# NEW FAMILIES OF CENTERS AND LIMIT CYCLES FOR POLYNOMIAL DIFFERENTIAL SYSTEMS WITH HOMOGENEOUS NONLINEARITIES*

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*Dedicated to Professor Yanqian Ye on the occasion of his 80th birthday*

**Abstract**

We consider the class of polynomial differential equations $\dot{x} = -y + P_n(x,y)$, $\dot{y} = x + Q_n(x,y)$, where $P_n$ and $Q_n$ are homogeneous polynomials of degree $n$. Inside this class we identify a new subclass of systems having a center at the origin. We show that this subclass contains at least two subfamilies of isochronous centers. By using a method different from the classical ones, we study the limit cycles that bifurcate from the periodic orbits of such centers when we perturb them inside the class of all polynomial differential systems of the above form. In particular, we present a function whose simple zeros correspond to the limit cycles which bifurcate from the periodic orbits of Hamiltonian systems.

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# 1 Introduction

Two of the main problem in the qualitative theory of real planar differential systems are the determination of centers and of limit cycles. Limit cycles of planar vector fields were defined by Poincaré [44], and started to be studied intensively at the end of 1920s by van der Pol [45], Liénard [34] and Andronov [1]. A *limit cycle* is a periodic orbit of the planar differential system isolated in the set of all periodic orbits.

One of the classical ways to produce limit cycles is by perturbing a system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the original system, see for instance Pontrjagin [46]. As usual a *center* is a singular point having a neighbourhood filled of periodic orbits.

In this paper we deal with the class of real planar polynomial differential systems of the form

$$\dot{x} = -y + P_n(x,y), \qquad \dot{y} = x + Q_n(x,y), \tag{1}$$

where $P_n$ and $Q_n$ are homogeneous polynomials of degree $n$. Inside this class we will consider a new subclass having a center at the origin, and we give a new method to study the limit cycles which bifurcate from their periodic orbits when we perturb this subclass inside the class of all systems (1), see Theorem 4 and Propositions 5 and 6.

There are essentially three well–known methods for studying the limit cycles which bifurcate from a center. The first is based on the Poincaré return map (see for instance [3] and [14]); the second on the Poincaré–Melnikov integral or Abelian integral which are equivalent in the plane (see [3], Section 6 of Chapter 4 of [26], and Section 5 of Chapter 6 of [2]); and the third is based on Theorem 9 of [22] (see Section 6 of that paper and [23]). These methods have been used by several authors for studying the limit cycles which bifurcate from a center of a Hamiltonian system, or of a system which can be reduced after a change of variables to a Hamiltonian one. In general these methods are difficult to apply for studying the limit cycles that bifurcate from the periodic orbits of a center when the system is integrable but not Hamiltonian. As far as we know there are few papers studying these class of non–Hamiltonian centers, [14, 17, 24, 32, 33]. The method that we have found in this paper is different. We used it by first time in [31]. The difference of this method with the classical ones consists in reducing the problem to a one–dimensional differential equation where explicit integral expressions for the solution of the first variational equation can be computed. Clearly, for the class of polynomial differential equations studied here our method is easier than the method based on the Abelian integrals for these class of problems.

This method was inspired in ideas that appear in the paper [39] of Lins Neto when he studied a class of Abel differential equations.

When $n = 2$ systems (1) are quadratic polynomial differential systems. Many authors have studied the limit cycles which bifurcate from periodic orbits of a center for a quadratic system, see for instance, [21, 27, 35, 38, 41, 42, 47, 50].

Some other results about limit cycles of systems (1) for arbitrary $n \geq 2$ have been given in [4], [5], [6], [11], [18] and [36]. The same systems with $P_n(x,y) = (ax+by)R_{n-1}(x,y)$, $Q_n(x,y) = (cx+dy)R_{n-1}(x,y)$, where $R_{n-1}$ is a homogeneous polynomial of degree $n-1$ have been studied in [12], [19] and [20].

We remark that the study that we make about centers and limit cycles which bifurcate from these centers for polynomial differential systems with homogeneous nonlinearities can be extended to differential systems defined by the sum of two quasi–homogeneous vector fields, see [16].

## 2 Statement of the results

In order to be more precise we need some preliminary notation and results. Thus, in polar coordinates $(r, \theta)$, defined by $x = r\cos\theta$, $y = r\sin\theta$, system (1) becomes

$$\dot{r} = f(\theta)r^n, \qquad \dot{\theta} = 1 + g(\theta)r^{n-1}, \tag{2}$$

where

$$\begin{aligned} f(\theta) &= \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta), \\ g(\theta) &= \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta). \end{aligned} \tag{3}$$

We remark that $f$ and $g$ are homogeneous trigonometric polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region

$$R = \{(r, \theta) : 1 + g(\theta)r^{n-1} > 0\} \tag{4}$$

the differential system (2) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{f(\theta)r^n}{1 + g(\theta)r^{n-1}}. \tag{5}$$

It is known that the periodic orbits surrounding the origin of system (2) do not intersect the curve $\dot{\theta} = 0$ (see, for instance, the Appendix of [6]). Therefore, these periodic orbits are contained in the region $R$, and consequently also they are periodic orbits of the differential equation (5).

The transformation $(r, \theta) \mapsto (\rho, \theta)$ defined by

$$\rho = \frac{r^{n-1}}{1 + g(\theta) r^{n-1}} \tag{6}$$

is a diffeomorphism from the region $R$ into its image. As far as we know the first in use this transformation was Cherkas in [11]. If we write equation (5) in the variable $\rho$, we obtain the following Abel differential equation

$$\frac{d\rho}{d\theta} = -(n-1) f(\theta) g(\theta) \rho^3 + [(n-1) f(\theta) - g'(\theta)] \rho^2 = A(\theta) \rho^3 + B(\theta) \rho^2. \tag{7}$$

These kind of differential equations appeared in the studies of Abel on the theory of elliptic functions. For more details on Abel differential equations, see [28] or [13].

Here we shall consider the Abel differential equation (7) defined on the cylinder $(\rho, \theta) \in \mathbb{R} \times \mathbb{S}^1$, where as usual $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{S}^1$ denotes the circle. Of course, only the orbits of the half–cylinder $\rho > 0$ can come from the orbits of system (5). Note that the origin of system (1) plays the role of the periodic orbit $\rho = 0$ for the Abel differential equation (7).

In short, we have proved the following result.

**Proposition 1** *The function $r = r(\theta)$ is a periodic solution of system (2) surrounding the origin if and only if $\rho(\theta) = r(\theta)^{n-1}/(1 + g(\theta) r(\theta)^{n-1})$ is a periodic solution of the Abel differential equation (7).*

We define the *greatest common divisor* of two homogeneous trigonometric polynomials $A(\theta)$ and $B(\theta)$ in the variables $\cos\theta$ and $\sin\theta$, and we write $\gcd\{A(\theta), B(\theta)\}$, as follows. First, we remove from $A(\theta)$ and $B(\theta)$ the factors of the form $(\cos\theta^2 + \sin\theta^2)^r$ if they appear, obtaining the polynomials $\bar{A}(\theta)$ and $\bar{B}(\theta)$, respectively. Since the polynomials $\bar{A}(\theta)$ and $\bar{B}(\theta)$ are homogeneous of degree $k$ and $l$ respectively, we write them as $\bar{A}(\theta) = \cos^k\theta\, a(\tan\theta)$ and $\bar{B}(\theta) = \cos^l\theta\, b(\tan\theta)$, where $a$ and $b$ are polynomials in the variable $\tan\theta$. Let $c(\tan\theta)$ be the usual greatest common divisor of $a(\tan\theta)$ and $b(\tan\theta)$ as polynomials of one variable. Then, by definition $\gcd\{A(\theta), B(\theta)\} = \cos^m\theta\, c(\tan\theta)$ if $m$ is the degree of the polynomial $c(\tan\theta)$ in the variable $\tan\theta$.

We shall say that all systems (1) define the class $HN$, we use the letters $HN$ for homogeneous nonlinearities. A system of the class $HN$ belongs to the subclass $HN_*$ if and only if its functions $A(\theta)$ and $B(\theta)$ (defined in (7)) are such that

(i) either $A(\theta)$ changes of sign, or $f(\theta) \equiv 0$, or $g(\theta) \equiv 0$ and $\int_0^{2\pi} f(\theta) d\theta = 0$;

(ii) either $B(\theta)/\gcd\{A(\theta), B(\theta)\}$ does not vanish, or $B(\theta) \equiv 0$; and

(iii) for some $a \in \mathbb{R}$ the following equality holds:

$$A'(\theta)B(\theta) - A(\theta)B'(\theta) = aB(\theta)^3. \tag{8}$$

We shall prove that all systems (1) of the subclass $HN_*$ have a center at the origin.

**Remark 2** *A system of $HN$ such that its funcions $f(\theta)$ and $g(\theta)$ satisfy (i) and*

$$\text{either} \quad \frac{g'(\theta)}{f(\theta)} = \text{constant}, \quad \text{or} \quad \frac{f(\theta)}{g'(\theta)} = \text{constant},$$

*belongs to the subclass $HN_*$.*

We consider three subfamilies in $HN_*$. The family $F$ defined by all systems of $HN$ such that their function $f(\theta) \equiv 0$ is identically zero. We note that the orbits of those systems have the polar coordinate $r$ constant.

The family $G$ defined by all systems of $HN$ such that their function $g(\theta)$ is identically zero and $\int_0^{2\pi} f(\theta)d\theta = 0$. We note that if the degree $n$ of system (1) is even, then the condition $\int_0^{2\pi} f(\theta)d\theta = 0$ is always satisfied. Also we remark that $g(\theta) \equiv 0$ means that the infinity of system (1) in the Poincaré compactification is filled of singular points. For more details on the Poincaré compactification, see for instance [25].

The family $C$ defined by all systems of $HN$ such that their function $B(\theta)$ is identically zero.

The subfamilies $F$, $G$ and $C$ were studied in [31], and there it is proved that the origin of system (1), when such system belongs to one of these three subfamilies, is a center. Moreover, in [31], we also study the limit cycles that bifurcate from the periodic orbits of these centers. Hence, in this paper we mainly deal with the $HN^* = HN_* \setminus \{F \cup G \cup C\}$.

We must mention that we have found the subclass $HN_*$ thanks to the case (f) of Abel differential equations studied in page 26 of the book [28] of Kamke, which, in fact, goes back to Chiellini [15].

**Theorem 3** *The following statements hold.*

*(a) Systems (1) in the class $HN_*$ have a center at the origin and an analytic first integral defined in its neighbourhood.*

*(b) The following systems of the class $HN_*$ have a rational first integral:*

*(b.1) The systems of the subclass $F$, $G$ and $C$.*

*(b.2) The systems of the class $HN^*$ for $n$ odd and whose $a$ (defined in (8)) satisfies $a < 1/4$, $a \neq 0$ and $\sqrt{1-4a}$ is rational.*

Theorem 3 is proved in Section 3.

In the next theorem we define a function whose simple zeros provide the hyperbolic limit cycles which bifurcate from the periodic orbits of the center of a system in $HN^*$ when we perturb it inside the class $HN$. For a definition of hyperbolic limit cycle, see for instance [40].

**Theorem 4** *We consider the system*

$$\dot{x} = -y + P_n(x,y) + \epsilon \bar{P}_n(x,y), \qquad \dot{y} = x + Q_n(x,y) + \epsilon \bar{Q}_n(x,y), \tag{9}$$

*where $P_n$, $\bar{P}_n$, $Q_n$ and $\bar{Q}_n$ are homogeneous polynomials of degree $n$. We assume that the unperturbed system (9) with $\epsilon = 0$ has a center, and that the periodic orbits of this center for the associated Abel differential equation (7) are $\varphi(\theta, z)$ such that $\varphi(0, z) = z$ with $z \in (z_1, z_2)$. Then, the perturbed system (9) for $\epsilon$ sufficiently small has a hyperbolic limit cycle surrounding the origin for each root $z$ of the function*

$$U(z) = \exp\left(\int_0^{2\pi} \alpha(\theta, z) d\theta\right) \int_0^{2\pi} \beta(\theta, z) \exp\left(-\int_0^{\theta} \alpha(s, z) ds\right) d\theta, \tag{10}$$

*where*

$$\alpha(\theta, z) = -3(n-1)f(\theta)g(\theta)\varphi(\theta, z)^2 + 2[(n-1)f(\theta) - g'(\theta)]\varphi(\theta, z),$$

$$\beta(\theta, z) = -(n-1)(f(\theta)\bar{g}(\theta) + \bar{f}(\theta)g(\theta))\varphi(\theta, z)^3 + \\ + \left[(n-1)\bar{f}(\theta) - \bar{g}'(\theta)\right] \varphi(\theta, z)^2,$$

*such that $U'(z) \neq 0$ and $z \in (z_1, z_2)$. Moreover, the limit cycle associated to such a root $z$ tends to the periodic solution $\varphi(\theta, z)$ when $\epsilon \to 0$.*

Theorem 4 is proved in Section 4.

The next proposition shows that the subclass of Hamiltonian systems of $HN$ is contained into the subclass $HN^*$.

**Proposition 5** *Any Hamiltonian system (1) belongs to the subclass $HN^*$ with $a = (n^2-1)/(4n^2)$.*

Proposition 5 is proved in Section 5.

In Section 6 we studied the centers of systems (1) for $n = 2, 3, 4$ which belongs to the class $HN_*$.

The class $HN_*$ contains some isochronous centers as it is shown in the next proposition.

**Proposition 6** *There are two subclasses of $HN_*$ which have isochronous centers.*

*(a) The subclass $G$; i.e. the systems of $HN_*$ such that $g(\theta) \equiv 0$ and $\int_0^{2\pi} f(\theta)d\theta = 0$.*

*(b) The center of system (1) in the class $C$ is isochronous if and only if $\int_0^{2\pi} g(\theta)^{2m+1}d\theta = 0$ for $m = 0, 1, 2, \ldots$. We note that this last condition always holds if $n$ is even.*

For the subclass $G$ note that $g(\theta) \equiv 0$ implies that $\dot{\theta} = 1$, and consequently the center located at the origin (see Theorem 4) is isochronous. So statement (a) of Proposition 6 is trivial. In Section 7 we prove statement (b) of Proposition 6. Moreover, in that section we also study the isochronous centers of systems (1) for $n = 2, 3, 4$ which belongs to the two classes of Proposition 6.

**Remark 7** *Let $z = x + yi \in \mathbb{C}$. Then, the system $\dot{x} = Re(iz + z^n)$, $\dot{y} = Im(iz + z^n)$, has an isochronous center at the origin, see for more details [10]. The functions $f(\theta)$ and $g(\theta)$ corresponding to this system are $\cos(n-1)\theta$ and $\sin(n-1)\theta$, respectively. Therefore, this isochronous center appears in system (1) satisfying the assumptions of Proposition 6(b). So, such a class of isochronous centers is not empty. On the other hand, all systems of the class $C$ for which $g(\theta)$ is an odd function also satisfy the assumptions of Proposition 6(b) for any $n$.*

# 3 Proof of Theorem 3

First we prove statement (a) but only for the class $HN^*$, for the classes $F$, $G$ and $C$ see [31]. We do the change of variables $(\rho, \theta) \to (v, \theta)$ defined by $\rho = B(\theta)v/A(\theta)$. For systems of $HN^*$ condition (8) is equivalent to

$$\frac{d}{d\theta}\left(\frac{A(\theta)}{B(\theta)}\right) = aB(\theta). \tag{11}$$

Using this expression, we get that in the new variables $(v, \theta)$ the Abel differential equation (7) becomes

$$\frac{dv}{d\theta} = \frac{B(\theta)^2}{A(\theta)}(v^3 + v^2 + av). \tag{12}$$

We consider this differential equation defined on the cylinder $(v, \theta) \in \mathbb{R} \times \mathbb{S}^1$.

If $a \neq 0$, using (11) we have

$$\int \frac{B(\theta)^2}{A(\theta)} d\theta = \frac{1}{a} \int \frac{aB(\theta)}{\frac{A(\theta)}{B(\theta)}} d\theta = \frac{1}{a} \int \frac{\frac{d}{d\theta}\left(\frac{A(\theta)}{B(\theta)}\right)}{\frac{A(\theta)}{B(\theta)}} d\theta = \frac{1}{a} \ln\left|\frac{A(\theta)}{B(\theta)}\right|.$$

Then, since the differential equation (12) is of separable variables, integrating it, we obtain the first integral

$$H(v,\theta) = \begin{cases} \frac{B(\theta)v}{A(\theta)\sqrt{v^2+v+a}} \exp\left[-\frac{1}{\sqrt{4a-1}} \operatorname{Arctan}\left[\frac{1+2v}{\sqrt{4a-1}}\right]\right] & \text{if } a > \frac{1}{4}. \\ \frac{B(\theta)v}{A(\theta)(1+2v)} \exp\left(\frac{1}{1+2v}\right) & \text{if } a = \frac{1}{4}. \\ \frac{B(\theta)v|\sqrt{1-4a}+1+2v|^{\frac{1}{2}\left(-1+\frac{1}{\sqrt{1-4a}}\right)}}{A(\theta)|\sqrt{1-4a}-1-2v|^{\frac{1}{2}\left(1+\frac{1}{\sqrt{1-4a}}\right)}} & \text{if } a < \frac{1}{4} \text{ and } a \neq 0. \\ \frac{1+v}{v} \exp\left(-\frac{1}{v} - \int \frac{B(\theta)^2}{A(\theta)} d\theta\right) & \text{if } a = 0. \end{cases}$$

For $a \neq 0$ and using the first part of assumption (ii) for the class $HN_*$, it is easy to check that the first integral $H(v,\theta)$ is defined on a neighbourhood of the periodic orbit $v = 0$ of the differential equation (12). Therefore, the Abel differential equation (7) has also a first integral $\bar{H}(\rho,\theta)$ defined on a neighbourhood of the periodic orbit $\rho = 0$. Therefore, the Poincaré map of the Abel differential equation (7) in a neighbourhood of the periodic orbit $\rho = 0$ is the identity. Hence, the origin of system (1) is a center. This completes the proof of statement (a).

Since for the class $F$ we have that $\dot{r} = 0$, the polynomial $x^2 + y^2$ is a first integral. So statement (b.1) follows.

From the Abel differential equation (7) and for the class $G$ it is easy to obtain that

$$H(\rho,\theta) = (n-1) \int f(\theta) d\theta + \frac{1}{\rho}$$

is a first integral. Going back to the cartesian variables, statement (b.1) follows for the class $G$.

Again from the Abel differential equation (7) and for the class $C$ it is easy to obtain that

$$H(\rho,\theta) = \frac{1}{\rho^2} - g(\theta)^2,$$

is a first integral. Going back to the cartesian variables, statement (b.1) follows for the class $C$.

Finally, from the expression of the first integral $H(v,\theta)$ for $a<1/4$ and $a\neq 0$, and since $\sqrt{1-4a}$ is rational, it follows easily statement (b.2) when $n$ is odd. In short, Theorem 3 is proved.

# 4 Proof of Theorem 4

We need the following lemma, which essentially comes from [31].

**Lemma 8** *We assume that the unperturbed system (9) with $\epsilon=0$ has a center, and that the periodic orbits of this center for the associated Abel differential equation (7) are $\varphi(\theta,z)$ such that $\varphi(0,z)=z$ with $z\in(z_1,z_2)$. Let $\phi(\theta,z,\epsilon)$ be the solution of the perturbed system (9) such that $\phi(0,z,\epsilon)=z$, and let $u(\theta,z)=(\partial\phi/\partial\epsilon)(\theta,z,0)$. Then, the perturbed system (9) for $\epsilon$ sufficiently small has a hyperbolic limit cycle surrounding the origin for each root $z$ of the function $U(z)=u(2\pi,z)$ such that $U'(z)\neq 0$ and $z\in(z_1,z_2)$. Moreover, the limit cycle associated to such a root $z$ tends to the periodic solution $\varphi(\theta,z)$ when $\epsilon\to 0$.*

*Proof*: The Abel differential equation (7) for system (9) is

$$\begin{aligned}\frac{d\rho}{d\theta} &= -(n-1)(f+\epsilon\bar{f})(g+\epsilon\bar{g})\rho^3+\left[(n-1)(f+\epsilon\bar{f})-(g'+\epsilon\bar{g}')\right]\rho^2\\ &= -(n-1)fg\rho^3+\left[(n-1)f-g'\right]\rho^2-\\ &\qquad \epsilon\left\{(n-1)(f\bar{g}+\bar{f}g)\rho^3-\left[(n-1)\bar{f}-\bar{g}'\right]\rho^2\right\}+O(\epsilon^2).\end{aligned} \tag{13}$$

Therefore, by the theorem of analytic dependence on initial conditions and parameters for the solutions of an analytic differential equation, we have that for $\epsilon$ sufficiently small the solution $\phi(\theta,z,\epsilon)$ of system (13) such that $\phi(0,z,\epsilon)=z$ can be written in the form

$$\phi(\theta,z,\epsilon)=\varphi(\theta,z)+\epsilon u(\theta,z)+\epsilon^2R(\theta,z,\epsilon), \tag{14}$$

where

$$\varphi(\theta,z)=\phi(\theta,z,0),\quad u(\theta,z)=\frac{\partial\phi}{\partial\epsilon}(\theta,z,0),\quad \varphi(0,z)=z,\quad u(0,z)=0.$$

We have that $U(z)=u(2\pi,z)$, and we assume that $U(z)$ has a root $z=z_0$ for which $U'(z_0)\neq 0$. Then, by the Implicit Function Theorem, for some $\epsilon_0>0$ and all $\epsilon$ with $|\epsilon|<\epsilon_0$ there exists a point $\bar{z}=\bar{z}(\epsilon)$ such that $\bar{z}(0)=z_0$ and $U(\bar{z})+\epsilon R(2\pi,\bar{z},\epsilon)=0$. So, from (14) we obtain

$$\phi(2\pi,\bar{z},\epsilon)-\bar{z}=\epsilon[U(\bar{z})+\epsilon R(2\pi,\bar{z},\epsilon)]=0.$$

Hence, the solution $\phi(\theta, \bar{z}, \epsilon)$ is periodic of period $2\pi$. Moreover, since $U'(z_0) \neq 0$, this periodic solution provides a hyperbolic limit cycle. ∎

Now we shall prove Theorem 4. By Lemma 8 we only need to show that the function $U(z)$ has the expression given in (10). First we compute $u(\theta, z)$ in terms of the coefficients of $\rho^2$ and $\rho^3$ in equation (13). For the solution $\phi(\theta, z, \epsilon)$ of system (13), we have

$$\begin{aligned} \dot{\phi} &= -(n-1)fg\phi^3 + [(n-1)f - g']\,\phi^2 - \\ &\quad \epsilon\left\{(n-1)(f\bar{g} + \bar{f}g)\phi^3 - \left[(n-1)\bar{f} - \bar{g}'\right]\phi^2\right\} + O(\epsilon^2). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial\epsilon}\right) = \frac{\partial\dot{\phi}}{\partial\epsilon} &= -3(n-1)fg\phi^2\frac{\partial\phi}{\partial\epsilon} + 2\,[(n-1)f - g']\,\phi\frac{\partial\phi}{\partial\epsilon} - \\ &\quad (n-1)(f\bar{g} + \bar{f}g)\phi^3 + \left[(n-1)\bar{f} - \bar{g}'\right]\phi^2 + O(\epsilon). \end{aligned} \tag{15}$$

From (14) and (15), it follows for $\epsilon = 0$ that

$$\begin{aligned} \frac{\partial u}{\partial\theta} = \dot{u} &= \left(-3(n-1)fg\varphi^2 + 2\,[(n-1)f - g']\,\varphi\right)u - \\ &\quad (n-1)(f\bar{g} + \bar{f}g)\varphi^3 + \left[(n-1)\bar{f} - \bar{g}'\right]\varphi^2. \end{aligned}$$

or equivalently

$$\dot{u} = \alpha(\theta, z)u + \beta(\theta, z).$$

Integrating this linear differential equation we obtain $u(\theta, z)$, and evaluating it at $\theta = 2\pi$ we get the expression (10) for the function $U(z)$. This completes the proof of Theorem 4.

# 5 Proof of Proposition 5

Suppose that we have a system (1) which is Hamiltonian; i.e.

$$\frac{\partial P_n(x,y)}{\partial x} + \frac{\partial Q_n(x,y)}{\partial y} = 0. \tag{16}$$

In this section we will write $P_n(x, y)$ and $Q_n(x, y)$ simply as $P(x, y)$ and $Q(x, y)$; and as usual we denote by $P_x$ the partial derivative with respect to the variable $x$ of the function $P$. Similar notation for $P_y$.

We claim that if $f(\theta)$ and $g(\theta)$ are the functions (3) associated to a Hamiltonian system (1), then $g'(\theta) = -(n+1)f(\theta)$. So, by Remark 2, the Hamiltonian systems (1) are contained into the class $HN^*$, and the proof of Proposition 5 is completed. The claim follows from the next equalities, where $P$, $P_x$ and $P_y$ will denote $P(\cos\theta, \sin\theta)$, $P_x(\cos\theta, \sin\theta)$ and $P_y(\cos\theta, \sin\theta)$, respectively. Of course, we use the same notation for $Q$. We have

$$
\begin{aligned}
g'(\theta) &= -\sin\theta Q + \cos\theta[-\sin\theta Q_x + \cos\theta Q_y] - \cos\theta P \\
&\qquad - \sin\theta[-\sin\theta P_x + \cos\theta P_y] \\
&= -\sin\theta Q - \sin\theta\cos\theta Q_x + [1-\sin^2\theta]Q_y - \cos\theta P \\
&\qquad + [1-\cos^2\theta]P_x - \sin\theta\cos\theta P_y \\
&= -\sin\theta Q - \sin\theta[\cos\theta Q_x + \sin\theta Q_y] - \cos\theta P \\
&\qquad - \cos\theta[\cos\theta P_x + \sin\theta P_y] \\
&= -\sin\theta Q - n\sin\theta Q - \cos\theta P - n\cos\theta P \\
&= -(n+1)[\sin\theta Q + \cos\theta P] \\
&= -(n+1)f(\theta),
\end{aligned}
$$

where in the third equality we have used (16), and in the fourth we have applied the Euler theorem for homogeneous functions to the homogeneous polynomials $P(x, y)$ and $Q(x, y)$ of degree $n$.

# 6 Centers

The centers of the polynomial differential systems (1) have been studied for the degrees $n = 2, 3, 4, 5$. They are completely classified for $n = 2, 3$, and partially for $n = 4, 5$. In this section we recall these results for $n = 2, 3, 4$, and we study what of these centers belong to our class of centers $HN_*$.

For $n = 2$ system (1) corresponds to a quadratic polynomial differential system with a weak focus at the origin, see for more details [49], which in polar coordinates can be written as system (2) with $n = 2$ and

$$
\begin{aligned}
f(\theta) &= R_3\cos(3\theta+\theta_3) + R_1\cos(\theta+\theta_1), \\
g(\theta) &= -R_3\sin(3\theta+\theta_3) + r_1\sin(\theta+\bar\theta_1).
\end{aligned}
$$

**Theorem 9** *The origin of system (1) for $n = 2$ is a center if and only if one of the following statements holds.*

*(i)* $\bar\theta_1 = \theta_1$, $r_1 = -3R_1$ *(Hamiltonian centers).*

*(ii)* $\bar\theta_1 = \theta_1$, $\theta_3 = 3\theta_1$ *(time–reversible).*

*(iii)* $\bar{\theta}_1 = \theta_1$, $R_1 = 3r_1$, $|R_3| = |r_1|$ *(critical centers).*

*(iv)* $\bar{\theta}_1 = \theta_1$ *(mod.* $\pi$*),* $r_1 = R_1$ *(straight line centers).*

Theorem 9 is due to Kapteyn [29, 30], and its presentation in polar coordinates can be found in [7].

In the next corollary we present the quadratic centers which belong to our class $HN_*$.

**Corollary 10** *The following statements hold for the centers of system (1) with* $n = 2$.

*(i) The Hamiltonian centers of Theorem 9(i) satisfy* $g'(\theta) = -3f(\theta)$*, and consequently they belong to the class* $HN^*$ *with* $a = 3/16$.

*(ii) The centers of statement (ii) of Theorem 9 which do not belong to any other statement of that theorem and belong to the class* $HN_*$ *are:*

*(ii.1)* $r_1 \neq R_1$ *and* $R_3 = 0$ *with* $a = -r_1R_1/(r_1 - R_1)^2$.

*(ii.2)* $r_1 = R_1$ *and* $R_3 = 0$.

*(ii.3)* $3r_1 + R_1 - 8R_3 = 0$ *with* $a = 3/16$.

*(ii.4)* $R_3 = R_1 = 0$ *with* $f(\theta) = 0$ *and* $a = 0$.

*(ii.5)* $R_3 = r_1 = 0$ *with* $g(\theta) = 0$ *and* $a = 0$.

*(iii) There are no centers of Theorem 9(iii) which belong to* $HN_*$.

*(iv) The centers of statement (iv) of Theorem 9 which do not belong to any other statement of that theorem and belong to the class* $HN^*$ *are either* $R_1 = -2R_3$ *and* $\theta_3 = 3\theta_1 + \pi$ *with* $a = 3/16$*; or* $R_3 = 0$.

*Proof*: It follows from direct computations. ■

For $n = 3$ system (1) in polar coordinates can be written as system (2) with $n = 3$ and

$$\begin{aligned} f(\theta) &= R_4\cos(4\theta + \theta_4) + R_2\cos(2\theta + \theta_2) + R_0, \\ g(\theta) &= -R_4\sin(4\theta + \theta_4) + r_2\sin(2\theta + \bar{\theta}_2) + r_0. \end{aligned}$$

**Theorem 11** *The origin of system (1) for* $n = 3$ *is a center if and only if one of the following statements holds.*

*(i)* $R_0 = 0$, $\bar{\theta}_2 = \theta_2$, $r_2 = -2R_2$ *(Hamiltonian centers).*

*(ii)* $R_0 = 0$, $\bar{\theta}_2 = \theta_2$, $\theta_4 = 2\theta_2 + \pi/2$ *(time–reversible centers).*

(iii) $R_0 = 0$, $\bar{\theta}_2 = \theta_2$, $r_0 = 0$, $R_2 = 2r_2$, $|R_4| = |R_2|$ *(critical centers). They have a rational first integral.*

Theorem 11 is due to Sibirskii [48], and its presentation in polar coordinates can be found in [7].

In the next corollary we present the centers of Theorem 11 which belong to our class $HN_*$.

**Corollary 12** *The following statements hold for the centers of system (1) with* $n = 3$.

(i) *The Hamiltonian centers of Theorem 11(i) satisfy* $g'(\theta) = -4f(\theta)$, *and consequently they belong to the class* $HN^*$ *with* $a = 2/9$.

(ii) *The time–reversible centers of Theorem 11(ii) which are not Hamiltonian and belong to the class* $HN^*$ *are:*

(ii.1) $r_2 \neq R_2$ *and* $R_4 = 0$ *with* $a = -r_2R_2/(r_2 - R_2)^2$.

(ii.2) $r_2 = R_2$ *and* $R_4 = 0$.

(ii.3) $2r_2^2 - r_2R_2 - R_2^2 - 18r_0R_4 + 18R_4^2 = 0$ *with* $a = 2/9$.

(ii.4) $R_4 = R_2 = R_0 = 0$ *with* $f(\theta) = 0$ *and* $a = 0$.

(ii.5) $R_4 = r_2 = r_0 = R_0 = 0$ *with* $g(\theta) = 0$ *and* $a = 0$.

(iii) *There are no centers of Theorem 11(iii) which belong to* $HN_*$.

*Proof*: Again it follows from direct computations. ∎

For $n = 4$ system (1) in polar coordinates can be written as system (2) with $n = 4$ and

$$
\begin{aligned}
f(\theta) &= R_5\cos(5\theta + \theta_5) + R_3\cos(3\theta + \theta_3) + R_1\cos(\theta + \theta_1),\\
g(\theta) &= -R_5\sin(5\theta + \theta_5) + r_3\sin(3\theta + \bar{\theta}_3) + r_1\sin(\theta + \bar{\theta}_1).
\end{aligned}
$$

**Theorem 13** *The origin of system (1) for* $n = 4$ *is a center if one of the following statements holds.*

(i) $\bar{\theta}_1 = \theta_1$, $\bar{\theta}_3 = \theta_3$, $r_3 = -5R_3/3$ *and* $r_1 = -5R_1$ *(Hamiltonian centers).*

(ii) $\bar{\theta}_1 = \theta_1$, $\bar{\theta}_3 = \theta_3$, $\theta_5 = 5\theta_1$ *and* $\theta_3 = 3\theta_1$ *(time–reversible centers).*

(iii) $\bar{\theta}_1 = \theta_1$, $\bar{\theta}_3 = \theta_3$, $\theta_5 = 2\theta_1 + \theta_3$, $r_3 = 3R_3$, $r_1 = 2R_1$, $R_5 = R_3$ *and* $|R_1| = 2|R_3|$.

(iv) $\bar{\theta}_1 = \theta_1$, $\bar{\theta}_3 = \theta_3$, $R_5 = 0$, *and* *(iv.1)* *either* $R_3 r_1 - 3r_3 R_1 = 0$,

*(iv.2)* *or* $r_1 = 3R_1$ *and* $r_3 = -3R_3$,

*(iv.3)* *or* $r_1 = R_1 = 3|R_3|$ *and* $r_3 = -3R_3$.

Theorem 13 is due to Chavarriga and Giné [7].

In the next corollary we present the centers of Theorem 13 which belong to our class $HN_*$.

**Corollary 14** *The following statements hold for the centers of system (1) with* $n = 4$.

*(i) The Hamiltonian centers of Theorem 13(i) satisfy* $g'(\theta) = -5f(\theta)$, *and consequently they belong to the class* $HN^*$ *with* $a = 15/64$.

*(ii) The time–reversible centers of Theorem 13(ii) which are neither Hamiltonian nor of Theorem 13(iv) and belong to the class* $HN^*$ *are:*

*(ii.1)* $16r_1 + 23r_3 - 15R_3 = 0$, $16R_1 + 5r_3 + 19R_3 = 0$ *and* $16R_5 - 5r_3 - 3R_3 = 0$ *with* $a = 15/64$.

*(ii.2)* $R_5 = 0$, $r_3 + 3R_3 = 0$ *and* $r_1 + R_1 - 8R_3 = 0$ *with* $a = 3/16$.

*(iii) There are no centers of Theorem 13(iii) which belong to* $HN_*$.

*(iv) The centers of Theorem 13(iv) which belong to* $HN_*$ *are*

*(iv.1.1)* $r_3 \neq R_3$ *with* $a = -r_3R_3/(r_3 - R_3)^2$.

*(iv.1.2)* $r_3 = R_3$ *and* $r_1 = 3R_1$.

*(iv.1.3)* $R_3 = 0$ *and* $r_3 = 0$ *with* $a = -3r_1R_1/(r_1 - 3R_1)^2$.

*(iv.1.4)* $R_3 = 0$ *and* $R_1 = 0$ *with* $f(\theta) = 0$ *and* $a = 0$.

*(iv.1.5)* $r_3 = r_1 = 0$ *with* $g(\theta) = 0$ *and* $a = 0$.

*(iv.2.1)* $R_1 = -2R_3$ *and* $\theta_3 = 3\theta_1 + \pi$ *with* $a = 3/16$.

*(iv.3) in this case there are no centers in* $HN_*$.

*Proof*: It follows from direct computations. ∎

Readers interested in the centers for $n = 5$ can look the paper [8].

## 7 Isochronous centers

First we shall end the proof of Proposition 6.
*Proof of Proposition 6(b)*: From (2) we have that

$$dt = \frac{d\theta}{1 + g(\theta) r^{n-1}} = (1 - g(\theta)\rho) d\theta,$$

where we have used the transformation (6). Now we integrate this equality along a periodic orbit $\rho(\theta, z)$ of period $T$ for a center of the class $C$ and we get

$$T = \int_0^T dt = \int_0^{2\pi} (1 - g(\theta)\rho(\theta, z)) d\theta = 2\pi - \int_0^{2\pi} g(\theta)\rho(\theta, z) d\theta. \quad (17)$$

As usual $\rho(\theta, z)$ denotes the solution such that $\rho(0, z) = z$.

In Proposition 2(c) of [31] it is proved that systems of class $C$ have a center at the origin, and that in coordinates $(\rho, \theta)$ their periodic solutions contained in the region (4) are given by

$$\rho(\theta, z) = \frac{z}{\sqrt{1 + 2(n-1)G(\theta) z^2}}, \quad (18)$$

where $z \in \left(0, \min_\theta |G(\theta)|^{-1}\right)$ and $G(\theta) = \int_0^\theta f(s) g(s) ds$. Therefore, since $f(\theta) = g'(\theta)/(n-1)$, we have that

$$\rho(\theta, z) = \frac{z}{\sqrt{1 + z^2 (g(\theta)^2 - g(0)^2)}}.$$

Therefore, for $z$ sufficiently small, from (18) and using the assumptions of Proposition 6(b) we can write

$$\begin{aligned} \int_0^{2\pi} g(\theta)\rho(\theta, z) d\theta &= \int_0^{2\pi} g(\theta) z \left( \sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} z^{2m} (g(\theta)^2 - g(0)^2)^m \right) d\theta \\ &= \sum_{m=0}^{\infty} a_m p_m(z) \int_0^{2\pi} g(\theta)^{2m+1} d\theta = 0, \end{aligned}$$

where $a_m$ and $p_m(z)$ are adequate constants and polynomials, respectively. Hence, from (17) we obtain that the period $T$ of the periodic orbits of the center sufficiently close to the origin is constant and equal to $2\pi$. By analyticity, the period is contant in the whole period annulus surrounding the origin. So, the center is isochronous. ∎

The isochronous centers of the polynomial differential systems (1) have been studied for the degrees $n = 2, 3, 4, 5$. They are completely classified for $n = 2, 3$, and partially for $n = 4, 5$. In this section we recall these results for $n = 2, 3, 4$, and we study which of these centers belong to the classes of isochronous centers given in Proposition 6. We use the notation in polar coordinates introduced in Section 6.

The next theorem is due to Loud [37].

**Theorem 15** *The origin of system (1) for $n = 2$ is an isochronous center if and only if after a linear change of coordinates and a rescaling of the time variable it can be written as one of the following systems:*

*(i)* $\dot{r} = r^2 \cos\theta$, $\dot{\theta} = 1 + r\sin\theta$;

*(ii)* $\dot{r} = r^2 \cos\theta$, $\dot{\theta} = 1$;

*(iii)* $\dot{r} = r^2(\cos 3\theta - \frac{7}{3}\cos\theta)$, $\dot{\theta} = 1 - r(\sin 3\theta + \sin\theta)$;

*(iv)* $\dot{r} = r^2(\cos 3\theta + \frac{13}{3}\cos\theta)$, $\dot{\theta} = 1 - r(\sin 3\theta - \frac{1}{3}\sin\theta)$.

We note that the isochronous centers of statements (i) and (ii) of Theorem 15 are of the kind given in statements (b) and (a) of Proposition 6, respectively.

The following theorem was proved by Pleshkan [43].

**Theorem 16** *The origin of system (1) for $n = 3$ is an isochronous center if and only if after a linear change of coordinates and a rescaling of the time variable it can be written as one of the following systems:*

*(i)* $\dot{r} = r^3 \cos 2\theta$, $\dot{\theta} = 1 + r^2 \sin 2\theta$;

*(ii)* $\dot{r} = r^3 \cos 2\theta$, $\dot{\theta} = 1$;

*(iii)* $\dot{r} = r^3(-\cos 4\theta + \frac{5}{2}\cos 2\theta)$, $\dot{\theta} = 1 - r^2(\sin 4\theta + \sin 2\theta)$;

*(iv)* $\dot{r} = r^3(\cos 4\theta - \frac{5}{2}\cos 2\theta)$, $\dot{\theta} = 1 + r^2(\sin 4\theta + \sin 2\theta)$.

We note that the isochronous centers of statements (i) and (ii) of Theorem 16 are of the kind given in statements (b) and (a) of Proposition 6, respectively.

The following theorem due to Chavarriga, Giné and García [9] studies the reversible isochronous centers of system (1) for $n = 4$.

**Theorem 17** *A necessary condition in order that the origin of system (1) to be a reversible isochronous center is that the system after a linear change of variables and a rescaling of the time variable can be written in one of the following forms:*

*(i)* $\dot{r} = r^4(R_3 \cos 3\theta + R_1 \cos\theta)$, $\dot{\theta} = 1$;

*(ii)* $\dot{r} = r^4(\frac{1}{3}\cos 3\theta - \frac{7}{9}\cos\theta)$, $\dot{\theta} = 1 + r^3(\sin 3\theta - \frac{1}{3}\sin\theta)$;

*(iii)* $\dot{r} = -r^4(\frac{1}{3}\cos 3\theta + \frac{13}{9}\cos\theta)$, $\dot{\theta} = 1 + r^3(\sin 3\theta - \frac{1}{3}\sin\theta)$;

*(iv)* $\dot{r} = r^4(-\frac{1}{3}\cos 3\theta + \frac{7}{9}\cos\theta)$, $\dot{\theta} = 1 + r^3(\sin 3\theta + \sin\theta)$;

*(v)* $\dot{r} = r^4(R_3 \cos 3\theta + R_1 \cos\theta)$, $\dot{\theta} = 1 + r^3(R_3 \sin 3\theta + 3R_1 \sin\theta)$;

*(vi)* $\dot{r} = r^4(-\cos 3\theta + \cos\theta)$, $\dot{\theta} = 1 + r^3(\sin 3\theta + \sin\theta)$;

*(vii)* $\dot{r} = r^4(\cos 5\theta + \frac{48+5r_3}{45}\cos 3\theta - \frac{1197-30r_3+25r_3}{540}\cos\theta)$,
$\dot{\theta} = 1 + r^3(-\sin 5\theta + r_3 \sin 3\theta - \frac{9-6r_3+5r_3^2}{36}\sin\theta)$.

*The condition is sufficient for systems (i)–(vi).*

In [9] it is written that for system (vii) it is unknown how to prove the isocronicity of the center at the origin for all values of $r_3$.

We note that the isochronous centers of statements (i) and (v) of Theorem 17 are of the kind given in statements (a) and (b) of Proposition 6, respectively.

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