_{\max} < 0$	$\beta_{\min} > 0$	$\lvert z(0)\rvert < \underline{z}_0$	$\lvert z(t)\rvert \leq \max\left(\lvert z(0)\rvert, \bar{z}_1\right)$
			$\beta_{\max} \leq 0$	Not a physical model	
			$\beta_{\min} \leq 0 < \beta_{\max}$	Refine the identification procedure	
		$\beta_{\min} + \lambda_{\min} < 0 \leq \beta_{\max} + \lambda_{\max}$		Refine the identification procedure	
	$\beta_{\max} - \lambda_{\min} \leq 0$	Not a physical model			
	$\beta_{\min} - \lambda_{\max} \leq 0 <$ $< \beta_{\max} - \lambda_{\min}$	Refine the identification procedure			
$A_{\min} \leq 0 \leq$ $\leq A_{\max}$	$\beta_{\min} + \lambda_{\min} > 0$	$\beta_{\min} - \lambda_{\max} > 0$		$z(0)$ arbitrary	$\lvert z(t)\rvert \leq \max\left(\lvert z(0)\rvert, \bar{z}_0, \bar{z}_1\right)$
		$\beta_{\max} - \lambda_{\min} < 0$	$\beta_{\min} > 0$	Refine the identification procedure	
			$\beta_{\max} \leq 0$	Not a physical model	
			$\beta_{\min} \leq 0 < \beta_{\max}$	Refine the identification procedure	
		$\beta_{\min} - \lambda_{\max} \leq 0 \leq \beta_{\max} - \lambda_{\min}$		Refine the identification procedure	
	$\beta_{\max} + \lambda_{\max} < 0$	Not a physical model			
	$\beta_{\min} + \lambda_{\min} \leq 0 \leq$ $\leq \beta_{\max} + \lambda_{\max}$	Refine the identification procedure			

Table 1: Boundedness of the Bouc-Wen model

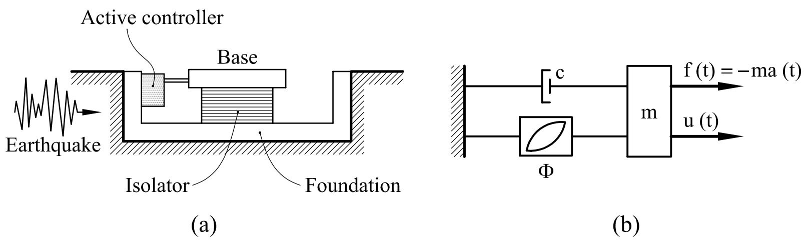

Figure 1: Base isolation system (a) and physical model (b).

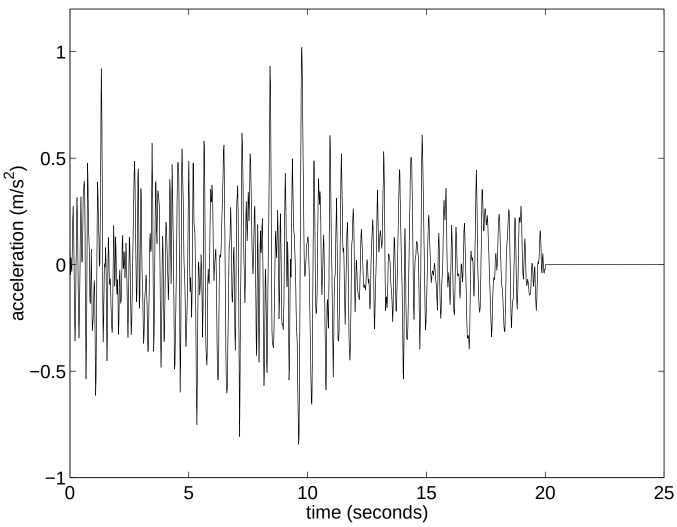

Figure 2: Earthquake ground acceleration.

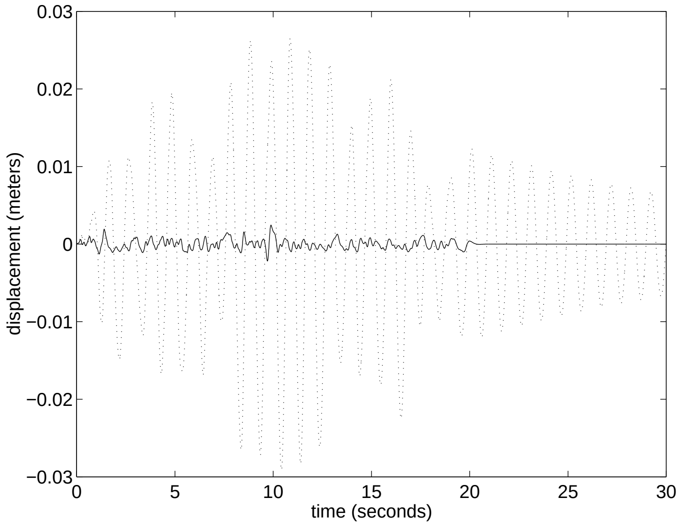

Figura 3: Closed loop displacement (solid) and open loop displacement (dashed).

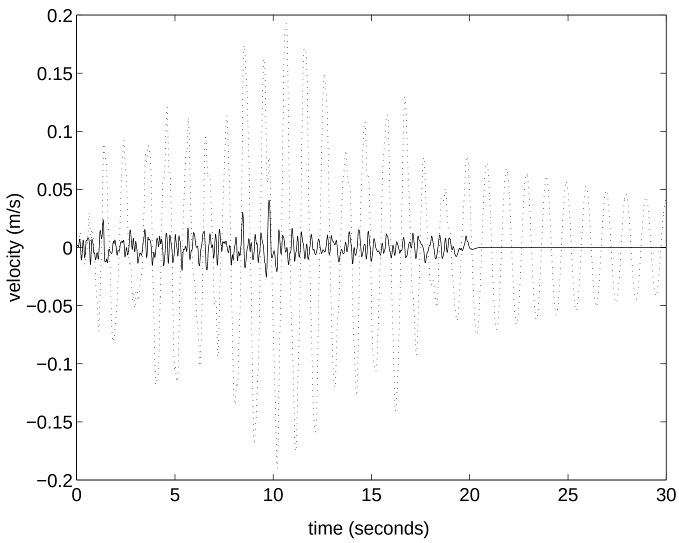

Figure 4: Closed loop velocity (solid) and open loop velocity (dashed).

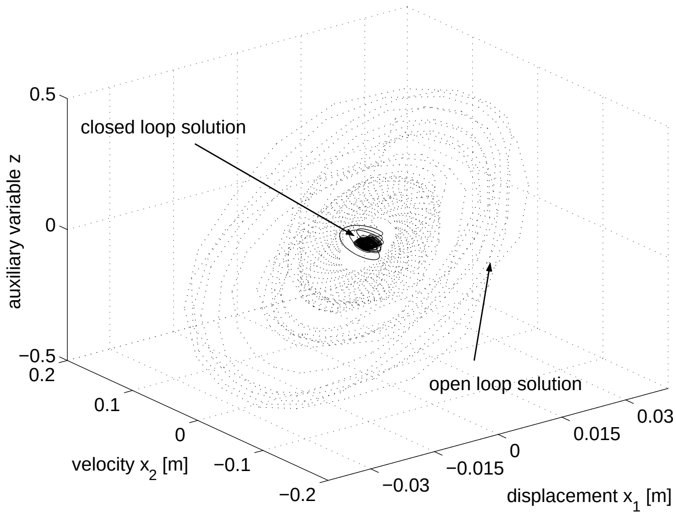

Figure 5: Phase portrait of the closed loop system (solid) and the open loop system (dashed).

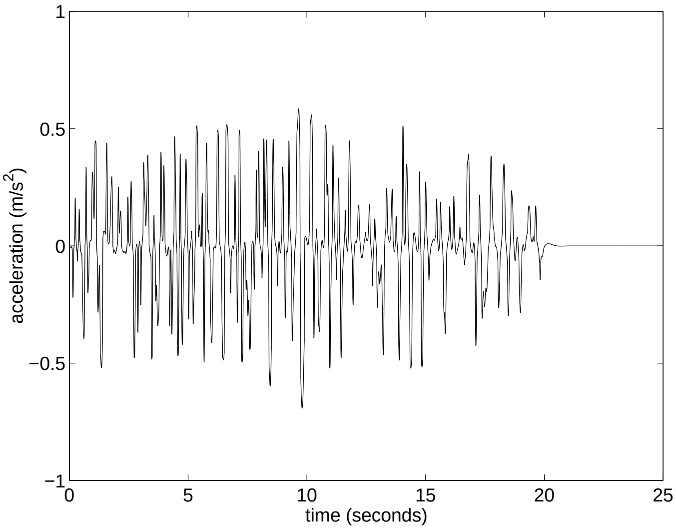

Figure 6: Control signal (acceleration, $u(t)/m_e$).